On supersingular representations of p-adic reductive groups

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Abstract. If G is a p-adic reductive group with connected center we study the universal spherical Hecke module M of G associated with a weight V in characteristic p. We show that the space of invariants of M under a fixed pro-p Iwahori subgroup of G is free over the spherical Hecke algebra and that its rank is equal to the order of the Weyl group. Our proof relies on an acyclicity result for coefficient systems of representations of finite groups of Lie type in natural characteristic. We then study the action of the spherical Hecke algebra on suitable spaces of coinvariants of the universal spherical Hecke module. For the general linear group we obtain that any supersingular quotient of M is supercuspidal and has trivial smooth dual.

Contents

1 Universal spherical Hecke modules

Let F be a non-archimedean local field with valuation val, valuation ring $\mathfrak{o}$ and residue class field k. Let p and q denote the characteristic and the cardinality of k, respectively. Throughout the article E will denote an algebraically closed field containing k.

Let $\mathbb{G}$ be an F-split connected reductive F-group with connected center $\mathbb{Z}$, $\mathbb{T}$ a maximal F-split F-torus of $\mathbb{G}$, $\mathbb{B}$ a Borel F-subgroup of $\mathbb{G}$ containing $\mathbb{T}$ and $\mathbb{N}$ the unipotent radical of $\mathbb{B}$. Let Φ denote the root system of $(\mathbb{G}, \mathbb{T})$ and denote by Φ^+ the set of positive roots corresponding to $\mathbb{B}$. Let Δ denote the set of simple roots inside Φ^+, and let W denote the Weyl group of Φ.

Let x be a fixed hyperspecial point of the Bruhat-Tits building X of $G := \mathbb{G}(F)$ which is contained in the apartment corresponding to $T := \mathbb{T}(F)$ (cf.

[20], §1.10.2). By [20], §3.8.1, there is a smooth $\mathfrak{o}$-group scheme $\mathcal{G} = \mathcal{G}_x$ whose generic fibre $\mathcal{G}_F$ is isomorphic to $\mathbb{G}$ and whose special fibre $\mathcal{G}_k$ is a k-split connected reductive k-group. Identifying the F-groups $\mathbb{G}$ and $\mathcal{G}_F$ we may view $K := \mathcal{G}(\mathfrak{o})$ as a maximal compact open subgroup of G (cf. [20], §3.2). For the natural action of G on X we then have

$$K = \{g \in G \mid g \cdot x = x\}$$

by [20], §3.4.1.

There is a closed $\mathfrak{o}$-subgroup scheme $\mathcal{B}$ of $\mathcal{G}$ with $\mathcal{B}_F = \mathbb{B}$ and whose special fibre $\mathcal{B}_k$ is a Borel k-subgroup of $\mathcal{G}_k$. As an $\mathfrak{o}$-scheme $\mathcal{B} \cong \mathcal{N} \times_{\mathfrak{o}} \mathcal{T}$ with closed $\mathfrak{o}$-subgroup schemes $\mathcal{N}$ and $\mathcal{T}$ of $\mathcal{G}$ such that $\mathcal{N}_F = \mathbb{N}$ and $\mathcal{T}_F = \mathbb{T}$. Further, the special fibers $\mathcal{T}_k$ and $\mathcal{N}_k$ are, respectively, a maximal k-split k-torus of $\mathcal{G}_k$ and the unipotent radical of $\mathcal{B}_k$. The root systems $\Phi = \Phi(\mathbb{G}, \mathbb{T})$ and $\Phi(\mathcal{G}_k, \mathcal{T}_k)$ are isomorphic and will henceforth be identified.

By construction, the $\mathfrak{o}$-scheme $\mathcal{N}$ is the direct product of one dimensional $\mathfrak{o}$-subgroup schemes $\mathcal{N}_\alpha$, $\alpha \in \Phi$, whose fibres over F and k are the root groups of $\mathbb{N}$ and $\mathcal{N}_k$, respectively, corresponding to α. The former will be denoted by $\mathbb{N}_\alpha$.

For any subset I of Δ we denote by W_I the subgroup of W generated by the simple reflections s_α with $\alpha \in I$. Further, $\mathbb{P}_I$ (resp. $\mathcal{P}_I$) denotes the parabolic subgroup of $\mathbb{G}$ (resp. $\mathcal{G}_k$) generated by $\mathbb{B}$ (resp. $\mathcal{B}_k$) and W_I (cf. [2], §14.17). Let $\mathbb{N}_I$ (resp. $\mathcal{N}_I$) denote the unipotent radical of $\mathbb{P}_I$ (resp. $\mathcal{P}_I$). Let $\mathbb{M}_I$ (resp. $\mathcal{M}_I$) denote the Levi subgroup of $\mathbb{P}_I$ (resp. $\mathcal{P}_I$) containing $\mathbb{T}$ (resp. $\mathcal{T}_k$).

Working with the Borel subgroup $\overline{\mathbb{B}}$ of $\mathbb{G}$ corresponding to the basis $-\Delta$ of Φ, the opposite versions of the various group schemes above will be marked with a bar on top, as well.

Let $X_*(\mathbb{T})$ and $X^*(\mathbb{T})$ denote the group of cocharacters and characters of $\mathbb{T}$, respectively. They are in perfect duality with respect to a natural pairing $\langle \cdot, \cdot \rangle$. There is an epimorphism $T \to X_*(\mathbb{T})$ of abelian groups which is characterized by the condition that $\langle t, \chi \rangle = val(\chi(t))$ for all $t \in T$ and $\chi \in X^*(\mathbb{T})$. Its kernel is the maximal compact subgroup $T_0 = T \cap K$ of T.

We let

$$X_*(\mathbb{T})^+ := \{\lambda \in X_*(\mathbb{T}) \mid \langle \lambda, \alpha \rangle \geq 0 \text{ for all } \alpha \in \Delta\}$$

denote the monoid of dominant cocharacters of $\mathbb{T}$ with respect to Δ and denote by T^+ its preimage in T. We have $T^+ = \{t \in T \mid t\mathcal{N}(\mathfrak{o})t^{-1} \subseteq \mathcal{N}(\mathfrak{o})\} = \{t \in T \mid t\overline{\mathcal{N}}(\mathfrak{o})t^{-1} \supseteq \overline{\mathcal{N}}(\mathfrak{o})\}$, as can be seen by choosing a T-equivariant isomorphism of group schemes $\mathbb{G}_a \to \mathbb{N}_\alpha$ for any root $\alpha \in \Phi$ (cf. [2], Remark 14.4). Similarly, $X_*(\mathbb{T})^-$ denotes the monoid of antidominant cocharacters of $\mathbb{T}$ with preimage $T^- = \{t \in T \mid t\mathcal{N}(\mathfrak{o})t^{-1} \supseteq \mathcal{N}(\mathfrak{o})\} = \{t \in T \mid t\overline{\mathcal{N}}(\mathfrak{o})t^{-1} \subseteq \overline{\mathcal{N}}(\mathfrak{o})\}$ in T.

The F-group $\mathbb{G}/\mathbb{Z}$ is connected and semisimple of adjoint type with $\mathbb{T}/\mathbb{Z}$ as a maximal F-split torus. There is a natural identification $\Phi = \Phi(\mathbb{G}, \mathbb{T}) = \Phi(\mathbb{G}/\mathbb{Z}, \mathbb{T}/\mathbb{Z})$ of root systems. The monoid $X_*(\mathbb{T}/\mathbb{Z})^+$ of dominant cocharacters of $\mathbb{T}/\mathbb{Z}$ is freely generated by the fundamental dominant coweights $(\lambda_\alpha)_{\alpha\in\Delta}$, i.e. the fundamental dominant weights of the dual root system $\check{\Phi}$.

Since $\mathbb{Z}$ is connected there is an F-subtorus $\mathbb{T}'$ of $\mathbb{T}$ such that the multiplication map $\mathbb{Z} \times \mathbb{T}' \to \mathbb{T}$ is an isomorphism (cf. [2], Corollary 8.5). It induces an isomorphism $X_*(\mathbb{T}') \cong X_*(\mathbb{T}/\mathbb{Z})$ of groups. Consequently, any fundamental dominant coweight λ_α, $\alpha \in \Delta$, can be represented by an element $t_\alpha \in T' := \mathbb{T}'(F)$ such that $t_\alpha \in \bigcap_{\beta\in\Delta\setminus\{\alpha\}} \ker(\beta)$. Fixing such representatives once and for all, any element $t \in T$ can be written as $t = z \cdot t_0' \cdot \prod_{\alpha\in\Delta} t_\alpha^{n_\alpha}$ with uniquely determined elements $z \in Z := \mathbb{Z}(F)$, $t_0' \in T' \cap K$ and integers n_α, $\alpha \in \Delta$. Choosing $t_0' = 1$ and fixing a set of representatives of Z/Z_0 in Z, $Z_0 := Z \cap K$, we obtain a fixed set of representatives of $X_*(\mathbb{T})^+$ in T^+.

Remark 1.1. The fixed representatives chosen above have the following important property. If I is a subset of Δ and if $t = z \cdot \prod_{\alpha\in I} t_\alpha^{n_\alpha}$ then the centralizer of t in G contains the groups $\mathbb{N}_\beta(F)$ for any root β which is a linear combination of the elements of $\Delta \setminus I$. This follows from $t \in \cap_{\beta\in\Delta\setminus I} \ker(\beta)$ and [2], Remark 14.4.

Let V be an E-vector space carrying an irreducible E-linear representation of $\mathcal{G}(k)$. We view V as a representation of K via inflation along the natural reduction homomorphism

$$red : K = \mathcal{G}(\mathfrak{o}) \longrightarrow \mathcal{G}(k).$$

Let $M := \mathrm{ind}_K^G(V)$ denote the E-vector space of all compactly supported maps $f : G \to V$ which satisfy $f(gh) = h^{-1}f(g)$ for all $g \in G$ and $h \in K$. If g runs through the elements of G and if v runs through an E-basis of V then a basis of the E-vector space $\mathrm{ind}_K^G(V)$ is given by the collection of functions $[g, v]$ with support gK and value v at g. The space $M = \mathrm{ind}_K^G(V)$ carries an E-linear smooth action of G via $(g \cdot f)(g') := f(g^{-1}g')$ for all $g, g' \in G$.

The E-algebra

$$\mathcal{H} = \mathcal{H}(G, K; V) := \mathrm{End}_G(M) = \mathrm{End}_G(\mathrm{ind}_K^G(V))$$

of G-equivariant E-linear endomorphisms of $\mathrm{ind}_K^G(V)$ is called the *spherical Hecke algebra* of G associated with V. Note that M is a module over $\mathcal{H}$ in a natural way.

We let $\mathcal{H}_0$ denote the image of the natural homomorphism $E[Z] \to \mathcal{H}$ of E-algebras. Note that $\mathrm{End}_{\mathcal{G}(k)}(V)$ is a skew field which is of finite dimension over E. Since E is algebraically closed, we have $\mathrm{End}_{\mathcal{G}(k)}(V) = E$. As a consequence, the group $Z_0 = Z \cap K$ acts on V via a character $\zeta_V : Z_0 \to E^\times$.

For any dominant cocharacter $\lambda \in X_*(\mathbb{T})^+$ there is a specific element $T_\lambda \in \mathcal{H}$, the so-called *Hecke operator associated with* λ (cf. [11], proof of Theorem 1.2). In order to define it let

$$\Delta(\lambda) := \{\alpha \in \Delta \mid \langle \lambda, \alpha \rangle > 0\}.$$

We set $\mathcal{P}_\lambda := \mathcal{P}_{\Delta \setminus \Delta(\lambda)}$ and $\mathcal{P}_{-\lambda} := \overline{\mathcal{P}}_{\Delta \setminus \Delta(\lambda)}$ with common Levi subgroup $\mathcal{M}_\lambda = \mathcal{M}_{-\lambda} := \mathcal{M}_{\Delta \setminus \Delta(\lambda)}$.

The unipotent radicals of $\mathcal{P}_\lambda$ and $\mathcal{P}_{-\lambda}$ extend to closed $\mathfrak{o}$-subgroup schemes $\mathcal{N}_\lambda$ and $\mathcal{N}_{-\lambda}$ of $\mathcal{N}$ and $\overline{\mathcal{N}}$, respectively. As an $\mathfrak{o}$-scheme $\mathcal{N}_\lambda$ (resp. $\mathcal{N}_{-\lambda}$) is the direct product of the root groups $\mathcal{N}_\alpha$ (resp. $\mathcal{N}_{-\alpha}$) with $\alpha \in \Phi^+ \setminus [\Delta \setminus \Delta(\lambda)]^+$. Here $[\Delta \setminus \Delta(\lambda)]^+$ denotes the set of all positive roots which are linear combinations of the elements of $\Delta \setminus \Delta(\lambda)$.

According to [11], Lemma 2.5 (ii), the natural map

$$V^{\mathcal{N}_\lambda(k)} \hookrightarrow V \twoheadrightarrow V_{\mathcal{N}_{-\lambda}(k)}$$

is an isomorphism of irreducible representations of $\mathcal{M}_\lambda(k)$. Using the inverse of this isomorphism we obtain the $\mathcal{M}_\lambda(k)$-equivariant map

$$\xi_\lambda : V \twoheadrightarrow V_{\mathcal{N}_{-\lambda}(k)} \cong V^{\mathcal{N}_\lambda(k)} \hookrightarrow V.$$

If $t \in T$ denotes an arbitrary element of T mapping to λ under the natural map $T \to X_*(\mathbb{T})$ then the Hecke operator $T_\lambda \in \mathcal{H}$ associated with λ is defined by

$$(1) \qquad T_\lambda([g,v]) := \sum_{gKtK = \coprod gxtK} [gxt, \xi_\lambda(x^{-1}v)] \text{ for any } g \in G, v \in V.$$

That T_λ is well-defined, i.e. is independent of the choice of t and of the representatives x, is due to the relation $[gx, v] = [g, xv]$ for $g \in G$, $x \in K$ and $v \in V$, as well as to the relation $t^{-1}xt \cdot \xi_\lambda(v) = \xi_\lambda(xv)$ for $x \in K \cap tKt^{-1}$ and $v \in V$ (cf. [11], proof of Theorem 1.2).

The following fundamental result is due to Schneider, Teitelbaum, Herzig, Henniart and Vignéras, in varying degrees of generality (cf. [10], Proposition 2.1). It is a characteristic p version of the classical isomorphism of Satake. Keep in mind that we assume the center $\mathbb{Z}$ of $\mathbb{G}$ to be connected.

Theorem 1.2. *If $Z_0 = Z \cap K$ denotes the maximal compact subgroup of Z and if $\tilde{\zeta}_V : Z \to E^\times$ extends the central character ζ_V of V then the homomorphism $E[Z] \to \mathcal{H}_0$, sending z to $\tilde{\zeta}_V^{-1}(z)z$, induces an isomorphism $E[Z/Z_0] \to \mathcal{H}_0$ of E-algebras. The $\mathcal{H}_0$-algebra $\mathcal{H}$ is commutative and freely generated by the Hecke operators T_α, $\alpha \in \Delta$. In particular, $\mathcal{H}$ is an integral domain.*

Proof. For the structure of $\mathcal{H}$ over $\mathcal{H}_0$ see [10], Proposition 2.1. Note that Z/Z_0 is a free abelian group of finite rank because $\mathbb{Z}$ is an F-split torus. Thus, $\mathcal{H}_0 \cong E[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ so that $\mathcal{H}_0$ and $\mathcal{H}$ are integral domains. $\square$

If $t \in T$ maps to $\lambda \in X_*(\mathbb{T})^+$ we put

$$K_t := K \cap tKt^{-1} \quad \text{and} \quad I_t := \mathit{red}^{-1}(\mathcal{P}_{-\lambda}(k)).$$

Denoting by $K_1 := \ker(\mathit{red}) \subset K$ the first congruence subgroup of K, Proposition 3.8 of [11] shows that we have the decomposition

$$I_t = K_1 K_t.$$

We shall also set $I := \mathit{red}^{-1}(\overline{\mathcal{B}}(k))$ which is an Iwahori subgroup of G.

Proposition 1.3. *Let $\alpha \in \Delta$, and let $\lambda \in X_*(\mathbb{T})^+$ be represented by $t \in T^+$. Letting W_λ and W_{λ_α} denote the stabilizers of λ and λ_α in W, respectively, we have*

$$ItKt_\alpha K \cap Ktt_\alpha K = \coprod_{w \in W_\lambda W_{\lambda_\alpha}/W_{\lambda_\alpha}} ItIwt_\alpha K = \coprod_{w \in W_\lambda W_{\lambda_\alpha}/W_{\lambda_\alpha}} Itwt_\alpha K.$$

More precisely, if $w \in W_\lambda W_{\lambda_\alpha}$ then $ItIwt_\alpha K = Itwt_\alpha K \subseteq Ktt_\alpha K$. If $w \in W$ with $w \notin W_\lambda W_{\lambda_\alpha}$ then $ItIwt_\alpha K \cap Ktt_\alpha K = \emptyset$.

Proof. The disjointness of the required decompositions follows from the Cartan-Iwahori decomposition $G = \coprod_{\mu \in X_*(\mathbb{T})} I\mu K$.

If $\mu \in X_*(\mathbb{T})^+$ is represented by $s \in T^+$ then $K = \coprod_{w \in W/W_\mu} IwK_s$, as follows from the Bruhat decomposition $\mathcal{G}(k) = \coprod_{w \in W/W_\mu} \overline{\mathcal{B}}(k)w\mathcal{P}_{-\mu}(k)$ by applying red^{-1}. Note that $\mathit{red}^{-1}(\mathcal{P}_{-\mu}(k)) = I_s = K_1K_s$ where K_1 is normal in K. As a consequence, $KsK = \coprod_{w \in W/W_\mu} IwsK$.

Let C be the chamber of X which is pointwise fixed by I. Further, let $A = X_*(\mathbb{T})_\mathbb{R}$ be the apartment of X corresponding to T, and let $\rho_C : X \to A$ denote the retraction of X to A centered at C. Note that our fixed vertex x corresponds to the origin of the real vector space $X_*(\mathbb{T})_\mathbb{R}$. The restriction of the metric d from X to A is given by $d(\mu, \nu) = ||\mu - \nu||$ where $|| \cdot ||$ is the norm associated with a W-invariant scalar product $\omega(\cdot,\cdot)$ on $X_*(\mathbb{T})_{\mathbb{R}}$. If $\alpha \in \Delta$ and if $\check{\alpha}$ denotes the corresponding coroot then

$$X_*(\mathbb{T})^+ = \{\lambda \in X_*(\mathbb{T}) \mid \forall \alpha \in \Delta : \omega(\lambda, \check{\alpha}) \geq 0\}.$$

If $k \in K$ then there is an element $s \in T$ and an element $b \in I$ with $tkt_\alpha K = bsK$. Let μ denote the image of s in $X_*(\mathbb{T})$, and note that $\rho_C^{-1}(\mu) = I\mu = Isx$. By [7], Proposition 7.4.20, we have

$$\begin{aligned} ||\lambda - \mu|| &= d(tx, sx) = d(\rho_C(tx), \rho_C(bsx)) \leq d(tx, bsx) \\ &= d((tk^{-1}t^{-1})tx, (tk^{-1}t^{-1})bsx) = d(tx, tt_\alpha x) = ||\lambda_\alpha||. \end{aligned}$$

If $||\lambda - \mu|| < ||\lambda_\alpha||$ then $s \notin Ktt_\alpha K = \cup_{w\in W} Iwtt_\alpha K$. Indeed, if $w \in W$ then $\lambda - w^{-1}(\lambda)$ is a non-negative real linear combination of the coroots $\check{\alpha}$ with $\alpha \in \Delta$ (cf. [5], VI.1.6 Proposition 18). Therefore,

$$\begin{aligned} ||w(\lambda + \lambda_\alpha) - \lambda||^2 &= ||\lambda_\alpha + \lambda - w^{-1}(\lambda)||^2 \\ &= ||\lambda_\alpha||^2 + ||\lambda - w^{-1}(\lambda)||^2 + 2\omega(\lambda_\alpha, \lambda - w^{-1}(\lambda)) \\ &\geq ||\lambda_\alpha||^2 > ||\mu - \lambda||. \end{aligned}$$

Thus, $\mu \neq w(\lambda + \lambda_\alpha)$. On the other hand, assume $||\lambda - \mu|| = ||\lambda_\alpha||$. There is a labelling of X by $\Delta \cup \{0\}$ with the following property. The vertex tx is of type 0 and if $w \in W$ and $\beta \in \Delta$ then the unique neighbor of tx in twC which is of type β is contained in the line segment $[tx, tt_\beta^w x]$ in A. Let y be the neighbor of tx in tC which is of type α. Setting $z := \rho_C(tkt^{-1}y)$ and using [7], Proposition 7.4.20 again, we have

$$\begin{aligned} d(tx, sx) &\leq d(tx, z) + d(z, sx) \\ &= d(\rho_C(tkt^{-1}tx), \rho_C(tkt^{-1}y)) + d(\rho_C(tkt^{-1}y), \rho_C(tkt^{-1}tt_\alpha x)) \\ &\leq d(tx, y) + d(y, tt_\alpha x) = d(tx, tt_\alpha x) = ||\lambda_\alpha|| = d(tx, sx). \end{aligned}$$

It follows from [7], Proposition 7.4.20 (iii), that $z \in [tx, sx]$. However, $\rho_C \circ tkt^{-1}$ is a label preserving simplicial map. Therefore, z is a neighbor of tx in A which is of type α. This implies $z \in [tx, tt_\alpha^w x]$ for some $w \in W$. Since $z \neq tx$ it follows that sx is contained in the half line in A through $tt_\alpha^w x$ with endpoint tx. The equation $||sx - tx|| = ||\lambda_\alpha|| = ||w(\lambda_\alpha)|| = ||tt_\alpha^w x - tx||$ then implies that $sx = tt_\alpha^w x$ and hence $sK = tt_\alpha^w K$.

Now if $w \in W$ is such that $Itwt_\alpha K \subseteq Ktt_\alpha K = \cup_{v\in W} Ivtt_\alpha K$ then there is an element $v \in W$ such that $t^v t_\alpha^{vw} K = tt_\alpha K$. Computing in $X_*(\mathbb{T})$ this implies $\lambda - v(\lambda) = vw(\lambda_\alpha) - \lambda_\alpha$. By [5], VI.1.6 Proposition 18, both $\lambda - v(\lambda)$ and $\lambda_\alpha - vw(\lambda_\alpha)$ are non-negative real linear combinations of the coroots $\check{\beta}$ with $\beta \in \Delta$. Therefore, $v \in W_\lambda$, $vw \in W_{\lambda_\alpha}$ and hence $w = v^{-1}vw \in W_\lambda W_{\lambda_\alpha}$. Conversely, if $w = \sigma\tau$ with $\sigma \in W_\lambda$ and $\tau \in W_{\lambda_\alpha}$ then $tt_\alpha^w = t^\sigma t_\alpha^{\sigma\tau} = (tt_\alpha^\tau)^\sigma = (tt_\alpha)^\sigma \in Ktt_\alpha K$ and hence $Itwt_\alpha K \subseteq ItKt_\alpha K \cap Ktt_\alpha K$. Thus,

$ItKt_\alpha K \cap Ktt_\alpha K = \coprod_{w \in W_\lambda W_{\lambda_\alpha}/W_{\lambda_\alpha}} Itwt_\alpha K.$

If $w = \sigma\tau \in W_\lambda W_{\lambda_\alpha}$ then the above equation $tt_\alpha^w = (tt_\alpha)^\sigma$ also implies that t and t_α^w lie in the closure of a common Weyl chamber. Therefore, their lengths in the affine Weyl group add up to the length of tt_α^w. It then follows from [21], Theorem 1, that $Itt_\alpha^w I = ItIt_\alpha^w I$. Therefore, $\coprod_{w \in W_\lambda W_{\lambda_\alpha}/W_{\lambda_\alpha}} ItIwt_\alpha K = \coprod_{w \in W_\lambda W_{\lambda_\alpha}/W_{\lambda_\alpha}} Itwt_\alpha K$, proving the desired decompositions.

It remains to see that $ItIwt_\alpha K \cap Ktt_\alpha K = \emptyset$ if $w \notin W_\lambda W_{\lambda_\alpha}$. If this intersection is non-empty then the above decompositions imply the existence of an element $v \in W_\lambda W_{\lambda_\alpha}$ such that $ItIwt_\alpha K \cap Itvt_\alpha K \neq \emptyset$. Hence, there are elements $b, b' \in I$ and $k, k' \in K$ with $tbwt_\alpha k = b'tvt_\alpha k'$, i.e.

$$\begin{aligned} bwv^{-1} = t^{-1}b't \cdot t_\alpha^v k'k^{-1}v^{-1}t_\alpha^{-v} &\in K \cap (t^{-1}It \cdot t_\alpha^v K t_\alpha^{-v}) \\ &= (K \cap t^{-1}It) \cdot (K \cap t_\alpha^v K t_\alpha^{-v}). \end{aligned}$$

The last equality comes from [7], Corollaire 4.3.2, applied to $\Omega = \{x\}$, $\Omega' = t^{-1}C$ and $\Omega'' = t_\alpha^v x$. As a consequence, $wv^{-1} \in I(K \cap t^{-1}It)(K \cap t_\alpha^v K t_\alpha^{-v})$.

It follows from the Iwahori decomposition $I = \overline{\mathcal{N}}(\mathfrak{o})T_0(\mathcal{N}(\mathfrak{o}) \cap K_1)$ that $K \cap t^{-1}It = \overline{\mathcal{N}}(\mathfrak{o})T_0(t^{-1}(\mathcal{N}(\mathfrak{o}) \cap K_1)t \cap K)$. Therefore, $red(I(K \cap t^{-1}It)) = \overline{\mathcal{B}}(k)\mathcal{N}_{\Delta\setminus\Delta(\lambda)}(k) = \mathcal{P}_{-\lambda}(k)$. Since $red(K \cap t_\alpha^v K t_\alpha^{-v}) = v\mathcal{P}_{-\lambda_\alpha}(k)v^{-1}$, we obtain that the image of wv^{-1} in $\mathcal{G}(k)$ is contained in $\mathcal{P}_{-\lambda}(k)v\mathcal{P}_{-\lambda_\alpha}(k)v^{-1} = \mathcal{P}_{-\lambda}(k)\mathcal{P}_{-\lambda_\alpha}(k)v^{-1}$. Thus, the image of w in $\mathcal{G}(k)$ is contained in the double coset $\mathcal{P}_{-\lambda}(k)\mathcal{P}_{-\lambda_\alpha}(k)$. Since $\mathcal{G}(k) = \coprod_{\sigma \in W_\lambda\backslash W/W_{\lambda_\alpha}} \mathcal{P}_{-\lambda}(k)\sigma\mathcal{P}_{-\lambda_\alpha}(k)$, we obtain $w \in W_\lambda W_{\lambda_\alpha}$, in contradiction with our assumption. □

Remark 1.4. Let $\alpha \in \Delta$. The coweight λ_α is minuscule, i.e. satisfies $\langle \lambda_\alpha, \beta \rangle \in \{0, \pm 1\}$ for all $\beta \in \Phi$, if and only if $\lambda_\alpha = t_\alpha x$ is a neighbor of x in the apartment $A = X_*(\mathbb{T})_\mathbb{R}$ of X. In this case $I_{t_\alpha} = K_{t_\alpha}$ by [12], Sublemma 6.8. This gives $K = \cup_{w \in W}\overline{\mathcal{N}}(\mathfrak{o})wK_{t_\alpha}$ and hence

$$ItKt_\alpha K = \cup_{w \in W} It\overline{\mathcal{N}}(\mathfrak{o})t^{-1}twt_\alpha K = \cup_{w \in W} Itwt_\alpha K$$

for all $t \in T^-$. This decomposition does generally not hold if λ_α is not minuscule. For the root system $\Phi = G_2$, for example, let α be the short positive simple root, and let C be the chamber of X which is pointwise fixed by I. There is a unique chamber C' of X in A containing λ_α and sharing a face with C of codimension one (cf. [5], page 276, where $\lambda_\alpha = \omega_1$). Thus, $t_\alpha^{-1}C'$ is the unique chamber of X in A containing x and sharing a face with $t_\alpha^{-1}C$ which is of codimension one. Since $t_\alpha^{-1}It_\alpha$ fixes $t_\alpha^{-1}C$ pointwise, we obtain $\rho_C(t_\alpha^{-1}It_\alpha x) \subseteq \{x, t_\alpha^{-1}x\}$ because the retraction ρ_C of X to A centered at C is a simplicial map. However, we cannot have $\rho_C(t_\alpha^{-1}It_\alpha x) = \{x\}$ because this would imply $t_\alpha^{-1}It_\alpha \subseteq K$ and hence that λ_α

was a neighbor of x. As a consequence, $It_\alpha^{-1}K \subseteq It_\alpha^{-1}It_\alpha K \subseteq It_\alpha^{-1}Kt_\alpha K$ where apparently $-\lambda_\alpha \notin -\lambda_\alpha + W(\lambda_\alpha)$.

Corollary 1.5. *Let $\alpha \in \Delta$, and let $\lambda \in X_*(\mathbb{T})^+$ be represented by $t \in T$. We have*

$$ItKt_\alpha K \cap Ktt_\alpha K = \coprod_{y \in I/(I \cap tKt^{-1})} (ytKt_\alpha K \cap Ktt_\alpha K).$$

If $y \in I$ then $ytKt_\alpha K \cap Ktt_\alpha K = yK_t tt_\alpha K = \coprod_{x \in K_t/K_{tt_\alpha}} yxtt_\alpha K$.

Proof. For the disjointness of the first decomposition we need to see that if $x, y \in I$ with $xtKt_\alpha K \cap ytKt_\alpha K \cap Ktt_\alpha K \neq \emptyset$ then $x^{-1}y \in I \cap tKt^{-1}$. It follows from Proposition 1.3 that $tKt_\alpha K \cap Ktt_\alpha K = \cup_{w \in W_\lambda} tIwt_\alpha K$.

First we claim that if $w \in W_\lambda$ then $tIwt_\alpha K \subseteq K_t tt_\alpha K$. Indeed, the Iwahori decomposition $I = (\mathcal{N}(\mathfrak{o}) \cap K_1)T_0\overline{\mathcal{N}}(\mathfrak{o})$, as well as the root group decompositions of $\mathcal{N}(\mathfrak{o})$ and $\overline{\mathcal{N}}(\mathfrak{o})$, imply that $w^{-1}Iw = (w^{-1}Iw \cap \mathcal{N}(\mathfrak{o}))T_0(w^{-1}Iw \cap \overline{\mathcal{N}}(\mathfrak{o}))$. Since $T_0(w^{-1}Iw \cap \overline{\mathcal{N}}(\mathfrak{o}))t_\alpha K = t_\alpha K$, we obtain

$$tIwt_\alpha K = wt(w^{-1}Iw \cap \mathcal{N}(\mathfrak{o}))t_\alpha K \subseteq wt\mathcal{N}(\mathfrak{o})t^{-1}tt_\alpha K \subseteq K_t tt_\alpha K,$$

because $w \in W_\lambda$ possesses a representative in K_t. As a consequence of this claim and the above decomposition we obtain

$$tKt_\alpha K \cap Ktt_\alpha K = K_t tt_\alpha K = \coprod_{x \in K_t/K_{tt_\alpha}} xtt_\alpha K,$$

using that $K_{tt_\alpha} \subseteq K_t$ by [7], Proposition 4.4.4 (iv). Multiplying through by $y \in I$, we obtain the second assertion of the corollary. Coming back to our initial disjointness assertion,

$$xtKt_\alpha K \cap ytKt_\alpha K \cap Ktt_\alpha K = xK_t tt_\alpha K \cap yK_t tt_\alpha K \neq \emptyset$$

implies that $x^{-1}y \in (K_t \cdot tt_\alpha K(tt_\alpha)^{-1} \cdot K_t) \cap K = K_t K_{tt_\alpha} K_t = K_t$. Therefore, $x^{-1}y \in I \cap tKt^{-1}$. □

Remark 1.6. The proof of Corollary 1.5 shows that there is in fact no need to restrict to the elements y of the subgroup I of K. We have $Ktt_\alpha K = \coprod_{y \in K/K_t}(ytKt_\alpha K \cap Ktt_\alpha K)$ with $ytKt_\alpha K \cap Ktt_\alpha K = yK_t tt_\alpha K$ for any element $y \in K$. The above weaker formulation of Corollary 1.5 is simply adjusted to our later needs.

For the following result see also [15], §3.3 Fact 2.

Proposition 1.7. *Let $\lambda \in X_*(\mathbb{T})^+$ be arbitrary. If we view the $\mathcal{M}_\lambda(k)$-representation $V^{\mathcal{N}_\lambda(k)}$ as a representation of $\mathcal{P}_{-\lambda}(k)$ by letting $\mathcal{N}_{-\lambda}(k)$ act trivially, then there is an isomorphism*

$$M^{K_1} = \mathrm{ind}_K^G(V)^{K_1} \cong \bigoplus_{\lambda \in X_*(\mathbb{T})^+} \mathrm{ind}_{\mathcal{P}_{-\lambda}(k)}^{\mathcal{G}(k)}(V^{\mathcal{N}_\lambda(k)})$$

of E-linear representations of $\mathcal{G}(k) \cong K/K_1$.

Proof. If a general element $\lambda \in X_*(\mathbb{T})^+$ is represented by $t \in T^+$ then the Cartan decomposition $G = \coprod_{\lambda \in X_*(\mathbb{T})^+} KtK$ induces a K-equivariant decomposition

$$M = \mathrm{ind}_K^G(V) \cong \bigoplus_{\lambda \in X_*(\mathbb{T})^+} M_\lambda,$$

where M_λ denotes the K-subrepresentation of M consisting of all functions supported on KtK. The K-representation M_λ is isomorphic to $\mathrm{ind}_{K_t}^K(V^t)$ by sending $f \in M_\lambda$ to the function $(h \mapsto f(ht)) : K \to V^t$. Here V^t denotes the E-linear representation of K_t whose underlying E-vector space is V and on which $h \in K_t$ acts via $v \mapsto t^{-1}ht \cdot v$. We point out that the isomorphism $M_\lambda \cong \mathrm{ind}_{K_t}^K(V^t)$ depends on the choice of the representative $t \in T$ of λ. Recall, however, that we chose fixed representatives as in the paragraph before Remark 1.1.

Since K_1 is a normal subgroup of K we have

$$\mathrm{ind}_{K_t}^K(V^t)^{K_1} \cong \mathrm{ind}_{K_1K_t}^K((V^t)^{K_1 \cap K_t}) \cong \mathrm{ind}_{I_t/K_1}^{K/K_1}((V^t)^{K_1 \cap K_t}),$$

where $K/K_1 \cong \mathcal{G}(k)$ and $K_1K_t/K_1 = I_t/K_1 \cong \mathcal{P}_{-\lambda}(k)$. Note that $\mathcal{N}_{-\lambda}(\mathfrak{o})$ acts trivially on $(V^t)^{K_1 \cap K_t}$ because $t^{-1}\mathcal{N}_{-\lambda}(\mathfrak{o})t \subset K_1$. Thus, the action of $\mathcal{P}_{-\lambda}(k)$ on $(V^t)^{K_1 \cap K_t}$ factors through $\mathcal{M}_\lambda(k)$.

Further, $\mathcal{M}_\lambda(k)$ is generated by the images in $\mathcal{G}(k)$ of $T_0 = T \cap K$ and $\mathcal{N}_\alpha(\mathfrak{o})$, $\alpha \in \pm(\Delta \setminus \Delta(\lambda))$, all of which centralize t (cf. Remark 1.1). Therefore, the action of $\mathcal{M}_\lambda(k)$ on V^t agrees with that on V. Now consider the Iwahori type decomposition

$$K_1 = (\overline{\mathcal{N}}(\mathfrak{o}) \cap K_1)(T \cap K_1)(\mathcal{N}(\mathfrak{o}) \cap K_1)$$

in which the two factors on the left are contracted under conjugation with t^{-1}. As a consequence,

$$t^{-1}K_1t \cap K = t^{-1}(\overline{\mathcal{N}}(\mathfrak{o}) \cap K_1)t(T \cap K_1)(t^{-1}(\mathcal{N}(\mathfrak{o}) \cap K_1)t \cap \mathcal{N}(\mathfrak{o})).$$

The two factors on the left are contained in K_1 and hence act trivially on V. It remains to note that $(V^t)^{K_1 \cap K_t} = V^{t^{-1}K_1t \cap K}$ and that the image of $t^{-1}(\mathcal{N}(\mathfrak{o}) \cap K_1)t \cap \mathcal{N}(\mathfrak{o})$ in $\mathcal{G}(k)$ under the reduction homomorphism is precisely $\mathcal{N}_\lambda(k)$. $\square$

Let $\lambda \in X_*(\mathbb{T})^+$ be represented by $t \in T^+$, and let $\alpha \in \Delta$. Since $\Delta(\lambda) \subseteq \Delta(\lambda + \lambda_\alpha)$ we have $\mathcal{P}_\lambda(k) \supseteq \mathcal{P}_{\lambda+\lambda_\alpha}(k)$. This also follows from the relation $K_{tt_\alpha} \subseteq K_t$ (cf. [7], Proposition 4.4.4 (iv)). We let

$$pr_\lambda : M = \bigoplus_{\mu \in X_*(\mathbb{T})^+} M_\mu \longrightarrow M_\lambda$$

be the projection onto the component corresponding to λ.

Since the Hecke operators T_α are G-equivariant, the K-subrepresentation $M^{K_1} = \mathrm{ind}_K^G(V)^{K_1}$ of M is an $\mathcal{H}$-submodule. The action of the operators T_α on this space can partially be made explicit.

Proposition 1.8. *If $\lambda \in X_*(\mathbb{T})^+$ and $\alpha \in \Delta$ then the diagram*

$$\begin{array}{ccccc}
M_\lambda^{K_1} & \hookrightarrow & M^{K_1} & \xrightarrow{pr_{\lambda+\lambda_\alpha} \circ T_\alpha} & M_{\lambda+\lambda_\alpha}^{K_1} \\
\cong \downarrow & & & & \downarrow \cong \\
\mathrm{ind}_{\mathcal{P}_{-\lambda}(k)}^{\mathcal{G}(k)}(V^{\mathcal{N}_\lambda(k)}) & & \xrightarrow{f \mapsto \xi_\alpha \circ f} & & \mathrm{ind}_{\mathcal{P}_{-\lambda-\lambda_\alpha}(k)}^{\mathcal{G}(k)}(V^{\mathcal{N}_{\lambda+\lambda_\alpha}(k)})
\end{array}$$

is commutative. Here the vertical isomorphisms come from Proposition 1.7.

In order to prove this result we need a property of the endomorphisms ξ_λ which is partially responsible for the commutativity of the E-algebra $\mathcal{H}$.

Lemma 1.9. *If $\lambda, \lambda' \in X_*(\mathbb{T})^+$ then $\xi_\lambda \circ \xi_{\lambda'} = \xi_{\lambda+\lambda'}$. In particular, the E-linear endomorphisms ξ_λ and $\xi_{\lambda'}$ of V commute with each other and*

$$\xi_{\lambda'}(V^{\mathcal{N}_\lambda(k)}) = V^{\mathcal{N}_\lambda(k)} \cap V^{\mathcal{N}_{\lambda'}(k)} = V^{\mathcal{N}_{\lambda+\lambda'}(k)}.$$

Proof. We will first show that ξ_λ and $\xi_{\lambda'}$ commute with each other. Since both of them are projections it suffices to show that $\ker(\xi_\lambda) = \ker(V \to V_{\mathcal{N}_{-\lambda}(k)})$ and $\mathrm{im}(\xi_\lambda) = V^{\mathcal{N}_\lambda(k)}$ are stable under $\xi_{\lambda'}$. This, however, follows from the fact that $\xi_{\lambda'}$ is $\mathcal{M}_{\lambda'}(k)$-equivariant. Indeed, this equivariance implies that $\xi_{\lambda'}(V^{\mathcal{N}_\lambda(k)})$ is invariant under the subgroup of $\mathcal{G}(k)$ generated by $\mathcal{N}_{\lambda'}(k)$ and $\mathcal{M}_{\lambda'}(k) \cap \mathcal{N}_\lambda(k)$. However, this subgroup contains $\mathcal{N}_\lambda(k)$. More precisely, the product map

$$(\mathcal{M}_{\lambda'}(k) \cap \mathcal{N}_\lambda(k)) \times \mathcal{N}_{\lambda'}(k) \longrightarrow \mathcal{N}_{\lambda+\lambda'}(k)$$

is bijective, whence $\xi_{\lambda'}(V^{\mathcal{N}_\lambda(k)}) \subseteq V^{\mathcal{N}_{\lambda+\lambda'}(k)} = V^{\mathcal{N}_\lambda(k)} \cap V^{\mathcal{N}_{\lambda'}(k)}$.

Finally, the kernel of ξ_λ is the E-subspace of V generated by all elements of the form $v - nv$ with $v \in V$ and $n \in \mathcal{N}_{-\lambda}(k)$. As above, we can write $n = mn'$

with $m \in \mathcal{M}_{\lambda'}(k) \cap \mathcal{N}_{-\lambda}(k)$ and $n' \in \mathcal{N}_{-\lambda'}(k)$. The $\mathcal{M}_{\lambda'}(k)$-equivariance of $\xi_{\lambda'}$ then implies

$$\xi_{\lambda'}(v - nv) = \xi_{\lambda'}(v) - m\xi_{\lambda'}(n'v) = \xi_{\lambda'}(v) - m\xi_{\lambda'}(v) \in \ker(\xi_\lambda),$$

where the second equality uses $v - n'v \in \ker(\xi_{\lambda'})$.

Thus, $\xi_\lambda \circ \xi_{\lambda'} = \xi_{\lambda'} \circ \xi_\lambda$. In particular, $\xi_\lambda \circ \xi_{\lambda'}$ is again a projection. Since it is the identity on $V^{\mathcal{N}_\lambda(k)} \cap V^{\mathcal{N}_{\lambda'}(k)} = V^{\mathcal{N}_{\lambda+\lambda'}(k)} = \mathrm{im}(\xi_{\lambda+\lambda'})$, it remains to show that $\xi_\lambda \circ \xi_{\lambda'}$ is zero on the kernel of $\xi_{\lambda+\lambda'}$. However, $\mathcal{N}_{-\lambda-\lambda'}(k)$ is generated by $\mathcal{N}_{-\lambda}(k)$ and $\mathcal{N}_{-\lambda'}(k)$ so that the relation $\xi_\lambda \circ \xi_{\lambda'} = \xi_{\lambda'} \circ \xi_\lambda$ implies $(\xi_\lambda \circ \xi_{\lambda'})(nv) = (\xi_\lambda \circ \xi_{\lambda'})(v)$ for all $n \in \mathcal{N}_{-\lambda-\lambda'}(k)$ and $v \in V$. □

Proof of Proposition 1.8. Note first that the lower map of the diagram is well defined, i.e. if $f \in \mathrm{ind}_{\mathcal{P}_{-\lambda}(k)}^{\mathcal{G}(k)}(V^{\mathcal{N}_\lambda(k)})$ then the map $\xi_\alpha \circ f : \mathcal{G}(k) \to V$ has image in $V^{\mathcal{N}_{\lambda+\lambda_\alpha}(k)}$ and satisfies $(\xi_\alpha \circ f)(xy) = y^{-1}(\xi_\alpha \circ f)(x)$ for all $x \in \mathcal{G}(k)$ and all $y \in \mathcal{P}_{-\lambda-\lambda_\alpha}(k)$.

The first assertion follows from Lemma 1.9. For the latter assertion it suffices to show that $\xi_\alpha : V^{\mathcal{N}_\lambda(k)} \to V^{\mathcal{N}_{\lambda+\lambda_\alpha}(k)}$ is $\mathcal{P}_{-\lambda-\lambda_\alpha}(k)$-equivariant for the actions of $\mathcal{P}_{-\lambda-\lambda_\alpha}(k) \subseteq \mathcal{P}_{-\lambda}(k)$ as described in Proposition 1.7. The map ξ_α is equivariant with respect to $\mathcal{M}_{\lambda_\alpha}(k) \supseteq \mathcal{M}_{\lambda+\lambda_\alpha}(k)$. Further, the group $\mathcal{N}_{-\lambda-\lambda_\alpha}(k)$ is generated by $\mathcal{N}_{-\lambda_\alpha}(k)$ and $\mathcal{N}_{-\lambda}(k) \cap \mathcal{M}_{\lambda_\alpha}(k)$ (confer the proof of Lemma 1.9 with $\lambda' = \lambda_\alpha$). It remains to show that the restriction of ξ_α to $V^{\mathcal{N}_\lambda(k)}$ is equivariant for the action of $\mathcal{N}_{-\lambda_\alpha}(k)$.

Let $x \in \mathcal{N}_{-\lambda_\alpha}(k)$ and $v \in V^{\mathcal{N}_\lambda(k)}$. If $\alpha \in \Delta(\lambda)$ then $\mathcal{N}_{-\lambda_\alpha}(k) \subseteq \mathcal{N}_{-\lambda}(k)$ and therefore $xv = v$ for the action as in Proposition 1.7. Since $\mathcal{N}_{\lambda_\alpha}(k) \subseteq \mathcal{N}_{\lambda+\lambda_\alpha}(k)$ we also have $\xi_\alpha(v) \in V^{\mathcal{N}_{\lambda_\alpha}(k)}$, whence $x\xi_\alpha(v) = \xi_\alpha(v) = \xi_\alpha(xv)$. If $\alpha \notin \Delta(\lambda)$ then $x \in \mathcal{M}_\lambda(k)$, and the action of x on V is the usual one. In this case, the general properties of ξ_α yield $\xi_\alpha(xv) = \xi_\alpha(v)$ because $x \in \mathcal{N}_{-\lambda_\alpha}(k)$. As seen above, we also have $x\xi_\alpha(v) = \xi_\alpha(v)$, proving that the map $f \mapsto \xi_\alpha \circ f$ is well-defined.

Let $\mu \in X_*(\mathbb{T})^+$, $x \in \mathcal{P}_{-\mu}(k)$ and $v \in V^{\mathcal{N}_\mu(k)}$. In analogy with our previous notation we let $[1, v]_{\mathcal{P}_{-\mu}(k)} \in \mathrm{ind}_{\mathcal{P}_{-\mu}(k)}^{\mathcal{G}(k)}(V^{\mathcal{N}_\mu(k)})$ denote the function with support $x\mathcal{P}_{-\mu}(k)$ and value v at x.

Let λ be represented by $t \in T^+$, and let $v \in V^{\mathcal{N}_\lambda(k)}$. The lower horizontal map of the diagram is determined by sending $[1, v]_{\mathcal{P}_{-\lambda}(k)}$ to the function $\sum_{x \in \mathcal{P}_{-\lambda}(k)/\mathcal{P}_{-\lambda-\lambda_\alpha}(k)} [x, \xi_\alpha(x^{-1}v)]_{\mathcal{P}_{-\lambda-\lambda_\alpha}(k)}$. Under the identifications of Proposition 1.7 the function $[1, v]_{\mathcal{P}_{-\lambda}(k)}$ correponds to the element of $\mathrm{ind}_K^G(V)^{K_1}$ with support $I_t tK = K_1 tK$ and value v at t, i.e. to $\sum_{y \in I_t/K_t} [yt, v]$. Similarly, for any $x \in \mathcal{P}_{-\lambda(k)}/\mathcal{P}_{-\lambda-\lambda_\alpha}(k) \cong I_t/I_{tt_\alpha}$, $[x, \xi_\alpha(x^{-1}v)]_{\mathcal{P}_{-\lambda-\lambda_\alpha}(k)}$

corresponds to $\sum_{y\in I_{tt_\alpha}/K_{tt_\alpha}}[xytt_\alpha, \xi_\alpha(t^{-1}x^{-1}tv)]$ in $\mathrm{ind}_K^G(V)^{K_1}$ with support in $xI_{tt_\alpha}tt_\alpha K$. Thus, $\sum_{x\in\mathcal{P}_{-\lambda}(k)/\mathcal{P}_{-\lambda-\lambda_\alpha}(k)}[x,\xi_\alpha(x^{-1}v)]_{\mathcal{P}_{-\lambda-\lambda_\alpha}(k)}$ corresponds to the function $F := \ sum_{x\in I_t/I_{tt_\alpha}}\sum_{y\in I_{tt_\alpha}/K_{tt_\alpha}}[xytt_\alpha, \xi_\alpha(t^{-1}x^{-1}tv)]$.

On the other hand, since $I_t tK = ItK$, Corollary 1.5 implies that

$$G := pr_{\lambda+\lambda_\alpha}(T_\alpha(\sum_{y\in I_t/K_t}[yt,v])) = \sum_{y\in I_t/K_t}\sum_{x\in K_t/K_{tt_\alpha}}[yxtt_\alpha, \xi_\alpha(t^{-1}x^{-1}tv)].$$

Let x_1, x_2 and y run through systems of representatives for $K_t/(K_t\cap I_{tt_\alpha})$, $(K_t\cap I_{tt_\alpha})/K_{tt_\alpha}$ and $I_{tt_\alpha}/(K_t\cap I_{tt_\alpha})$, respectively. Note that

$$\begin{aligned} I_t &= K_1K_t = K_1K_{tt_\alpha}K_t = I_{tt_\alpha}K_t \\ &= K_tK_1 = K_tK_{tt_\alpha}K_1 = K_tI_{tt_\alpha}, \end{aligned}$$

so that the natural maps $K_t/(K_t\cap I_{tt_\alpha}) \to I_t/I_{tt_\alpha}$ and $I_{tt_\alpha}/(K_t\cap I_{tt_\alpha}) \to I_t/K_t$ are bijective. As a consequence, we may also regard x_1 and y as running through systems of representatives for I_t/I_{tt_α} and I_t/K_t, respectively. Therefore, we have the simultaneous decompositions

$$\begin{aligned} I_t &= \coprod_{y,x_1,x_2} yx_1x_2K_{tt_\alpha}, \quad \text{where} \quad \coprod_{x_1,x_2} x_1x_2K_{tt_\alpha} = K_t, \\ &= \coprod_{y,x_1,x_2} x_1yx_2K_{tt_\alpha}, \quad \text{where} \quad \coprod_{y,x_2} yx_2K_{tt_\alpha} = I_{tt_\alpha}. \end{aligned}$$

Since $I_t/K_t \cong K_1/(K_1\cap K_t)$, we may assume all y to be contained in K_1.

Now keep x_1 fixed and note that x_1 acts on $K_1/(K_1\cap K_t)$ by conjugation. This implies that also $\coprod_{y,x_2}(x_1^{-1}yx_1)x_2K_{tt_\alpha} = I_{tt_\alpha}$. Thus, for any y and x_2 there are y' and x_2' such that

$$yx_1x_2K_{tt_\alpha} = x_1(x_1^{-1}yx_1)x_2K_{tt_\alpha} = x_1y'x_2'K_{tt_\alpha}.$$

Now $(K_t\cap I_{tt_\alpha})/K_{tt_\alpha} \subseteq I_{tt_\alpha}/K_{tt_\alpha} \cong K_1/(K_1\cap K_{tt_\alpha})$, so that we may assume all x_2 to be contained in K_1, as well. By assumption, $v\in V^{t^{-1}K_1t\cap K}$ is fixed by $t^{-1}x_2t$. Since $t^{-1}K_1t\cap K$ is normalized by $t^{-1}x_1t \in t^{-1}Kt\cap K$, also $t^{-1}x_1tv$ is fixed by $t^{-1}x_2t$. This yields

$$G(yx_1x_2tt_\alpha) = \xi_\alpha(t^{-1}x_2^{-1}x_1^{-1}tv) = \xi_\alpha(t^{-1}x_1^{-1}tv) = F(x_1y'x_2'tt_\alpha).$$

By construction, $z := (x_2')^{-1}(y')^{-1}(x_1^{-1}yx_1)x_2 \in K_1\cap K_{tt_\alpha}$. The relation $K_{tt_\alpha}\subseteq K_t$ then implies $t^{-1}zt\in K_{t_\alpha}$ so that

$$\begin{aligned} F(yx_1x_2tt_\alpha) &= F(x_1y'x_2'tt_\alpha(tt_\alpha)^{-1}ztt_\alpha) = (tt_\alpha)^{-1}ztt_\alpha F(x_1y'x_2'tt_\alpha) \\ &= t_\alpha^{-1}(t^{-1}zt)t_\alpha\cdot\xi_\alpha(t^{-1}x_1tv) = \xi_\alpha(t^{-1}zt\cdot t^{-1}x_1tv) \\ &= \xi_\alpha(t^{-1}x_1tv) = F(x_1y'x_2'tt_\alpha) = G(yx_1x_2tt_\alpha). \end{aligned}$$

□

Corollary 1.10. *If $\alpha \in \Delta$ and $\lambda \in X_*(\mathbb{T})^+$ then the map $pr_{\lambda+\lambda_\alpha} \circ T_\alpha : M_\lambda \to M_{\lambda+\lambda_\alpha}$ is injective.*

Proof. Since $pr_{\lambda+\lambda_\alpha} \circ T_\alpha$ is K_1-equivariant and since K_1 is a pro-p group it suffices to see that the induced map $pr_{\lambda+\lambda_\alpha} \circ T_\alpha : M_\lambda^{K_1} \to M_{\lambda+\lambda_\alpha}^{K_1}$ is injective (cf. [16], Lemma 2.1). By Proposition 1.8 this is equivalent with the injectivity of the map $\mathrm{ind}_{\mathcal{P}_{-\lambda}(k)}^{\mathcal{G}(k)}(V^{\mathcal{N}_\lambda(k)}) \to \mathrm{ind}_{\mathcal{P}_{-\lambda-\lambda_\alpha}(k)}^{\mathcal{G}(k)}(V^{\mathcal{N}_{\lambda+\lambda_\alpha}(k)})$, sending f to $\xi_\alpha \circ f$. Note, however, that if $f \neq 0$ then its image as a map $\mathcal{G}(k) \to V^{\mathcal{N}_\lambda(k)}$ generates all of $V^{\mathcal{N}_\lambda(k)}$ over E. Indeed, by its equivariance for the action of $\mathcal{P}_{-\lambda}(k)$ its image generates a subrepresentation of the irreducible $\mathcal{M}_\lambda(k)$-representation $V^{\mathcal{N}_\lambda(k)}$. Thus, $\xi_\alpha \circ f = 0$ implies $f = 0$ because ξ_α is E-linear and non-zero. □

Under certain conditions one even has the following bijectivity result.

Corollary 1.11. *If $\alpha \in \Delta$ and $\lambda \in X_*(\mathbb{T})^+$ with $\alpha \in \Delta(\lambda)$ then the map $pr_{\lambda+\lambda_\alpha} \circ T_\alpha : M_\lambda^{K_1} \to M_{\lambda+\lambda_\alpha}^{K_1}$ is bijective. In particular, the $\mathcal{H}$-module M^{K_1} is generated by the sum of all $M_\mu^{K_1}$ for which $\mu = \sum_{\beta\in\Delta} n_\beta \lambda_\beta$ with $n_\beta \in \{0,1\}$ for all $\beta \in \Delta$.*

Proof. If $\alpha \in \Delta(\lambda)$ then $\Delta(\lambda) = \Delta(\lambda + \lambda_\alpha)$, $\mathcal{P}_{-\lambda} = \mathcal{P}_{-\lambda-\lambda_\alpha}$ and ξ_α is the identity on $V^{\mathcal{N}_\lambda(k)} = V^{\mathcal{N}_{\lambda+\lambda_\alpha}(k)}$. Therefore, the first claim follows from Proposition 1.8. The second assertion is an immediate consequence of the first one by induction on $\sum_{\beta\in\Delta} n_\beta$. Note that if $M_\lambda^{K_1}$ is contained in the $\mathcal{H}$-submodule of M^{K_1} generated by the above set of components $M_\mu^{K_1}$ then so is $M_{\lambda+\lambda'}^{K_1}$ for any $\lambda' \in X_*(\mathbb{Z}) \subseteq X_*(\mathbb{T})^+$. Indeed, if λ' is represented by $z \in Z$ then $M_{\lambda+\lambda'}^{K_1} = zM_\lambda^{K_1} \subseteq \mathcal{H}_0 M_\lambda^{K_1}$. □

Remark 1.12. We warn the reader that the natural inclusion $V^{\mathcal{N}_{\lambda+\lambda_\alpha}(k)} \subseteq V^{\mathcal{N}_\lambda(k)}$ need not be equivariant for the actions of $\mathcal{P}_{-\lambda-\lambda_\alpha}(k) \subseteq \mathcal{P}_{-\lambda}(k)$ described in Proposition 1.7 unless $\alpha \in \Delta(\lambda)$. Indeed, if $\alpha \notin \Delta(\lambda)$ then $\mathcal{M}_\lambda(k) \cap \mathcal{N}_{-\lambda-\lambda_\alpha}(k)$ contains the root subgroup $\mathcal{N}_{-\alpha}(k)$ which acts trivially on $V^{\mathcal{N}_{\lambda+\lambda_\alpha}(k)}$ but possibly non-trivially on $V^{\mathcal{N}_\lambda(k)}$.

Using the identification made in Proposition 1.8 there is yet another natural description of the map $pr_{\lambda+\lambda_\alpha} \circ T_\alpha : M_\lambda \to M_{\lambda+\lambda_\alpha}$.

Lemma 1.13. *Let $\alpha \in \Delta$ and $\lambda \in X_*(\mathbb{T})^+$. The diagram*

$$\begin{array}{ccc}
\mathrm{ind}_{\mathcal{P}_{-\lambda}(k)}^{\mathcal{G}(k)}(V^{\mathcal{N}_\lambda(k)}) & \xrightarrow{f\mapsto\xi_\alpha\circ f} & \mathrm{ind}_{\mathcal{P}_{-\lambda-\lambda_\alpha}(k)}^{\mathcal{G}(k)}(V^{\mathcal{N}_{\lambda+\lambda_\alpha}(k)}) \\
\downarrow\cong & & \cong\downarrow \\
\mathrm{ind}_{\mathcal{P}_{-\lambda}(k)}^{\mathcal{G}(k)}(V_{\mathcal{N}_{-\lambda}(k)}) & \longrightarrow & \mathrm{ind}_{\mathcal{P}_{-\lambda-\lambda_\alpha}(k)}^{\mathcal{G}(k)}(V_{\mathcal{N}_{-\lambda-\lambda_\alpha}(k)})
\end{array}$$

of $\mathcal{G}(k)$-equivariant maps is commutative. Here the actions of $\mathcal{P}_{-\lambda}(k)$ and $\mathcal{P}_{-\lambda-\lambda_\alpha}(k)$ on $V_{\mathcal{N}_{-\lambda}(k)}$ and $V_{\mathcal{N}_{-\lambda-\lambda_\alpha}(k)}$, respectively, are the natural ones. The lower horizonal arrow composes a function with the natural projection $V_{\mathcal{N}_{-\lambda}(k)} \to V_{\mathcal{N}_{-\lambda-\lambda_\alpha}(k)}$. Finally, the left (resp. right) vertical arrow is induced by the $\mathcal{M}_\lambda(k)$-equivariant isomorphism $V^{\mathcal{N}_\lambda(k)} \to V \to V_{\mathcal{N}_{-\lambda}(k)}$ (resp. by the $\mathcal{M}_{\lambda+\lambda_\alpha}(k)$-equivariant isomorphism $V^{\mathcal{N}_{\lambda+\lambda_\alpha}(k)} \to V \to V_{\mathcal{N}_{-\lambda-\lambda_\alpha}(k)}$).

Proof. It suffices to show that the two maps

$$V^{\mathcal{N}_\lambda(k)} \hookrightarrow V \twoheadrightarrow V_{\mathcal{N}_{-\lambda}(k)} \twoheadrightarrow V_{\mathcal{N}_{-\lambda-\lambda_\alpha}(k)}$$

and

$$V^{\mathcal{N}_\lambda(k)} \xrightarrow{\xi_\alpha} V^{\mathcal{N}_{\lambda+\lambda_\alpha}(k)} \hookrightarrow V \twoheadrightarrow V_{\mathcal{N}_{-\lambda-\lambda_\alpha}(k)}$$

coincide. The first one simply sends $v \in V^{\mathcal{N}_\lambda(k)}$ to its residue class modulo $\ker(V \to V_{\mathcal{N}_{-\lambda-\lambda_\alpha}})$. The second one sends v to the residue class of $\xi_\alpha(v)$. However, $v = \xi_\alpha(v) + v - \xi_\alpha(v)$ where $v - \xi_\alpha(v) \in \ker(\xi_\alpha) = \ker(V \to V_{\mathcal{N}_{-\lambda_\alpha}(k)})$ because ξ_α is a projection. Now $\mathcal{N}_{-\lambda_\alpha}(k) \subseteq \mathcal{N}_{-\lambda-\lambda_\alpha}(k)$ so that $\ker(\xi_\alpha) \subseteq \ker(V \to V_{\mathcal{N}_{-\lambda-\lambda_\alpha}})$. □

2 Freeness of the pro-p Iwahori invariants

The E-linear smooth G-representation $M = \mathrm{ind}_K^G(V)$ is a module over its endomorphism ring $\mathcal{H}$, the latter being freely generated by the Hecke operators T_α, $\alpha \in \Delta$, over the subalgebra $\mathcal{H}_0$ (cf. Theorem 1.2).

The structure of M over $\mathcal{H}$ was studied in detail by Bellaïche-Otwinowska and Große-Klönne. For $\mathbb{G} = \mathrm{PGL}_3$ and $V = E$ the trivial representation, the $\mathcal{H}$-module M is free for any ring E (cf. [1], Théorème 1.5). In a much wider class of examples, yet assuming $F = \mathbb{Q}_p$, Große-Klönne showed that $M \otimes_{\mathcal{H}_0,\theta} E$ is a free module over $\mathcal{H}/\ker(\theta)\mathcal{H}$ for any homomorphism $\theta : \mathcal{H}_0 \to E$ of E-algebras (cf. [10], Theorem 1.1).

The group $I_1 := \mathit{red}^{-1}(\mathcal{N}(k))$ is a pro-p Sylow subgroup of K, a so-called *pro-p Iwahori subgroup.* The aim of this section is to show that the $\mathcal{H}$-submodule M^{I_1} of I_1-invariants of M is finitely generated and free without any restriction on V or F. The structure of M^{I_1} as a module over the so-called *pro-p Iwahori-Hecke algebra* was determined by Ollivier (cf. [15], Lemma 3.6)

Theorem 2.1. *For any irreducible E-linear representation V of $\mathcal{G}(k)$ the module $M^{I_1} = ind_K^G(V)^{I_1}$ is finitely generated and free over $\mathcal{H} = \mathrm{End}_G(M)$.*

Proof. By Corollary 1.11 the $\mathcal{H}$-module M^{K_1} is finitely generated. Since $\mathcal{H}$ is a noetherian ring the submodule M^{I_1} is finitely generated, as well. By a theorem of Quillen-Suslin (cf. [19], Corollary 7.4) and by [4], II.5.3 Corollaire 2, it now suffices to see that M^{I_1} is flat over $\mathcal{H}$. By [4], II.3.4 Proposition 15 and II.3.2 Corollaire 2, it suffices to show that the torsion group $\mathrm{Tor}_1^{\mathcal{H}_\mathfrak{m}}(\mathcal{H}_\mathfrak{m} \otimes_\mathcal{H} M^{I_1}, \mathcal{H}_\mathfrak{m}/\mathfrak{m}\mathcal{H}_\mathfrak{m})$ vanishes for any maximal ideal $\mathfrak{m}$ of $\mathcal{H}$. Here $\mathcal{H}_\mathfrak{m}$ denotes the localization of $\mathcal{H}$ at $\mathfrak{m}$. Further,

$$\mathrm{Tor}_1^{\mathcal{H}_\mathfrak{m}}(\mathcal{H}_\mathfrak{m} \otimes_\mathcal{H} M^{I_1}, \mathcal{H}_\mathfrak{m}/\mathfrak{m}\mathcal{H}_\mathfrak{m}) \cong \mathrm{Tor}_1^{\mathcal{H}}(M^{I_1}, \mathcal{H}/\mathfrak{m})$$

by [3], X.6.6 Proposition 8.

Since E is algebraically closed and $\mathcal{H}$ is an E-algebra of finite type any maximal ideal $\mathfrak{m}$ of $\mathcal{H}$ is the kernel of a uniquely determined homomorphism $\chi : \mathcal{H} \to E$ of E-algebras. We denote by $\theta : \mathcal{H}_0 \to E$ its restriction to $\mathcal{H}_0$ and set $\mathcal{H}_\theta := \mathcal{H}/\ker(\theta)\mathcal{H}$, as well as $M_\theta^{I_1} := M^{I_1}/\ker(\theta)M^{I_1}$.

Putting $M' := \bigoplus_{\lambda \in X_*(\mathbb{T}')^+} M_\lambda \subseteq M$ we claim that the E-linear map

$$\mathcal{H}_0 \otimes_E M'^{I_1} \longrightarrow M^{I_1}, \quad \varphi \otimes m \mapsto \varphi(m), \tag{2}$$

is bijective. Indeed, any set of representatives of Z/Z_0 gives rise to an E-basis of $\mathcal{H}_0$ (cf. Theorem 1.2). Now if $z \in Z$ then (2) maps $Ez \otimes_E M'^{I_1}$ bijectively onto the subspace of M^{I_1} consisting of all functions supported on $zKT'K$. Since $G = \coprod_{z \in Z/Z_0} zKT'K$ the claim follows. In particular, the $\mathcal{H}_0$-module M^{I_1} is free. As a consequence, if $P^\bullet \to M^{I_1}$ is a free resoluton of M^{I_1} over $\mathcal{H}$ then $P^\bullet \otimes_\mathcal{H} \mathcal{H}_\theta \to M_\theta^{I_1}$ is a free resolution of $M_\theta^{I_1}$ over $\mathcal{H}_\theta$. Therefore, we obtain an isomorphism

$$\mathrm{Tor}_1^{\mathcal{H}}(M^{I_1}, \mathcal{H}/\mathfrak{m}) \cong \mathrm{Tor}_1^{\mathcal{H}_\theta}(M_\theta^{I_1}, \mathcal{H}_\theta/\mathfrak{m}\mathcal{H}_\theta).$$

We fix an enumeration $\Delta = \{\alpha_1, \ldots, \alpha_d\}$ of Δ. Since the ordered family $(T_{\alpha_1} - \chi(T_{\alpha_1}), \ldots, T_{\alpha_d} - \chi(T_{\alpha_d}))$ is a regular sequence of the ring $\mathcal{H}_\theta \cong E[T_\alpha \mid \alpha \in \Delta]$, the groups $\mathrm{Tor}_\bullet^{\mathcal{H}_\theta}(M_\theta^{I_1}, \mathcal{H}_\theta/\mathfrak{m}\mathcal{H}_\theta)$ are the homology groups of the Koszul complex $M_\theta^{I_1} \otimes_E \bigwedge^\bullet E^\Delta$ associated with the above regular sequence (cf. [3], X.9.4, page 155). We will show that this complex is acyclic, i.e. has trivial homology in positive degrees. In an essential way this relies on an acyclicity result for coefficient systems of representations of the finite group $\mathcal{G}(k)$ of Lie type which will be proved in the following section.

It follows from (2) that that the inclusion $M'^{I_1} \subseteq M^{I_1}$ induces an E-linear bijection $M'^{I_1} \cong M_\theta^{I_1}$. Further, if $\lambda \in X_*(\mathbb{T}')$ and $\mu \in X_*(\mathbb{Z})$ then $M_\lambda^{I_1}$ and $M_{\mu+\lambda}^{I_1}$ map isomorphically to the same subspace of $M_\theta^{I_1}$. We will therefore write $M_\theta^{I_1} = \bigoplus_{\lambda \in X_*(\mathbb{T}/\mathbb{Z})^+} M_\lambda^{I_1}$ and denote the projections $M_\theta^{I_1} \to M_\lambda^{I_1}$ by

pr_λ as before.

We choose a total ordering $\leq$ on $X_*(\mathbb{T}/\mathbb{Z})$ which refines the dominance relation and which is compatible with the group structure, i.e. satisfies $\mu + \lambda' \leq \lambda + \lambda'$ for all $\mu, \lambda, \lambda' \in X_*(\mathbb{T}/\mathbb{Z})$ with $\mu \leq \lambda$. To give an explicit example, write $X_*(\mathbb{T}/\mathbb{Z}) \subseteq \sum_{\alpha\in\Delta} \mathbb{R}\check{\alpha}$ and choose the lexicographical ordering for the usual ordering of the real numbers on the right. Here $\check{\alpha}$ denotes the coroot associated with $\alpha \in \Delta$, and the usual dominance relation is defined by $\mu \leq \lambda$ if and only if $\lambda - \mu \in \sum_{\alpha\in\Delta} \mathbb{R}_{\geq 0}\check{\alpha}$.

For $\alpha \in \Delta$ let $e_\alpha \in E^\Delta$ be the corresponding standard unit vector. For any subset $J = \{\alpha_{j_1}, \ldots, \alpha_{j_r}\} \subseteq \Delta$ with $j_1 < \ldots < j_r$ set $e_J := e_{\alpha_{j_1}} \wedge \ldots \wedge e_{\alpha_{j_r}} \in \bigwedge^r E^\Delta$ and $\rho_J := \sum_{\alpha\in J} \lambda_\alpha$. We endow the Koszul complex $M_\theta^{I_1} \otimes_E \bigwedge^\bullet E^\Delta$ with the following filtration indexed by $X_*(\mathbb{T}/\mathbb{Z})^+$. For any $\lambda \in X_*(\mathbb{T}/\mathbb{Z})^+$ set

$$\text{(3)} \qquad \mathrm{Fil}^\lambda_\bullet := \mathrm{Fil}^\lambda(M_\theta^{I_1} \otimes_E \bigwedge^\bullet E^\Delta) := \bigoplus_{\substack{J\subseteq\Delta, |J|=\bullet \\ \mu\in X_*(\mathbb{T}/\mathbb{Z})^+, \mu+\rho_J\leq\lambda}} M_\mu^{I_1} \otimes_E Ee_J.$$

Further, set

$$\mathrm{Fil}^{\lambda-}_\bullet := \mathrm{Fil}^{\lambda-}(M_\theta^{I_1} \otimes_E \bigwedge^\bullet E^\Delta) := \bigoplus_{\substack{J\subseteq\Delta, |J|=\bullet \\ \mu\in X_*(\mathbb{T}/\mathbb{Z})^+, \mu+\rho_J<\lambda}} M_\mu^{I_1} \otimes_E Ee_J.$$

If $f \in M_\lambda^{I_1}$ and $J \subseteq \Delta$ with $|J| = r$ then the boundary maps of the Koszul complex $M_\theta^{I_1} \otimes_E \bigwedge^\bullet E^\Delta$ are given by

$$\partial_r(f \otimes e_J) = \sum_{\alpha\in J} sgn(\alpha, J)(T_\alpha - \chi(T_\alpha))(f) \otimes e_{J\setminus\{\alpha\}},$$

where the sign $sgn(\alpha, J) := (-1)^i$ if $J = \{\alpha_{j_1}, \ldots, \alpha_{j_r}\}$ with $j_1 < \ldots < j_r$ and $\alpha = \alpha_{j_i}$.

Assume that $\mu \in X_*(\mathbb{T}/\mathbb{Z})^+$ is represented by $s \in T$ with $\mu + \rho_J \leq \lambda \in X_*(\mathbb{T}/\mathbb{Z})^+$ for some subset J of Δ. If $f \in M_\mu^{I_1}$ then (1) shows that $T_\alpha(f)$ is supported on $KsKt_\alpha K$ where the latter is a finite union of double cosets $Kt'K$ such that the image λ' of t' in $X_*(\mathbb{T}/\mathbb{Z})$ satisfies $\lambda' \leq \mu + \lambda_\alpha$ for the usual dominance relation (cf. [7], Proposition 4.4.4 (iii)). Therefore, the boundary maps $\partial_\bullet$ are filtered of degree zero, i.e. leave $\mathrm{Fil}^\lambda_\bullet$ and $\mathrm{Fil}^{\lambda-}_\bullet$ invariant. More precisely, $(T_\alpha - \chi(T_\alpha))(f) \otimes e_{J\setminus\{\alpha\}} \equiv pr_{\mu+\lambda_\alpha} \circ T_\alpha(f) \otimes e_{J\setminus\{\alpha\}}$ mod $\mathrm{Fil}^{\lambda-}_{|J|-1}$, so that the associated graded complex

$$\mathrm{gr}(M_\theta^{I_1} \otimes_E \bigwedge^\bullet E^\Delta) := \bigoplus_{\lambda\in X_*(\mathbb{T}/\mathbb{Z})^+} \underbrace{\mathrm{Fil}^\lambda_\bullet/\mathrm{Fil}^{\lambda-}_\bullet}_{=:\mathrm{gr}^\lambda_\bullet} \cong_E M_\theta^{I_1} \otimes_E \bigwedge^\bullet E^\Delta$$

is the Koszul complex associated with the family of commuting E-linear endomorphisms $\mathrm{gr}(T_\alpha)$, $\alpha \in \Delta$, of $M_\theta^{I_1}$ sending $f \in M_\lambda^{I_1}$ to $(pr_{\lambda+\lambda_\alpha} \circ T_\alpha)(f) \in M_{\lambda+\lambda_\alpha}^{I_1}$. In particular, this graded Koszul complex is independent of the character χ. We denote its boundary maps by $\mathrm{gr}(\partial_\bullet)$.

We claim that it suffices to show that this associated graded complex is acyclic. Since we are not in the rank one situation treated in [13], I.4.2 Theorem 4, this requires an extra argument. The technical problem is that there are usually infinitely many elements $\mu \in X_*(\mathbb{T}/\mathbb{Z})^+$ with $\mu \leq \lambda$ for a given dominant cocharacter λ of $\mathbb{T}/\mathbb{Z}$. Further, $\mathrm{Fil}_\bullet^{\lambda-}$ need not be equal to $\mathrm{Fil}_\bullet^{\mu}$ for any $\mu < \lambda$.

Let us now prove the claim. If $F \in \ker(\partial_r)$ is not equal to zero, write $F = \sum_{|J|=r} f_J \otimes e_J$. By abuse of notation we call the subset

$$\mathrm{supp}(F) := \{\lambda + \rho_J \mid |J| = r,\ \lambda \in X_*(\mathbb{T}/\mathbb{Z})^+,\ pr_\lambda(f_J) \neq 0\}$$

of $X_*(\mathbb{T}/\mathbb{Z})^+$ the support of F. Let $\{\mu_1, \ldots, \mu_N\}$ with $0 = \mu_1 < \ldots < \mu_N$ be the finite set of elements of $X_*(\mathbb{T}/\mathbb{Z})^+$ which are less than or equal to some element of $\mathrm{supp}(F)$ for the usual dominance order (not its refinement). Then μ_N is the maximal element of $\mathrm{supp}(F)$ so that $F \in \mathrm{Fil}_r^{\mu_N}$. Denoting by $\overline{F}$ the image of F in $\mathrm{gr}_r^{\mu_N}$ we have $\overline{F} \in \ker(\mathrm{gr}(\partial_r))$. By our assumption, there is $\overline{G} \in \mathrm{gr}_{r+1}^{\mu_N}$ with $\overline{F} = \mathrm{gr}(\partial_{r+1})(\overline{G})$. Choose $G \in \bigoplus_{|J|=r+1} M_{\mu_N-\rho_J}^{I_1} \otimes Ee_J$ whose image in $gr_{r+1}^{\mu_N}$ is $\overline{G}$. We have $F - \partial_{r+1}(G) \in \mathrm{Fil}_r^{\mu_N-}$.

On the other hand, we have seen above that the support of $\partial_{r+1}(G)$ is contained in the set of all $\mu \in X_*(\mathbb{T}/\mathbb{Z})^+$ for which $\mu \leq \mu_N$ with respect to the usual dominance relation (not its refinement). This implies that $F - \partial_{r+1}(G)$ is supported on $\{\mu_1, \ldots, \mu_{N-1}\}$ and hence is contained in $\mathrm{Fil}^{\mu_{N-1}}$. It is precisely for this conclusion that we require the set $\{\mu_1, \ldots, \mu_{N-1}\}$ to be totally ordered. Note that together with μ_j, $1 \leq j \leq N-1$, the set $\{\mu_1, \ldots, \mu_{N-1}\}$ contains all elements $\mu \in X_*(\mathbb{T}/\mathbb{Z})^+$ which are less than or equal to μ_j for the usual dominance relation. If $F' := F - \partial_{r+1}(G) \in \ker(\partial_r)$ is non-zero, the analogous procedure for F' yields a set $\{\mu'_1, \ldots, \mu'_{N'}\}$ contained in $\{\mu_1, \ldots, \mu_{N-1}\}$. We can thus proceed inductively and obtain $F \in \mathrm{im}(\partial_{r+1})$ after finitely many steps. Note that $\mathrm{Fil}_\bullet^{0-} = 0$. This proves the claim.

We will now show that the graded piece $\mathrm{gr}_\bullet^\lambda$ of our complex is acyclic for any $\lambda \in X_*(\mathbb{T}/\mathbb{Z})^+$. We have

$$\mathrm{gr}_\bullet^\lambda = [0 \longrightarrow \bigoplus_{\substack{J \subseteq \Delta(\lambda) \\ |J| = \bullet}} M_{\lambda-\rho_J}^{I_1} \otimes_E Ee_J \longrightarrow 0] \tag{4}$$

with boundary maps

$$\mathrm{gr}(\partial_r)(f_J \otimes e_J) = \sum_{\alpha \in J} sgn(\alpha, J)\mathrm{gr}(T_\alpha)(f_J) \otimes e_{J\setminus\{\alpha\}}.$$

We will distinguish two cases. First assume that there is $\alpha \in \Delta$ with $\lambda - 2\lambda_\alpha \in X_*(\mathbb{T}/\mathbb{Z})^+$. In particular, $\alpha \in \Delta(\lambda)$. For $0 \leq r < |\Delta(\lambda)|$ we define the E-linear map $\iota_r : \mathrm{gr}_r^\lambda \to \mathrm{gr}_{r+1}^\lambda$ as follows. If $J \subseteq \Delta(\lambda)$ with $|J| = r$ and if $f_J \in M_{\lambda-\rho_J}^{I_1}$ then

$$\iota_r(f_J \otimes e_J) := \begin{cases} sgn(\alpha, J \cup \{\alpha\})\mathrm{gr}(T_\alpha)^{-1}(f_J) \otimes e_{J\cup\{\alpha\}}, & \text{if } \alpha \notin J, \\ 0 & , \quad \text{if } \alpha \in J. \end{cases}$$

Note that $\mathrm{gr}(T_\alpha) : M_{\lambda-\rho_{J\cup\{\alpha\}}}^{I_1} \to M_{\lambda-\rho_J}^{I_1}$ is an E-linear isomorphism for any J not containing α since $\alpha \in \Delta(\lambda - \rho_{J\cup\{\alpha\}})$ by our assumption on λ and because of Corollary 1.11. We also set $\iota_{-1} := 0$ and $\iota_{|\Delta(\lambda)|} := 0$.

If $0 \leq r \leq |\Delta(\lambda)|$ and $J \subseteq \Delta(\lambda)$ with $|J| = r$ then $\mathrm{gr}(\partial_{r+1}) \circ \iota_r$ maps $f_J \otimes e_J$ to 0 if $\alpha \in J$ and to

$$\sum_{\beta \in J\cup\{\alpha\}} sgn(\beta, J \cup \{\alpha\})sgn(\alpha, J \cup \{\alpha\})\mathrm{gr}(T_\beta)\mathrm{gr}(T_\alpha)^{-1}(f_J) \otimes e_{(J\cup\{\alpha\})\setminus\{\beta\}}$$

if $\alpha \notin J$. On the other hand, $\iota_{r-1} \circ \mathrm{gr}(\partial_r)$ maps $f_J \otimes e_J$ to $f_J \otimes e_J$ if $\alpha \in J$ and to

$$\sum_{\beta \in J} sgn(\alpha, (J \setminus \{\beta\}) \cup \{\alpha\})sgn(\beta, J)\mathrm{gr}(T_\alpha)^{-1}\mathrm{gr}(T_\beta)(f_J) \otimes e_{(J\setminus\{\beta\})\cup\{\alpha\}}$$

if $\alpha \notin J$. Now $\mathrm{gr}(T_\alpha)^{-1}$ commutes with $\mathrm{gr}(T_\beta)$ whenever it is defined because $\mathrm{gr}(T_\alpha)$ commutes with $\mathrm{gr}(T_\beta)$. Moreover, if $\alpha \notin J$ and $\beta \in J$ one readily checks the sign relation

$$sgn(\alpha, (J \setminus \{\beta\}) \cup \{\alpha\})sgn(\beta, J) = -sgn(\beta, J \cup \{\alpha\})sgn(\alpha, J \cup \{\alpha\}).$$

As a consequence, we obtain $\mathrm{gr}(\partial_{r+1}) \circ \iota_r + \iota_{r-1} \circ \mathrm{gr}(\partial_r) = \mathrm{id}_{\mathrm{gr}_r^\lambda}$, so that $\iota_\bullet$ is a contracting homotopy of the complex $\mathrm{gr}_\bullet^\lambda$. Therefore, $\mathrm{gr}_\bullet^\lambda$ is even exact.

Now assume that $\lambda - 2\lambda_\alpha \notin X_*(\mathbb{T}/\mathbb{Z})^+$ for all $\alpha \in \Delta$. In this case we have $\lambda = \rho_{\Delta(\lambda)}$ and $\Delta(\lambda - \rho_J) = \Delta(\lambda) \setminus J$ for all subsets J of $\Delta(\lambda)$. By Proposition 1.7, Proposition 1.8 and Lemma 1.13, the graded piece $\mathrm{gr}_\bullet^\lambda$ can be identified with the natural complex

$$0 \longrightarrow \bigoplus_{\substack{J\subseteq\Delta(\lambda) \\ |J|=\bullet}} \mathrm{ind}_{\mathcal{P}_{-\lambda+\rho_J}(k)}^{\mathcal{G}(k)}(V_{\mathcal{N}_{-\lambda+\rho_J}})^{\mathcal{N}(k)} \longrightarrow 0.$$

Here $\mathcal{P}_{-\lambda+\rho_J}$ is opposite to the standard parabolic subgroup of $\mathcal{G}_k$ associated with the subset $\Delta \setminus (\Delta(\lambda) \setminus J)$. Therefore, we can rewrite this complex as

$$0 \longrightarrow \bigoplus_{\substack{\Delta\setminus\Delta(\lambda)\subseteq J\subseteq\Delta \\ |J|-|\Delta\setminus\Delta(\lambda)|=\bullet}} \operatorname{ind}_{\overline{\mathcal{P}}_J(k)}^{\mathcal{G}(k)}(V_{\overline{\mathcal{N}}_J(k)})^{\mathcal{N}(k)} \longrightarrow 0.$$

In Theorem 3.2 below, complexes of this type will be shown to be acyclic. □

The following consequence of the freeness assertion in Theorem 2.1 was first proved by Herzig, using different methods (cf. [12], Corollary 6.5).

Corollary 2.2. *For any irreducible E-linear representation V of $\mathcal{G}(k)$ the $\mathcal{H}$-module $M = \operatorname{ind}_K^G(V)$ is torsion free.*

Proof. Since I_1 is a pro-p group, it suffices to see that the $\mathcal{H}$-module M^{I_1} is torsion free (cf. [16], Lemma 2.1). By Theorem 2.1 the latter is even free. □

Corollary 2.3. *For any irreducible E-linear representation V of $\mathcal{G}(k)$ the rank of $M^{I_1} = \operatorname{ind}_K^G(V)^{I_1}$ as a module over the spherical Hecke algebra $\mathcal{H}$ is equal to the order of the Weyl group W.*

Proof. We take up the notation of the proof of Theorem 2.1. Choose an arbitrary character $\theta : \mathcal{H}_0 \to E$ of $\mathcal{H}_0$ and consider the Koszul complex $(C^\bullet, \partial_\bullet) = M_\theta^{I_1} \otimes_E \bigwedge^\bullet E^\Delta$ associated with the family of endomorphisms T_α, $\alpha \in \Delta$, of $M_\theta^{I_1} := \bigoplus_{\lambda \in X_*(\mathbb{T}/\mathbb{Z})^+} M_\lambda^{I_1}$. As above, we endow it with a filtration $\mathrm{Fil}_\bullet^\lambda$ indexed by $\lambda \in X_*(\mathbb{T}/\mathbb{Z})^+$. By the proof of Theorem 2.1 it suffices to show that the E-vector space $C_{-1} := M_\theta^{I_1} / \sum_{\alpha\in\Delta} T_\alpha(M_\theta^{I_1})$ has dimension $|W|$.

We endow C_{-1} with the quotient filtration induced by $\mathrm{Fil}_0(C_\bullet)$, i.e.

$$\begin{aligned} \mathrm{Fil}_{-1}^\lambda &:= (\mathrm{Fil}_0^\lambda + \operatorname{im}(\partial_0))/\operatorname{im}(\partial_0) \quad \text{and} \\ \mathrm{Fil}_{-1}^{\lambda-} &:= (\mathrm{Fil}_0^{\lambda-} + \operatorname{im}(\partial_0))/\operatorname{im}(\partial_0). \end{aligned}$$

All E-linear maps in the exact complex

$$C_2 \xrightarrow{\partial_1} C_1 \xrightarrow{\partial_0} C_0 \xrightarrow{\partial_{-1}} C_{-1} \longrightarrow 0$$

respect the filtrations so that by passing to the graded objects we obtain a complex

$$\mathrm{gr}(C_2) \xrightarrow{\mathrm{gr}(\partial_1)} \mathrm{gr}(C_1) \xrightarrow{\mathrm{gr}(\partial_0)} \mathrm{gr}(C_0) \xrightarrow{\mathrm{gr}(\partial_{-1})} \mathrm{gr}(C_{-1}) \longrightarrow 0$$

of E-vector spaces that we claim to be exact. The exactness at $\mathrm{gr}(C_1)$ was shown in the proof of Theorem 2.1. We claim that this implies that ∂_0 is strict, i.e. that it satisfies $\mathrm{im}(\partial_0) \cap \mathrm{Fil}_0^\lambda = \partial_0(\mathrm{Fil}_1^\lambda)$ for any $\lambda \in X_*(\mathbb{T}/\mathbb{Z})^+$. Since we are not in the rank one situation treated in [13], I.4.2 Theorem 4(2), this requires an argument close to the one given in the proof of Theorem 2.1.

Let $F = \sum_{\alpha\in\Delta} f_\alpha \otimes e_\alpha \in C_1$ be non-zero, define the support $\mathrm{supp}(F) \subseteq X_*(\mathbb{T}/\mathbb{Z})^+$ of F as in the proof of Theorem 2.1 and let $\{\mu_1, \ldots, \mu_N\}$ with $0 = \mu_1 < \ldots < \mu_N$ be the finite set of elements of $X_*(\mathbb{T}/\mathbb{Z})^+$ which are less than or equal to one of the elements of $\mathrm{supp}(F)$ for the usual dominance relation (not its refinement). We then have $F \in \mathrm{Fil}_1^{\mu_N}$ and $\partial_0(F) \in \mathrm{Fil}_0^{\mu_N}$. As seen before, the support of $\partial_0(F)$ is contained in $\{\mu_1, \ldots, \mu_N\}$, as well. Therefore, if $\partial_0(F) \in \mathrm{Fil}_0^{\mu_N-}$ then $\partial_0(F) \in \mathrm{Fil}_0^{\mu_{N-1}}$. Further, the image of F in $\mathrm{gr}_1^{\mu_N}$ is then contained in the kernel of $\mathrm{gr}(\partial_0)$. Hence, there is $G \in \bigoplus_{|J|=2} M^{I_1}_{\mu_N-\rho_J} \otimes_E Ee_J \subset \mathrm{Fil}_2^{\mu_N}$ whose image in $\mathrm{gr}_2^{\mu_N}$ maps to the image of F in $\mathrm{gr}_1^{\mu_N}$ under $\mathrm{gr}(\partial_1)$. The support of $\partial_1(G)$ is again contained in $\{\mu_1, \ldots, \mu_N\}$. As a consequence, the support of $F - \partial_1(G)$ is contained in $\{\mu_1, \ldots, \mu_{N-1}\}$. Since $\partial_0(F) = \partial_0(F - \partial_1(G))$ we obtain $\partial_0(F) \in \partial_0(\mathrm{Fil}_1^{\mu_{N-1}})$ and may replace F by $F - \partial_1(G)$. Proceeding inductively and using $\mathrm{Fil}_0^{0-} = 0$ the strictness of ∂_0 follows after finitely many steps.

Adjusting the proof of [13], I.4.2 Theorem 4(1), in a similar fashion, the strict exactness of the complex

$$C_1 \xrightarrow{\partial_0} C_0 \xrightarrow{\partial_{-1}} C_{-1} \longrightarrow 0$$

implies the exactness of the associated graded complex, as desired.

Since the filtration Fil_{-1} of C_{-1} is exhaustive and $\mathrm{Fil}_{-1}^{0-} = 0$, the E-vector spaces C_{-1} and $\mathrm{gr}(C_{-1})$ have the same dimension. By our above arguments, this dimension in turn is equal to the dimension of *coker*$(\mathrm{gr}(\partial_0))$. By the proof of Theorem 2.1 and by Corollary 3.3 below we have

$$\begin{aligned} \dim_E [\mathit{coker}(\mathrm{gr}(\partial_0))] &= \sum_{I\subseteq\Delta} \sum_{I\subseteq J} (-1)^{|J|-|I|} |W/W_J| \\ &= \sum_{J\subseteq\Delta} \Big(\sum_{I\subseteq J} (-1)^{|J|-|I|}\Big) |W/W_J|. \end{aligned}$$

The summand corresponding to $J = \emptyset$ gives $|W|$. If $J \neq \emptyset$ then

$$\sum_{I\subseteq J} (-1)^{|J|-|I|} = \sum_{n=0}^{|J|} \binom{|J|}{n} (-1)^{|J|-n} = (1-1)^{|J|} = 0.$$

□

3 An acyclicity result for finite reductive groups

In order to simplify our notation we will assume in this section that $\mathcal{G}$ is a split connected reductive group over the finite field k of characteristic p with a maximal k-split k-torus $\mathcal{T}$ and a Borel k-subgroup $\mathcal{B}$ with unipotent radical $\mathcal{N}$. As before we denote by Φ the root system of $(\mathcal{G}, \mathcal{T})$ with positive roots Φ^+ corresponding to $\mathcal{B}$, negative roots Φ^- and positive simple roots Δ.

For any subset J of Δ we denote by $\mathcal{P}_J$ the standard parabolic subgroup of $\mathcal{G}$ containing $\mathcal{B}$ and corresponding to J. In particular, $\mathcal{P}_\emptyset = \mathcal{B}$. Let further $\mathcal{N}_J$ denote the unipotent radical of $\mathcal{P}_J$ and $\mathcal{M}_J$ the Levi subgroup of $\mathcal{P}_J$ containing $\mathcal{T}$. We denote by W_J the subgroup of W generated by the simple reflections s_α with $\alpha \in J$. We denote by ${}^J W$ the set of minimal length coset representatives of W/W_J. Finally, we denote by $\overline{\mathcal{B}}$ the Borel subgroup of $\mathcal{G}$ opposite to $\mathcal{B}$ and by $\overline{\mathcal{N}}$ its unipotent radical.

We write $\leq$ for the Bruhat ordering of W in which $v \leq w$ if and only if v can be written as a subexpression of some reduced expression of w in terms of the simple reflections s_α, $\alpha \in \Delta$ (cf. [14], Theorem 5.10). If $J \subseteq \Delta$ and $w \in {}^J W$, for example, then $w \leq ww'$ for any $w' \in W_J$. This follows from the fact that the length of ww' is the sum of the lengths of w and w'.

Let E be any algebraically closed field containing k and let V be any E-linear irreducible representation of the finite group $\mathcal{G}(k)$ of Lie type. According to [11], Lemma 2.5 (i), the E-vector space $V^{\overline{\mathcal{N}}(k)}$ is one dimensional. We fix a non-zero element $v \in V^{\overline{\mathcal{N}}(k)}$. The crucial fact that we are going to need is that for any subset J of Δ the natural map $V^{\overline{\mathcal{N}}(k)} \to (V_{\mathcal{N}_J(k)})^{\mathcal{M}_J(k) \cap \overline{\mathcal{N}}(k)}$ is bijective (cf. [11], Lemma 2.5 (ii), noting that $\overline{\mathcal{N}}(k) = \overline{\mathcal{N}}_J(k)(M_J(k) \cap \overline{\mathcal{N}}(k))$. Here $V_{\mathcal{N}_J(k)}$ denotes the maximal quotient of V on which $\mathcal{N}_J(k)$ acts trivially, viewed as a representation of $\mathcal{P}_J(k)$.

For any subset $J \subseteq \Delta$ we have the induced E-linear $\mathcal{G}(k)$-representation $\mathrm{ind}_{\mathcal{P}_J(k)}^{\mathcal{G}(k)}(V_{\mathcal{N}_J(k)})$. We choose an enumeration $W = \{w_1, \ldots, w_N\}$ of W such that if $w_i \leq w_j$ for the Bruhat ordering then $i \leq j$. It gives rise to the following filtration $\mathrm{Fil}_J^\bullet$ on $\mathrm{ind}_{\mathcal{P}_J(k)}^{\mathcal{G}(k)}(V_{\mathcal{N}_J(k)})^{\overline{\mathcal{N}}(k)}$:

$$\mathrm{Fil}_J^j := \{f \in \mathrm{ind}_{\mathcal{P}_J(k)}^{\mathcal{G}(k)}(V_{\mathcal{N}_J(k)})^{\overline{\mathcal{N}}(k)} \mid f(\overline{\mathcal{B}}(k) w_i \mathcal{B}(k)) = 0 \text{ for all } i < j\}$$

if $1 \leq j \leq N$ and $\mathrm{Fil}_J^{N+1} := 0$.

Lemma 3.1. *Let J be a subset of Δ. For $1 \leq j \leq N$ the E-vector space $\mathrm{gr}_J^j := \mathrm{Fil}_J^j / \mathrm{Fil}_J^{j+1}$ is of dimension one if $w_j \in {}^J W$ and of dimension zero, otherwise. In particular, the E-dimension of $\mathrm{ind}_{\mathcal{P}_J(k)}^{\mathcal{G}(k)}(V_{\mathcal{N}_J(k)})^{\overline{\mathcal{N}}(k)}$ is equal to $|W/W_J|$.*

Proof. If $\mathrm{gr}_J^j \neq 0$ then there exists $f \in \mathrm{ind}_{\mathcal{P}_J(k)}^{\mathcal{G}(k)}(V_{\mathcal{N}_J(k)})^{\overline{\mathcal{N}}(k)}$ vanishing on $\overline{\mathcal{B}}(k)w_i\mathcal{B}(k)$ for $i < j$ but for which $f|\overline{\mathcal{B}}(k)w_j\mathcal{B}(k) \neq 0$. Assume $w_j \notin {}^JW$ and let $w_i \in {}^JW$ with $w_jW_J = w_iW_J$. Then $w_i \leq w_j$ with $w_i \neq w_j$ and therefore $i < j$. In particular, $f|\overline{\mathcal{B}}(k)w_i\mathcal{B}(k) = 0$. However, f is $\mathcal{P}_J(k)$-right equivariant so that even $f|\overline{\mathcal{B}}(k)w_i\mathcal{P}_J(k) = 0$. Since $\mathcal{P}_J(k)$ contains representatives of the elements of W_J we have $\overline{\mathcal{B}}(k)w_iP_J(k) \supseteq \overline{\mathcal{B}}(k)w_j\mathcal{B}(k)$ and arrive at a contradiction.

Conversely, assume $w_j \in {}^JW$. First of all, we claim that in this case there is a subgroup $\mathcal{N}_J^j(k)$ of $\mathcal{N}_J(k)$ such that

$$w_j^{-1}\overline{\mathcal{N}}(k)w_j \cap \mathcal{P}_J(k) = \mathcal{N}_J^j(k) \cdot (\mathcal{M}_J(k) \cap \overline{\mathcal{N}}(k)).$$

Indeed, $w_j^{-1}\overline{\mathcal{N}}(k)w_j = (w_j^{-1}\overline{\mathcal{N}}(k)w_j \cap \mathcal{N}(k)) \cdot (w_j^{-1}\overline{\mathcal{N}}(k)w_j \cap \overline{\mathcal{N}}(k))$. By [22], Lemma 3.1.2 (a), we have $w_j^{-1}\Phi^- \cap \Phi^+ \subseteq \Phi^+ \setminus [J]^+$, where $[J]^+$ denotes the set of all positive roots which are linear combinations of the elements of J. Since $\Phi^+ \setminus [J]^+$ is precisely the set of roots whose corresponding root groups appear in $\mathcal{N}_J$ we obtain $N_J^j(k) := w_j^{-1}\overline{\mathcal{N}}(k)w_j \cap \mathcal{N}(k) \subseteq \mathcal{N}_J(k) \subset \mathcal{P}_J(k)$. This also implies $w_j^{-1}\overline{\mathcal{N}}(k)w_j \cap \mathcal{P}_J(k) = \mathcal{N}_J^j(k) \cdot (w_j^{-1}\overline{\mathcal{N}}(k)w_j \cap \mathcal{P}_J(k) \cap \overline{\mathcal{N}}(k))$, where

$$w_j^{-1}\overline{\mathcal{N}}(k)w_j \cap \mathcal{P}_J(k) \cap \overline{\mathcal{N}}(k) = w_j^{-1}\overline{\mathcal{N}}(k)w_j \cap \mathcal{M}_J(k) \cap \overline{\mathcal{N}}(k).$$

However, by [22], Lemma 3.1.2 (a), we have $-[J]^+ \subseteq w_j^{-1}\Phi^-$. Since $\mathcal{M}_J \cap \overline{\mathcal{N}}$ is the direct product of the root subgroups corresponding to the elements of $-[J]^+$ we obtain $\mathcal{M}_J(k) \cap \overline{\mathcal{N}}(k) \subseteq w_j^{-1}\overline{\mathcal{N}}(k)w_j$, proving our claim.

Next we claim that setting $f(xw_jy) := y^{-1}v + V_{\mathcal{N}_J(k)}$ if $x \in \overline{\mathcal{N}}(k)$, $y \in \mathcal{P}_J(k)$, and $f(g) := 0$ if $g \notin \overline{\mathcal{N}}(k)w_j\mathcal{P}_J(k) = \overline{\mathcal{B}}(k)w_j\mathcal{P}_J(k)$, gives a well-defined element f of $\mathrm{ind}_{\mathcal{P}_J(k)}^{\mathcal{G}(k)}(V_{\mathcal{N}_J(k)})^{\overline{\mathcal{N}}(k)}$. To see this, note that $v + V_{\mathcal{N}_J(k)}$ is invariant under $\mathcal{N}_J^j(k) \cdot (\mathcal{M}_J(k) \cap \overline{\mathcal{N}}(k)) = w_j^{-1}\overline{\mathcal{N}}(k)w_j \cap \mathcal{P}_J(k)$ because of our above reasoning. Apparently, $f \in \mathrm{Fil}_J^j \setminus \mathrm{Fil}_J^{j+1}$.

On the other hand, the map $V^{\overline{\mathcal{N}}(k)} \to (V_{\mathcal{N}_J(k)})^{\mathcal{M}_J(k) \cap \overline{\mathcal{N}}(k)}$ is bijective, so that the latter space is of dimension one over E. The same arguments as above then show that the E-vector space gr_J^j is at most one dimensional. $\square$

For any subset J of Δ and any element $\alpha \in \Delta \setminus J$ let $pr : V_{\mathcal{N}_{J\cup\{\alpha\}}(k)} \to V_{\mathcal{N}_J(k)}$ be the natural $\mathcal{P}_J(k)$-equivariant projection and consider the $\mathcal{G}(k)$-equivariant map

$$T_\alpha : \mathrm{ind}_{\mathcal{P}_{J\cup\{\alpha\}}(k)}^{\mathcal{G}(k)}(V_{\mathcal{N}_{J\cup\{\alpha\}}(k)}) \longrightarrow \mathrm{ind}_{\mathcal{P}_J(k)}^{\mathcal{G}(k)}(V_{\mathcal{N}_J(k)}), \quad f \mapsto pr \circ f.$$

It is obvious that T_α respects the above filtrations, i.e. satisfies $T_\alpha(\mathrm{Fil}^j_{J\cup\{\alpha\}}) \subseteq \mathrm{Fil}^j_J$ for all $1 \leq j \leq N+1$. Therefore, it induces a $\mathcal{T}(k)$-equivariant E-linear map $\mathrm{gr}(T_\alpha)$ on the corresponding graded objects.

Let $w_j \in {}^{J\cup\{\alpha\}}W \subset {}^JW$. By the proof of Lemma 3.1 there is a commutative $\mathcal{T}(k)$-equivariant diagram

$$\begin{array}{ccc} \mathrm{gr}^j_{J\cup\{\alpha\}} & \xrightarrow{\mathrm{gr}(T_\alpha)} & \mathrm{gr}^j_J \\ \cong\downarrow & & \downarrow\cong \\ V_{\mathcal{N}_{J\cup\{\alpha\}}(k)}^{\mathcal{M}_{J\cup\{\alpha\}}(k)\cap\overline{\mathcal{N}}(k)} & \xrightarrow{pr} & V_{\mathcal{N}_J(k)}^{M_J(k)\cap\overline{\mathcal{N}}(k)}, \end{array}$$

in which the vertical arrows are induced by evaluation at w_j. Let $\varphi_J : V_{\mathcal{N}_J(k)}^{M_J(k)\cap\overline{\mathcal{N}}(k)} \to E$ be the E-linear isomorphism sending the class of v to 1. Since pr sends the class of v in $V_{\mathcal{N}_{J\cup\{\alpha\}}(k)}$ to that of v in $V_{\mathcal{N}_J(k)}$, there is a commutative diagram of $\mathcal{T}(k)$-equivariant E-linear maps

$$\begin{array}{ccc} \mathrm{gr}(\mathrm{Fil}^\bullet_{J\cup\{\alpha\}}) & \xrightarrow{\mathrm{gr}(T_\alpha)} & \mathrm{gr}(\mathrm{Fil}^\bullet_J) \\ \cong\downarrow & & \downarrow\cong \\ E[{}^{J\cup\{\alpha\}}W] & \longrightarrow & E[{}^JW] \end{array}$$

in which the lower horizontal arrow is induced by the inclusion ${}^{J\cup\{\alpha\}}W \subset {}^JW$. The right vertical arrow sends the class of a function $f \in \mathrm{Fil}^j_J$ in gr^j_J to $(\varphi_J \circ f)(w_j) \cdot w_j$. The definition of the left vertical arrow is analogous.

Now let I be any subset of Δ and consider the complex

$$(5) \qquad 0 \longrightarrow \bigoplus_{\substack{I\subseteq J\subseteq\Delta \\ |J|-|I|=\bullet}} \mathrm{ind}_{\mathcal{P}_J(k)}^{\mathcal{G}(k)}(V_{\mathcal{N}_J(k)}) \longrightarrow 0$$

for the alternating sum of the natural face maps T_α, $\alpha \in \Delta \setminus J$, $I \subseteq J \subseteq \Delta$, with the same sign conventions as in the previous section. For the induced complex of $\overline{\mathcal{N}}(k)$-invariants we have the following very general acyclicity result.

Theorem 3.2. *For any subset I of Δ the complex*

$$(6) \qquad 0 \longrightarrow \bigoplus_{\substack{I\subseteq J\subseteq\Delta \\ |J|-|I|=\bullet}} \mathrm{ind}_{\mathcal{P}_J(k)}^{\mathcal{G}(k)}(V_{\mathcal{N}_J(k)})^{\overline{\mathcal{N}}(k)} \longrightarrow 0$$

is acyclic, i.e. has trivial homology in positive degrees.

Proof. As seen above, the complex (6) admits a decreasing, separated and exhaustive filtration with only finitely many jumps. By a standard argument it suffices to prove that the associated graded complex is acyclic (cf. [13], I.4.2 Theorem 4(5)). By the above remarks, the latter is isomorphic to the complex

$$0 \longrightarrow \bigoplus_{\substack{I \subseteq J \subseteq \Delta \\ |J|-|I|=\bullet}} E[{}^J W] \longrightarrow 0$$

for the alternating sum of the inclusions $E[{}^{J\cup\{\alpha\}}W] \subset E[{}^J W]$ whenever $I \subseteq J \subseteq \Delta$ and $\alpha \in \Delta \setminus J$. We apply [18], §2 Proposition 6, to the abelian group $E[{}^I W]$ and its family of subgroups $E[{}^{I\cup\{\alpha\}}W]$, $\alpha \in \Delta \setminus I$.

If A and B are two subsets of W then we have the obvious relations

$$E[A \cup B] = E[A] + E[B] \quad \text{and} \quad E[A \cap B] = E[A] \cap E[B]$$

inside $E[W]$. Together with the usual associativity and distributivity rules for unions and intersections of sets this gives

$$\Big(\sum_{\alpha \in M} E[{}^{I\cup\{\alpha\}}W]\Big) \cap \Big(\bigcap_{\beta \in N} E[{}^{I\cup\{\beta\}}W]\Big) = \sum_{\alpha \in M} \Big(E[{}^{I\cup\{\alpha\}}W] \cap \Big(\bigcap_{\beta \in N} E[{}^{I\cup\{\beta\}}W]\Big)\Big)$$

for any two subsets M, N of $\Delta \setminus I$. According to the proof of [18], §2 Proposition 6, our complex is an acyclic resolution of $\sum_{\alpha \in \Delta \setminus I} E[{}^{I\cup\{\alpha\}}W]$ as a subgroup of $E[{}^I W]$. □

Corollary 3.3. *For any subset I of Δ the 0-th homology group of the complex (6) has dimension*

$$\sum_{I \subseteq J \subseteq \Delta} (-1)^{|J|-|I|} |W/W_J|$$

over E.

Proof. This follows from the dimension formula for exact sequences of finite dimensional E-vector spaces, as well as from Lemma 3.1 and Theorem 3.2. □

Without passing to the graded objects one can still prove the distributivity property needed to apply [18], §2 Proposition 6, to the proof of Theorem 3.2. This direct strategy is followed in [9] if V is the trivial representation (cf. [9], Proposition 3.2.9 and Theorem 7.1.10). It seems more elaborate than our filtered approach. Since the authors of [9] work over a field of characteristic zero they can even deduce the acyclicity of the original complex (5) from the result in Theorem 3.2. This is not possible in natural characteristic, as we shall now explain.

If $J \subseteq \Delta$ and $\alpha \notin J$ then $\mathcal{N}_{J\cup\{\alpha\}}(k) \subset \mathcal{N}_J(k)$, and we have the $\mathcal{P}_J(k)$-equivariant inclusion $V^{\mathcal{N}_J(k)} \subset V^{\mathcal{N}_{J\cup\{\alpha\}}(k)} \subset \operatorname{ind}_{\mathcal{P}_{J\cup\{\alpha\}}(k)}^{\mathcal{G}(k)}(V^{\mathcal{N}_{J\cup\{\alpha\}}(k)})$. By Frobenius reciprocity it gives rise to a $\mathcal{G}(k)$-equivariant map

$$S_\alpha : \operatorname{ind}_{\mathcal{P}_J(k)}^{\mathcal{G}(k)}(V^{\mathcal{N}_J(k)}) \longrightarrow \operatorname{ind}_{\mathcal{P}_{J\cup\{\alpha\}}(k)}^{\mathcal{G}(k)}(V^{\mathcal{N}_{J\cup\{\alpha\}}(k)}).$$

For any subset I of Δ we may then consider the complex

$$0 \longrightarrow \bigoplus_{\substack{I\subseteq J\subseteq\Delta \\ |J|-|I|=\bullet}} \operatorname{ind}_{\mathcal{P}_J(k)}^{\mathcal{G}(k)}(V^{\mathcal{N}_J(k)}) \longrightarrow 0 \tag{7}$$

for the alternating sum of the face maps S_α, $\alpha \in \Delta\setminus J$, $I \subseteq J \subseteq \Delta$, with the same sign conventions as in the previous section. Up to its numbering, this is the chain complex of the fixed-point sheaf on the spherical building of $\mathcal{G}$ associated with V, as studied by Ronan and Smith (cf. [17], page 322 and page 324). Although it is generally not acyclic (cf. [17], Section 1, Example 4), we have the following general acyclicity result for the associated complex of $\overline{\mathcal{N}}(k)$-coinvariants.

Theorem 3.4. *For any subset I of Δ the complex*

$$0 \longrightarrow \bigoplus_{\substack{I\subseteq J\subseteq\Delta \\ |J|-|I|=\bullet}} \operatorname{ind}_{\mathcal{P}_J(k)}^{\mathcal{G}(k)}(V^{\mathcal{N}_J(k)})_{\overline{\mathcal{N}}(k)} \longrightarrow 0 \tag{8}$$

is acyclic, i.e. has trivial homology in positive degrees. Its 0-th homology group has dimension $\sum_{I\subseteq J\subseteq\Delta}(-1)^{|J|-|I|}|W/W_J|$ over E.

Proof. As in Theorem 3.2 and Corollary 3.3 this can be proved by endowing the complex (8) with a suitable filtration and by analyzing the associated graded complex. One can also deduce the assertions directly from Theorem 3.2 and Corollary 3.3 by making use of the following duality argument.

For any finite group H and any finite dimensional E-linear representation W of H we denote by $W^* := \operatorname{Hom}_E(W, E)$ the E-linear dual of W endowed with the contragredient representation of H. Dualizing the inclusion $W^H \subseteq W$ gives rise to a natural E-linear surjection $(W^*)_H \to (W^H)^*$. It is in fact bijective because $((W^*)_H)^* \subseteq (W^{**})^H = W^H$. Further, if H is a subgroup of some finite group G then the map

$$\operatorname{ind}_H^G(W^*) \longrightarrow \operatorname{ind}_H^G(W)^*, \quad F \mapsto (f \mapsto \sum_{g\in G/H} F(g)(f(g)), \tag{9}$$

is a natural G-equivariant bijection. In fact, it is easily seen to be injective and hence bijective for dimension reasons.

Now note that the natural map from V into its E-linear double dual V^{**} is a $\mathcal{G}(k)$-equivariant bijection. Therefore, V^{**} is irreducible over $\mathcal{G}(k)$ and so must be V^*. Dualizing the complex (8), the above arguments show that we obtain a complex of the form

$$0 \longrightarrow \bigoplus_{\substack{I\subseteq J\subseteq \Delta \\ |J|-|I|=\bullet}} \operatorname{ind}_{\mathcal{P}_J(k)}^{\mathcal{G}(k)}((V^*)_{\mathcal{N}_J(k)})^{\overline{\mathcal{N}}(k)} \longrightarrow 0.$$

We claim that it coincides with the complex (6) associated with the irreducible $\mathcal{G}(k)$-representation V^*. This will follow once we can show that the diagram

$$\begin{array}{ccc}
\operatorname{ind}_{\mathcal{P}_{J\cup\{\alpha\}}(k)}^{\mathcal{G}(k)}(V^{\mathcal{N}_{J\cup\{\alpha\}}(k)})^* & \xrightarrow{\;S_\alpha^*\;} & \operatorname{ind}_{\mathcal{P}_J(k)}^{\mathcal{G}(k)}(V^{\mathcal{N}_J(k)})^* \\
\cong\uparrow & & \uparrow\cong \\
\operatorname{ind}_{\mathcal{P}_{J\cup\{\alpha\}}(k)}^{\mathcal{G}(k)}((V^*)_{\mathcal{N}_{J\cup\{\alpha\}}(k)}) & \xrightarrow{\;T_\alpha\;} & \operatorname{ind}_{\mathcal{P}_J(k)}^{\mathcal{G}(k)}((V^*)_{\mathcal{N}_J(k)})
\end{array}$$

is commutative for any $J \subseteq \Delta$ and $\alpha \notin J$. Here the vertical arrows are as in (9) and use the identifications $(V^*)_{\mathcal{N}_{J\cup\{\alpha\}}(k)} \cong (V^{\mathcal{N}_{J\cup\{\alpha\}}(k)})^*$ and $(V^*)_{\mathcal{N}_J(k)} \cong (V^{\mathcal{N}_J(k)})^*$. Further, S_α^* is the transpose of S_α. For the commutativity of the above diagram, it suffices to see that the diagram of natural maps

$$\begin{array}{ccc}
(V^{\mathcal{N}_{J\cup\{\alpha\}}(k)})^* & \longrightarrow & (V^{\mathcal{N}_J(k)})^* \\
\uparrow & & \uparrow \\
(V^*)_{\mathcal{N}_{J\cup\{\alpha\}}(k)} & \longrightarrow & (V^*)_{\mathcal{N}_J(k)}
\end{array}$$

is commutative, which is obvious. Since the functor $(\cdot)^*$ is an exact autoequivalence of the category of finite dimensional E-vector spaces, the assertions follow from Theorem 3.2 and Corollary 3.3. □

4 Supercuspidality and smooth duals

We turn back to the notation introduced in Section 1. In particular, $M = \operatorname{ind}_K^G(V)$ is a smooth E-linear representation of G and a module over the spherical Hecke algebra $\mathcal{H} = \operatorname{End}_G(M)$.

For any subgroup H of G we denote by M_H the space of H-coinvariants of M, i.e. the largest quotient of M on which H acts trivially. The kernel of the natural map $M \to M_H$ will be denoted by $M(H)$. For certain subgroups H of G, the action of the Hecke operators T_α, $\alpha \in \Delta$, on M_H simplifies as follows.

Lemma 4.1. *Let $\alpha \in \Delta$ and $t \in t_\alpha^{-1}T^-$. If $v \in V$ then*

$$T_\alpha([t,v]) \equiv \sum_{tIt_\alpha K = \coprod txt_\alpha K} [txt_\alpha, \xi_\alpha(v)] \mod M(K_1 \cap \overline{\mathbb{N}}_{\Delta\setminus\{\alpha\}}(F)).$$

Proof. Consider the generalized Bruhat decomposition

$$\mathcal{G}(k) = \coprod_{w \in W/W_{\Delta\setminus\{\alpha\}}} \overline{\mathcal{N}}(k) w \overline{\mathcal{P}}_{\Delta\setminus\{\alpha\}}(k)$$

of $\mathcal{G}(k)$. As in the previous section we denote by ${}^{\Delta\setminus\{\alpha\}}W$ the set of coset representatives of $W/W_{\Delta\setminus\{\alpha\}}$ which are of minimal length. We then have

$$\overline{\mathcal{N}}(k) w \overline{\mathcal{P}}_{\Delta\setminus\{\alpha\}}(k) = \coprod_{n \in \overline{\mathcal{N}}(k) \cap w\mathcal{N}(k)w^{-1}} n w \overline{\mathcal{P}}_{\Delta\setminus\{\alpha\}}(k)$$

for any $w \in {}^{\Delta\setminus\{\alpha\}}W$ (cf. [2], Lemma 21.14 and Proposition 21.29). Thus, we obtain

$$K = \coprod_{w \in {}^{\Delta\setminus\{\alpha\}}W} \coprod_{n \in \overline{\mathcal{N}}(k) \cap w\mathcal{N}(k)w^{-1}} n w I_{t_\alpha}$$

by applying red^{-1} to the above decomposition of $\mathcal{G}(k)$. This in turn yields

$$Kt_\alpha K = \coprod_{w \in {}^{\Delta\setminus\{\alpha\}}W} \coprod_{n \in \overline{\mathcal{N}}(k) \cap w\mathcal{N}(k)w^{-1}} n w I t_\alpha K,$$

because $I_{t_\alpha} = IK_{t_\alpha}$. For the rest of the proof we fix an arbitrary element $w \in {}^{\Delta\setminus\{\alpha\}}W$ with $w \neq 1$. As a set, $\overline{\mathcal{N}}(k) \cap w\mathcal{N}(k)w^{-1}$ is the direct product of the root groups $\mathcal{N}_\beta(k)$ with $\beta \in \Phi^- \cap w\Phi^+$. We claim that $w\Phi^+ \cap (\Phi^- \setminus [\Delta \setminus \{\alpha\}]^-) \neq \emptyset$ where $[\Delta \setminus \{\alpha\}]^-$ is the set of all roots which are negative linear combinations of the elements of $\Delta \setminus \{\alpha\}$.

To see this, let $\sigma \in W_{\Delta\setminus\{\alpha\}}$ be such that σw is of minimal length in its *right* coset $W_{\Delta\setminus\{\alpha\}}w$. By [22], Lemma 3.1.2 (a), we have $\sigma w \Phi^+ \cap \Phi^- \subseteq \Phi^- \setminus [\Delta \setminus \{\alpha\}]^-$. Since the right hand side is stable under $W_{\Delta\setminus\{\alpha\}}$ we obtain $w\Phi^+ \cap \sigma^{-1}\Phi^- \subseteq \Phi^- \setminus [\Delta \setminus \{\alpha\}]^-$, giving the claim unless the left hand side is empty. This is true if and only if

$$0 = |w\Phi^+ \cap \sigma^{-1}\Phi^-| = |\sigma w \Phi^+ \cap \Phi^-|,$$

which is the length of σw. This is zero if and only if $\sigma w = 1$ which is equivalent to $w = \sigma^{-1} \in W_{\Delta\setminus\{\alpha\}}$. This, however, implies $w = 1$ because w is of minimal length in $wW_{\Delta\setminus\{\alpha\}}$. Since we assumed $w \neq 1$, the claim follows.

Now choose $\beta \in w\Phi^+ \cap (\Phi^- \setminus [\Delta \setminus \{\alpha\}]^-)$ and put

$$\mathcal{N}'(k) := \prod_{\substack{\gamma \in \Phi^- \cap w\Phi^+ \\ \gamma \neq \beta}} \mathcal{N}_\gamma(k).$$

By (1) and the above decomposition of $Kt_\alpha K$ it suffices to show that

$$\sum_{n\in\mathcal{N}_\beta(k)}\sum_{x\in I/(I\cap t_\alpha K t_\alpha^{-1})}[tn'nwxt_\alpha,\xi_\alpha(w^{-1}n^{-1}(n')^{-1}v)]$$

is contained in $M(K_1\cap\overline{\mathbb{N}}_{\Delta\setminus\{\alpha\}}(F))$ for any $n'\in\mathcal{N}'(k)$. Note that the natural map $K_1/(K_1\cap t_\alpha K t_\alpha^{-1})\to I/(I\cap t_\alpha K t_\alpha^{-1})$ is bijective so that we may choose the representatives x to lie in K_1 and hence to act trivially on V. Since $tn't^{-1}$ is contained in $\overline{\mathcal{N}}(\mathfrak{o})$ and therefore stabilizes $M(K_1\cap\overline{\mathbb{N}}_{\Delta\setminus\{\alpha\}}(F))$, it suffices to prove the corresponding statement for

$$\begin{aligned}&\sum_{n\in\mathcal{N}_\beta(k)}\sum_{x\in I/(I\cap t_\alpha K t_\alpha^{-1})}[tnwxt_\alpha,\xi_\alpha(w^{-1}n^{-1}(n')^{-1}v)]\\ =&\sum_{n\in\mathcal{N}_\beta(k)}tnt^{-1}\sum_{x\in I/(I\cap t_\alpha K t_\alpha^{-1})}[twxt_\alpha,\xi_\alpha(w^{-1}n^{-1}(n')^{-1}v)].\end{aligned}$$

Now $t\mathcal{N}_\beta(\mathfrak{o})t^{-1}\subseteq K_1\cap\overline{\mathbb{N}}_{\Delta\setminus\{\alpha\}}(F)$ because $t\in t_\alpha^{-1}T^-$ and since the root β is contained in $\Phi^-\setminus[\Delta\setminus\{\alpha\}]^-$, hence has a negative contribution from α. Therefore, it suffices to show that

$$\begin{aligned}0&=\sum_{n\in\mathcal{N}_\beta(k)}[twxt_\alpha,\xi_\alpha(w^{-1}n^{-1}(n')^{-1}v)]\\ &=[twxt_\alpha,\xi_\alpha(\sum_{n\in\mathcal{N}_\beta(k)}(w^{-1}nw)w^{-1}(n')^{-1}v)]\end{aligned}$$

for any $x\in I$. In fact, we shall see that the E-linear endomorphism $\varphi:=\xi_\alpha\circ\sum_{n\in\mathcal{N}_\beta(k)}w^{-1}nw$ of V is zero. Note that

$$\sum_{n\in\mathcal{N}_\beta(k)}w^{-1}nw=\sum_{n\in\mathcal{N}_\gamma(k)}n$$

where $\gamma:=w^{-1}(\beta)\in\Phi^+\cap w^{-1}\Phi^-\subseteq\Phi^+\setminus[\Delta\setminus\{\alpha\}]^+$, the last inclusion coming from [22], Lemma 3.1.2 (a). Now $\Phi^+\setminus[\Delta\setminus\{\alpha\}]^+$ is precisely the set of roots whose corresponding root groups occur in $\mathcal{N}_{\Delta\setminus\{\alpha\}}$. Since the cardinality of $\mathcal{N}_\gamma(k)$ is a positive power of p and since E is of characteristic p, the endomorphism φ is zero on $V^{\mathcal{N}_{\Delta\setminus\{\alpha\}}(k)}$.

On the other hand, $V=V^{\mathcal{N}_{\Delta\setminus\{\alpha\}}(k)}\oplus\ker(\xi_\alpha)$ where the kernel of ξ_α is a sum of T_0-weight spaces of V (confer the proof of [11], Lemma 2.5). Since the group $\mathcal{N}_\gamma(k)$ is stable under T_0, the endomorphism $\sum_{n\in\mathcal{N}_\gamma(k)}n$ of V preserves the T_0-weight spaces of V. Therefore, it also preserves $\ker(\xi_\alpha)$. □

As an immediate consequence we obtain the following surjectivity result of T_α, provided that the fundamental dominant coweight λ_α is minuscule, i.e. satisfies $\lambda_\alpha(\check{\beta})\in\{0,1\}$ for any $\beta\in\Phi^+$.

Theorem 4.2. *If $\alpha \in \Delta$ is such that λ_α is minuscule then the endomorphism of $M_{\overline{\mathbb{N}}_{\Delta\setminus\{\alpha\}}(F)}$ induced by T_α is surjective. In particular, the $\overline{\mathbb{N}}_{\Delta\setminus\{\alpha\}}(F)$-coinvariants of $M/T_\alpha(M)$ and those of $M_0 := M/\sum_{\beta\in\Delta} T_\beta(M)$ are zero.*

Proof. Since $G = \overline{P}K$, the $\overline{P}$-representation M is generated by the E-subspace V of $M = \mathrm{ind}_K^G(V)$. Since the E-subspace $\mathrm{im}(T_\alpha)+M(\overline{\mathbb{N}}_{\Delta\setminus\{\alpha\}}(F))$ of M is $\overline{P}$-stable, it suffices to prove that it contains V. Letting $v \in V$, Lemma 4.1 implies that

$$\xi_\alpha(v) - T_\alpha([t_\alpha^{-1}, v]) \in M(\overline{\mathbb{N}}_{\Delta\setminus\{\alpha\}}(F))$$

because $t_\alpha^{-1} I t_\alpha \subseteq K$ by our assumption on λ_α (cf. [12], Sublemma 6.8). However, $v - \xi_\alpha(v) \in \ker(\xi_\alpha) = V(\overline{\mathcal{N}}_{\Delta\setminus\{\alpha\}}(k)) \subset M(\overline{\mathbb{N}}_{\Delta\setminus\{\alpha\}}(F))$, so that

$$v = \xi_\alpha(v) + v - \xi_\alpha(v) \equiv T_\alpha([t_\alpha^{-1}, v]) \mod M(\overline{\mathbb{N}}_{\Delta\setminus\{\alpha\}}(F)).$$

The final statements follow from the right exactness of the coinvariance functor. □

We call a (not necessarily irreducible) G-representation π *supercuspidal* if the space of coinvariants $\pi_{\mathbb{U}(F)}$ is equal to zero for the unipotent radical $\mathbb{U}$ of any proper parabolic F-subgroup $\mathbb{P}$ of $\mathbb{G}$. By Frobenius reciprocity such a representation does not admit any non-zero G-equivariant homomorphisms into representations of the form $\mathrm{ind}_P^G(\sigma)$ where σ is any smooth E-linear representation of $P = \mathbb{P}(F)$ on which $\mathbb{U}(F)$ acts trivially.

In a more precise form, the following result is due to Herzig (cf. [12], Corollary 1.2).

Theorem 4.3. *If the root system Φ is equal to A_d then the G-representation M_0 and all of its quotients are supercuspidal.*

Proof. Any proper parabolic F-subgroup of $\mathbb{G}$ is G-conjugate to one contained in $\overline{\mathbb{P}}_{\Delta\setminus\{\alpha\}}$ for some $\alpha \in \Delta$ (cf. [2], Proposition 21.12). By the right exactness of the coinvariance functor it suffices to show that the space of $\overline{\mathbb{N}}_{\Delta\setminus\{\alpha\}}(F)$-coinvariants of M_0 is zero. This follows from Theorem 4.2, using that in $\Phi = A_d$ any fundamental dominant coweight is minuscule. □

A surjectivity statement similar to Theorem 4.2 can be proved for the coinvariants modulo K_1.

Proposition 4.4. *If $\alpha \in \Delta$ is such that λ_α is minuscule then* $\mathrm{im}(T_\alpha) + M(K_1)$ *contains all functions in $M = \mathrm{ind}_K^G(V)$ whose support is contained in $K t_\alpha^{-1} T^- K$.*

Proof. Since $\mathrm{im}(T_\alpha) + M(K_1)$ is a K-subrepresentation of M, it suffices to see that $[t_\alpha^{-1}t, v]$ is contained in $\mathrm{im}(T_\alpha) + M(K_1)$ for any $t \in T^-$ and any $v \in V$.

By Lemma 4.1 the function $[t_\alpha^{-1}t, \xi_\alpha(v)] - T_\alpha([t_\alpha^{-2}t, v])$ is contained in $M(K_1)$, using once again that $t_\alpha^{-1} I t_\alpha \subseteq K$ because λ_α is minuscule. However, $[t_\alpha^{-1}t, v] - [t_\alpha^{-1}t, \xi_\alpha(v)] = [t_\alpha^{-1}t, v - \xi_\alpha(v)]$ where $v - \xi_\alpha(v) \in \ker(\xi_\alpha) = V(\overline{\mathcal{N}}_{\Delta\setminus\{\alpha\}}(k))$. Thus, there are elements $n_i \in \overline{\mathcal{N}}_{\Delta\setminus\{\alpha\}}(\mathfrak{o}) \subset K$ and vectors $v_i \in V$ such that $v - \xi_\alpha(v) = \sum_i (1 - n_i)v_i$. This yields

$$[t_\alpha^{-1}t, v - \xi_\alpha(v)] = \sum_i (1 - t_\alpha^{-1} t n_i t^{-1} t_\alpha)[t_\alpha^{-1}t, v_i].$$

Since $t_\alpha^{-1} t \overline{\mathcal{N}}_{\Delta\setminus\{\alpha\}}(\mathfrak{o}) t^{-1} t_\alpha \subset K_1$, we obtain $[t_\alpha^{-1}t, v] - [t_\alpha^{-1}t, \xi_\alpha(v)] \in M(K_1)$, proving the proposition. □

Recall that if π is a smooth E-linear representation of G then its smooth dual $\check{\pi}$ is the subrepresentation of the contragredient G-representation $\pi^* = \mathrm{Hom}_E(\pi, E)$ consisting of all vectors whose stabilizers in G are open.

For the group $\mathrm{GL}_2(\mathbb{Q}_p)$ the following statement seems to have first been proven in an unpublished work of Livné.

Theorem 4.5. *Assume that the root system Φ is equal to A_d. The space of K_1-coinvariants of the G-representation $M_0 = M / \sum_{\alpha\in\Delta} T_\alpha(M)$ and that of any of its quotients is zero. In particular, the smooth dual of any of these G-representations is zero.*

Proof. Since $\Phi = A_d$ all λ_α with $\alpha \in \Delta$ are minuscule. It follows from Proposition 4.4 by multiplication with the elements of W that $[g, v] \in \sum_{\alpha\in\Delta} \mathrm{im}(T_\alpha) + M(K_1)$ for all $g \in G$, $v \in V$ except possibly for the case $g \in K$. Since the K-representation V is irreducible we will have $M = \sum_{\alpha\in\Delta} \mathrm{im}(T_\alpha) + M(K_1)$ once we can show that the K-subrepresentation V of M intersects $\sum_{\alpha\in\Delta} \mathrm{im}(T_\alpha) + M(K_1)$ non-trivially. However, by Proposition 4.4 again, we have $\xi_\alpha(v) \in \mathrm{im}(T_\alpha) + M(K_1)$ for any $v \in V$. As a consequence, the space of K_1-coinvariants of M_0 and any of its quotients is zero.

Now let π be any quotient of the G-representation M_0. We claim that $\check{\pi}^{K_1} = 0$ which implies $\check{\pi} = 0$ because K_1 is a pro-p group (cf. [16], Lemma 2.1). Now $\check{\pi}^{K_1} = (\pi^*)^{K_1}$. However, any K_1-invariant linear form $\pi \to E$ factors through $\pi_{K_1} = 0$, hence is the zero map. □

Remark 4.6. Theorem 4.5 implies that the smooth duality functor is not an autoequivalence of the category of E-linear admissible smooth G-representations once the G-representation M_0 admits non-zero admissible quotients.

By [6], Theorem 1.5, this is true for $G = \mathrm{GL}_2(F)$, for example, and is in stark contrast to the situation over a field of characteristic zero (cf. [8], Proposition 2.1.10).

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