Bachelor thesis

Perfect closures of rings and schemes

Florian Leptien
Mathematics
3014072

Supervised by
Prof. Dr. Jan Kohlhaase

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Faculty of mathematics

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Introduction

Let p be a prime number. An $\mathbb{F}_p$-algebra A is called perfect if the Frobenius homomorphism $F_A = (x \mapsto x^p)$ on A is an isomorphism. Of course, not every $\mathbb{F}_p$-algebra is perfect. For example, $A := \mathbb{F}_p[t]$ is not perfect as the element t does not have a p-th root in A. In [3] Marvin J. Greenberg showed that one can always construct a so-called perfect closure A^{perf} of an $\mathbb{F}_p$-algebra A, coming with a ring homomorphism $A \to A^{\text{perf}}$ satisfying a certain universal property. If A is a domain, one can construct the perfect closure within an algebraic closure of the field of fraction of A. In this simple case one can obtain the perfect closure by adjoining all p^n-th roots of elements of A to A. The general case of the perfect closure as well as the dual notion, the so-called inverse perfection A_{perf}, will be discussed in detail in the first section of this work.

In contrast to the inverse perfection, we will show that the perfect closure commutes with localizations. This has the far reaching consequence that this functor can be globalized to the perfect closure of $\mathbb{F}_p$-schemes. In modern algebraic geometry the notion of a perfect $\mathbb{F}_p$-scheme is becoming increasingly important, as exemplified by the works [4] and [5] of Peter Scholze and Bhargav Bhatt. In the first section, we discuss properties of the perfect closure that we will need in the second part of our work. The perfect closure and the inverse perfection will be constructed explicitly and interpreted as adjunctions for the inclusion of the category of perfect $\mathbb{F}_p$-algebras into the category of all $\mathbb{F}_p$-algebras. In the second part of our work, we will construct the perfect closure of an $\mathbb{F}_p$-scheme. To this end, we briefly discuss the construction of projective limits in the category of schemes. Our final result will be an equivalence of categories between the small étale site of an $\mathbb{F}_p$-scheme and that of its perfect closure.

We point out that the perfect closure of $\mathbb{F}_p$-schemes is also the topic of the recent master's thesis [6] of Robin Suxdorf. However, in contrast to our work, it does not treat the above questions on étale coverings.

1 Perfect closure and inverse perfection of $\mathbb{F}_p$-algebras

Let us fix a prime p and let us assume that all rings in this work are commutative and unital.

Definition 1.1

(i) $\mathbb{F}_p$-alg denotes the category of $\mathbb{F}_p$-algebras.

(ii) For an $\mathbb{F}_p$-algebra A, the mapping $F_A := (x \mapsto x^p) : A \to A$ is called the Frobenius homomorphism of A.

(iii) An $\mathbb{F}_p$-algebra A is called perfect if and only if F_A is bijective. In this case, the inverse of F_A^n will usually be denoted by $\left(x \mapsto x^{1/p^n}\right)$.

(iv) $\mathbb{F}_p$-perf denotes the category of perfect $\mathbb{F}_p$-algebras.

Remark 1.2

(i) $\mathbb{F}_p$-perf is a full subcategory of $\mathbb{F}_p$-alg.

(ii) Perfect rings are reduced: If $a^n = 0$, then for suitable $m \in \mathbb{N}$ we have

$$a^{p^m} = \left(a^{p^m - n}\right) a^n = 0$$

and thus $a = 0$, i.e., A is reduced.

Definition 1.3
For an $\mathbb{F}_p$-algebra A, a perfect closure of A is a perfect $\mathbb{F}_p$-algebra A^{perf} together with a ring homomorphism $\varphi_A : A \to A^{\text{perf}}$, in short $(A^{\text{perf}}, \varphi_A)$, such that it satisfies the following universal property:
For any ring homomorphism $f : A \to B$ with B a perfect $\mathbb{F}_p$-algebra, there exists a unique ring homomorphism $\tilde{f} : A^{\text{perf}} \to B$ such that $f = \tilde{f} \circ \varphi_A$, i.e., the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\varphi_A} & A^{\text{perf}} \\ & {\scriptstyle f}\searrow & \Big\downarrow {\scriptstyle \exists! \tilde{f}} \\ & & B \end{array}$$

Sometimes, the terms *perfection* or *direct perfection* are used to denote the perfect closure A^{perf}. The notation A^{1/p^∞} is also frequently used.

Proposition 1.4
Every $\mathbb{F}_p$-algebra A has a perfect closure $(A^{\text{perf}}, \varphi_A)$.

Proof
Let A be an $\mathbb{F}_p$-algebra and set $A_n := A$ for $n \in \mathbb{N}$, $\varphi_{mn} := F_A^{n-m} : A_m \to A_n$ for $n, m \in \mathbb{N}$ such that $m \leq n$. This defines an inductive system (A_n, φ_{mn}) over $\mathbb{N}$:

- $\varphi_{nn} = F_A^0 = id_A$
- $m \leq n \leq k$, then $\varphi_{nk} \circ \varphi_{mn} = F_A^{k-n} \circ F_A^{n-m} = F_A^{k-m} = \varphi_{mk}$.

Set $A^{\text{perf}} := \varinjlim_{n\in\mathbb{N}} A_n = \left(\coprod_{n\in\mathbb{N}} A_n\right) / \sim$ coming with a canonical ring homomorphism

$\varphi_A = (a \mapsto [a]) : A = A_0 \to A^{\text{perf}}$.

Now let B be a perfect $\mathbb{F}_p$-algebra and let $f \in \operatorname{Hom}_{\mathbb{F}_p\text{-alg}}(A, B)$.
Set $f_n := (a \mapsto f(a)^{1/p^n}) : A_n \to B$ and note that f_n is well-defined since B is perfect, i.e., we find unique p^n-th roots for elements in B by the bijectivity of F_B.
Let $m, n \in \mathbb{N}$ with $m \leq n$, then for all $a \in A_m$ we have

$$(f_n \circ \varphi_{mn})(a) = f_n\left(F_A^{n-m}(a)\right) = f_n\left(a^{p^{n-m}}\right) = f\left(a^{p^{n-m}}\right)^{1/p^n} = f(a)^{1/p^m} = f_m(a)\,.$$

Thus, by the universal property of the inductive limit there exists a unique ring homomorphism $\tilde{f} : A^{\text{perf}} \to B$ s.t. $\tilde{f} \circ \varphi_A = f$.

Finally, we note that A^{perf} is perfect:

Now let $[a] \in A^{\text{perf}}$ be represented by $a \in A_n = A$ with $[0] = [a]^p = [a^p] = [F_A(a)]$. By definition of the inductive limit, we find $m \in \mathbb{N}$ with $n \leq m$ such that

$$0 = \varphi_{nm}(0) = \varphi_{nm}\left(F_A(a)\right) = F_A^{m-n}\left(F_A(a)\right) = F_A^{m+1-n}(a),$$

i.e., we obtain that $\varphi_{n,m+1}(a) = 0 = \varphi_{n,m+1}(0)$ and therefore $[a] = [0]$ in A^{perf}. This shows that $F_{A^{\text{perf}}}$ is injective.

To see that $F_{A^{\text{perf}}}$ is also surjective, let $[a] \in A^{\text{perf}}$ be represented by $a \in A_n = A$. Note that we can use $n \leq n+1$ and $n+1 \leq n+1$ to see that $\varphi_{n,n+1}(a) = \varphi_{n+1,n+1}\left(\varphi_{n,n+1}(a)\right)$. Thus $[a] = [F_A(a)] = [a^p] = [a]^p$, which shows the surjectivity of $F_{A^{\text{perf}}}$. Altogether, A^{perf} is a perfect $\mathbb{F}_p$-algebra.

□

The following strong uniqueness property comes directly out of the universal property of the perfect closure.

Corollary 1.5
Let (R, φ_R) be a perfect closure of A. Then there is a unique isomorphism $\widetilde{\varphi_A} : R \to A^{\text{perf}}$ of $\mathbb{F}_p$-algebras satisfying $\varphi_A = \widetilde{\varphi_A} \circ \varphi_R$.

Proof
Consider the ring homomorphism $\varphi_A : A \to A^{\text{perf}}$. By the universal property of R there exists a unique ring homomorphism $\widetilde{\varphi_A} : R \to A^{\text{perf}}$ such that $\varphi_A = \widetilde{\varphi_A} \circ \varphi_R$. Likewise, we can apply the universal property of A^{perf} to $\varphi_R : A \to R$ and find a unique ring homomorphism $\widetilde{\varphi_R} : A^{\text{perf}} \to R$ such that $\varphi_R = \widetilde{\varphi_R} \circ \varphi_A$.

We notice that $\widetilde{\varphi_A} \circ \widetilde{\varphi_R} \circ \varphi_A = \widetilde{\varphi_A} \circ \varphi_R = \varphi_A$ and thus, both $\widetilde{\varphi_A} \circ \widetilde{\varphi_R}$ and $id_{A^{\text{perf}}}$ make

$$\begin{array}{ccc} A & \xrightarrow{\varphi_A} & A^{\text{perf}} \\ & \underset{\varphi_A}{\searrow} & \downarrow \\ & & A^{\text{perf}} \end{array}$$

commutative, i.e. $\widetilde{\varphi_A} \circ \widetilde{\varphi_R} = id_{A^{\text{perf}}}$ by uniqueness.

Likewise $\widetilde{\varphi_R} \circ \widetilde{\varphi_A} \circ \varphi_R = \widetilde{\varphi_R} \circ \varphi_A = \varphi_R$ and therefore, both $\widetilde{\varphi_R} \circ \widetilde{\varphi_A}$ and id_R make

$$\begin{array}{ccc} A & \xrightarrow{\varphi_R} & R \\ & \underset{\varphi_R}{\searrow} & \downarrow \\ & & R \end{array}$$

commutative, i.e. we see that $\widetilde{\varphi_R} \circ \widetilde{\varphi_A} = id_R$ by uniqueness. Thus, $A^{\text{perf}} \cong R$.

$\square$

Example 1.6
Let K be a field of characteristic p and define $L := \left\{x \in K^{\text{alg}} : \exists n \geq 0 : x^{p^n} \in K\right\}$. Then L is a perfect field that contains K. We claim that $(L, \subseteq)$ has the universal property of K^{perf}. Let A be a perfect $\mathbb{F}_p$-algebra and let $f : K \to A$ be a ring homomorphism. For $x \in L$ arbitrary we find $n \geq 0$ such that $x^{p^n} \in K$ and we define $\tilde{f} = \left(x \mapsto f(x^{p^n})^{1/p^n}\right) : L \to A$. Note that if $m \geq 0$ such that $x^{p^m} \in K$, say $m = n + k$, then

$$f\left(x^{p^m}\right)^{1/p^m} = f\left(\left(x^{p^n}\right)^{p^k}\right)^{1/p^m} = f\left(x^{p^n}\right)^{p^k/p^m} = f\left(x^{p^n}\right)^{1/p^n}$$

and hence, $\tilde{f}$ is well-defined. One can check directly that $\tilde{f}$ is a ring homomorphism and it is clear by construction that

$$\begin{array}{ccc} K & \xrightarrow{\subseteq} & L \\ & \underset{f}{\searrow} & \downarrow \tilde{f} \\ & & A \end{array}$$

commutes. For the uniqueness of $\tilde{f}$ assume that $h : L \to A$ is another ring homomorphism that makes

$$\begin{array}{ccc} K & \xrightarrow{\subseteq} & L \\ & \underset{f}{\searrow} & \downarrow \\ & & A \end{array}$$

commutative. Let $x \in L$, then there exists $n \geq 0$ such that $x^{p^n} \in K$ and we calculate

$$h(x)^{p^n} = h\left(x^{p^n}\right) = f\left(x^{p^n}\right) = \left(f\left(x^{p^n}\right)^{1/p^n}\right)^{p^n} = \tilde{f}(x)^{p^n}$$

and see that $h = \tilde{f}$ since A is perfect. Thus, by Corollary 1.5 we have $L \cong K^{\text{perf}}$.

Lemma 1.7
$\varphi_A : A \to A^{\text{perf}}$ satisfies $\ker(\varphi_A) = \text{Rad}_A(0)$. In particular, φ_A is injective if and only if A is reduced, i.e. $\text{Rad}_A(0) = 0$.

Proof
Let $a \in A = A_0$ such that $\varphi_A(a) = 0$, i.e. $[a] = [0]$ in A^{perf}. By definition of A^{perf}, we find $m \in \mathbb{N}$ such that $0 = \varphi_{0m}(0) = \varphi_{0m}(a) = F_A^m(a) = a^{p^m}$, i.e. $a \in \text{Rad}_A(0)$.

Conversely, let $a \in \text{Rad}_A(0)$, then there exists $n \in \mathbb{N}$ such that $a^n = 0$ in A. Now choose $m \in \mathbb{N}$ such that $n \leq p^m$, then $a^{p^m} = a^{p^m - n} a^n = 0$ and with that we see $\varphi_A(a)^{p^m} = \varphi_A\left(a^{p^m}\right) = \varphi_A(0) = 0$, i.e. $\varphi_A(a) = 0$ by the bijectivity of $F_{A^{\text{perf}}}$. $\square$

Remark 1.8
The above proof also shows that $\ker(\varphi_A) = \bigcup_{n \geq 0} \ker(F_A^n)$.

Lemma 1.9
If F_A is injective, so is φ_A. If F_A is surjective, φ_A is surjective as well, in particular $A / \operatorname{Rad}_A(0) \cong A^{\text{perf}}$.

Proof
If F_A is injective, then so is φ_A by Remark 1.8. Now assume that F_A is surjective and let $[a] \in A^{\text{perf}}$ be represented by $a \in A_n = A$. Since F_A is surjective, there is $b \in A = A_0$ such that $F_A^n(b) = a$, i.e. $\varphi_{0n}(b) = \varphi_{nn}(a)$ and thus $[a] = [b] = \varphi_A(b)$. The final statement follows from Lemma 1.7. □

Example 1.10
Let $A := \mathbb{F}_p[t]$, we claim $A^{\text{perf}} \cong B := \bigcup_{n \geq 0} \mathbb{F}_p \left[t^{1/p^n}\right] \subseteq \operatorname{Frac}\left(\mathbb{F}_p[t]\right)^{\text{alg}}$.
Indeed, $(B, \subseteq)$ is a perfect closure of A. To see this, let C be a perfect $\mathbb{F}_p$-algebra and let $f \in \operatorname{Hom}_{\mathbb{F}_p\text{-alg}}(A, C)$. We define $\tilde{f} : B \to C$ in the obvious way by mapping t^{1/p^n} to $f(t)^{1/p^n}$ which is possible since C is perfect. We see that

$$\begin{array}{ccc} A & \xrightarrow{\subseteq} & B \\ & \searrow^{f} & \downarrow{\tilde{f}} \\ & & C \end{array}$$

commutes and an easy calculation shows that $\tilde{f}$ is the unique ring homomorphism with that property and thus, by Corollary 1.5, $A^{\text{perf}} \cong B$.

We will now consider the dual notion of the inverse perfection of an $\mathbb{F}_p$-algebra.

Definition 1.11
For an $\mathbb{F}_p$-algebra A, an inverse perfection of A is a perfect $\mathbb{F}_p$-algebra A_{perf} together with a ring homomorphism $\psi_A : A_{\text{perf}} \to A$, in short $(A_{\text{perf}}, \psi_A)$, such that it satisfies the following universal property:
For any ring homomorphism $f : B \to A$ with B a perfect $\mathbb{F}_p$-algebra there exists a unique ring homomorphism $\tilde{f} : B \to A_{\text{perf}}$ such that $f = \psi_A \circ \tilde{f}$, i.e., the following diagram commutes:

$$\begin{array}{ccc} B & \overset{\exists ! \tilde{f}}{\dashrightarrow} & A_{\text{perf}} \\ & \searrow^{f} & \downarrow{\psi_A} \\ & & A \end{array}$$

Proposition 1.12
Every $\mathbb{F}_p$-algebra A has an inverse perfection $(A_{\text{perf}}, \psi_A)$.

Proof
Let A be an $\mathbb{F}_p$-algebra and set $A_n := A$ and $\varphi_{mn} := F_A^{n-m} : A_n \to A_m$ for $m \leq n$. This defines a projective system (A_n, φ_{mn}) over $\mathbb{N}$:

- $\varphi_{nn} = F_A^0 = id_A$
- $m \leq n \leq k$, then $\varphi_{mn} \circ \varphi_{nk} = F_A^{n-m} \circ F_A^{k-n} = F_A^{k-m} = \varphi_{mk}$.

Set

$$\begin{aligned} A_{\text{perf}} &:= \varprojlim_{n\in\mathbb{N}} A_n \\ &= \left\{ (a_n)_n \in \prod_{n\in\mathbb{N}} A_n : \; a_m = \varphi_{mn}(a_n) \text{ for all } m \leq n \text{ in } \mathbb{N} \right\} \\ &= \left\{ (a_n)_n \in \prod_{n\in\mathbb{N}} A_n : \; F_A(a_{n+1}) = a_n \text{ for all } n \in \mathbb{N} \right\}, \end{aligned}$$

coming with a ring homomorphism $\psi_A : A_{\text{perf}} \to A_0 = A, (a_n)_n \mapsto a_0$.

Let B be a perfect $\mathbb{F}_p$-algebra and let $f \in \mathrm{Hom}_{\mathbb{F}_p\text{-alg}}(B, A)$.
Set $f_n := \left(b \mapsto f\left(b^{1/p^n}\right)\right) : B \to A_n$ and note that f_n is well defined since B is perfect. Let $m, n \in \mathbb{N}$ with $m \leq n$, then $\varphi_{mn} \circ f_n = f_m$ and, by the universal property of the projective limit, there exists a unique ring homomorphism $\tilde{f} : B \to A_{\text{perf}}$ such that $f = \psi_A \circ \tilde{f}$. In fact, $\tilde{f}(b) = \left(f\left(b^{1/p^n}\right)\right)_n$.

Finally, we note that A_{perf} is perfect:

For the surjectivity of $F_{A_{\text{perf}}}$ let $(a_n)_n \in A_{\text{perf}}$ and notice that

$$(a_n)_n = (a_0, a_1, a_2, a_3, ...) = (a_1^p, a_2^p, a_3^p, ...) = (a_1, a_2, a_3, ...)^p$$

where $(a_1, a_2, a_3, ...) \in A_{\text{perf}}$, i.e. $F_{A_{\text{perf}}}$ is surjective.

In order to see that $F_{A_{\text{perf}}}$ is injective as well, let $(a_n)_n \in \ker(F_{A_{\text{perf}}})$. Then

$$0 = (a_n)_n^p = (a_n^p)_n = (a_0^p, a_0, a_1, a_2, a_3, ...),$$

i.e. $(a_n)_n = 0$. Altogether, this shows that $F_{A_{\text{perf}}}$ is bijective. □

The following strong uniqueness property comes directly out of the universal property of the inverse perfection.

Corollary 1.13
Let (R, ψ_R) be an inverse perfection of A. Then there is a unique isomorphism $\widetilde{\psi_A} : A_{\text{perf}} \to R$ of $\mathbb{F}_p$-algebras satisfying $\psi_A = \psi_R \circ \widetilde{\psi_A}$.

Proof
The result follows from the application of the universal property to both ψ_A and ψ_R. A direct computation shows that the obtained ring homomorphisms are inverse to each other using the uniqueness statement in the universal property. □

Lemma 1.14
If F_A is surjective or if F_A is injective, it holds that $\mathrm{Im}(\psi_A) = \bigcap_{n\in\mathbb{N}} \mathrm{Im}\left(F_A^n\right)$.

Proof
Let $a \in \mathrm{Im}(\psi_A)$, then we have $a = \psi_A\left((a_n)_n\right) = a_0$ for some $(a_n)_n \in A_{\mathrm{perf}}$, i.e. $F_A(a_{n+1}) = a_n$ for all $n \in \mathbb{N}$. By induction on $n \in \mathbb{N}$ we see that $a = a_n^{p^n}$ for all $n \in \mathbb{N}$ and thus $a \in \bigcap_{n\in\mathbb{N}} \mathrm{Im}(F_A^n)$.

Conversely, let $a \in \bigcap_{n\in\mathbb{N}} \mathrm{Im}(F_A^n)$, then for all $n \in \mathbb{N}$ there exists some $a_n \in A$ such that $a = a_n^{p^n}$. If F_A is injective, it follows from

$$a_n^{p^n} = a = a_{n+1}^{p^{n+1}} = \left(a_{n+1}^p\right)^{p^n}$$

that $a_{n+1}^p = a_n$ and thus $(a_n)_n \in A_{\mathrm{perf}}$ such that $a = \psi_A((a_n)_n) \in \mathrm{Im}(\psi_A)$.

If F_A is surjective, we set $a_0 := a$ und use the surjectivity to define $(a_n)_n \in A_{\mathrm{perf}}$ inductively such that $a = \psi_A((a_n)_n) \in \mathrm{Im}(\psi_A)$. $\square$

Remark 1.15
One can show that the conclusion of the previous lemma holds more generally if the projective system $(\ker(F_A^n), F_A)_{n\in\mathbb{N}}$ satisfies the so-called Mittag-Leffler condition.

Lemma 1.16
If F_A is surjective, so is ψ_A. If F_A is injective, so is ψ_A and we have:

$$A_{\mathrm{perf}} \cong \bigcap_{n\in\mathbb{N}} \mathrm{Im}(F_A^n).$$

Proof
We only need to show that the injectivity of F_A implies the injectivity of ψ_A, the rest follows from the previous lemma. Therefore, let F_A be injective. Let $(a_n)_n \in \ker(\psi_A)$, i.e. $0 = \psi_A((a_n)_n) = a_0$. Using $F_A((a_{n+1})) = a_n$ for all $n \in \mathbb{N}$, an induction on $n \in \mathbb{N}$ shows that $(a_n)_n = 0$ because F_A is injective. This implies that ψ_A is injective as well. $\square$

Example 1.17
Let $A := \mathbb{F}_p[t]$. By the previous lemma we see that

$$A_{\mathrm{perf}} \cong \mathrm{Im}(\psi_A) = \bigcap_{n\in\mathbb{N}} \mathrm{Im}(F_A^n) = \bigcap_{n\in\mathbb{N}} \mathbb{F}_p\left[t^{p^n}\right] = \mathbb{F}_p \subseteq \mathbb{F}_p[t] = A\,.$$

We now show that both the direct perfection as well as the inverse perfection are functorial and that they give rise to certain adjunctions. This result is completely formal and follows from the universal properties in Definition 1.3 and Definition 1.11.

Proposition 1.18
The inclusion functor $\iota : \mathbb{F}_p\text{-perf} \to \mathbb{F}_p\text{-alg}$ has a left adjoint

$$(\cdot)^{\text{perf}} : \mathbb{F}_p\text{-alg} \to \mathbb{F}_p\text{-perf}$$

and a right adjoint

$$(\cdot)_{\text{perf}} : \mathbb{F}_p\text{-alg} \to \mathbb{F}_p\text{-perf},$$

i.e. functors such that for all $A \in \mathbb{F}_p$-alg and for all $B \in \mathbb{F}_p$-perf we have a functorial bijection in A and B:

$$\text{Hom}_{\mathbb{F}_p\text{-alg}}(A, B) \cong \text{Hom}_{\mathbb{F}_p\text{-perf}}(A^{\text{perf}}, B)$$

and

$$\text{Hom}_{\mathbb{F}_p\text{-alg}}(B, A) \cong \text{Hom}_{\mathbb{F}_p\text{-perf}}(B, A_{\text{perf}}).$$

Proof
We have done some of the work already by showing the existence of A^{perf} and A_{perf}. Now let us first focus on $(\cdot)^{\text{perf}} : \mathbb{F}_p\text{-alg} \to \mathbb{F}_p\text{-perf}$. Let $f \in \text{Hom}_{\mathbb{F}_p\text{-alg}}(A, B)$ and consider $\varphi_B \circ f : A \to B^{\text{perf}}$, since B^{perf} is perfect we get a commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\varphi_A} & A^{\text{perf}} \\ & \searrow^{\varphi_B \circ f} & \Big\downarrow {\scriptstyle \exists! f^{\text{perf}} := \widetilde{\varphi_B \circ f}} \\ & & B^{\text{perf}} \end{array}$$

Thus, $f^{\text{perf}} \circ \varphi_A = \varphi_B \circ f$.

Consider the diagram:

$$\begin{array}{ccc} A & \xrightarrow{\varphi_A} & A^{\text{perf}} \\ & \searrow^{\varphi_A \circ id_A} & \Big\downarrow {\scriptstyle \exists! id_A^{\text{perf}}} \\ & & A^{\text{perf}} \end{array}$$

Clearly $id_{A^{\text{perf}}} \circ \varphi_A = \varphi_A = \varphi_A \circ id_A$ and therefore $id_A^{\text{perf}} = id_{A^{\text{perf}}}$ by uniqueness.

Now let $f \in \text{Hom}_{\mathbb{F}_p\text{-alg}}(A, B)$ and let $g \in \text{Hom}_{\mathbb{F}_p\text{-alg}}(B, C)$ and consider $g \circ f : A \to C$. We apply the universal property of A^{perf} to $\varphi_C \circ (g \circ f)$ and get the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\varphi_A} & A^{\text{perf}} \\ & \searrow^{\varphi_C \circ (g \circ f)} & \Big\downarrow {\scriptstyle \exists! (g \circ f)^{\text{perf}}} \\ & & C^{\text{perf}} \end{array}$$

Now consider the diagram:

$$\begin{array}{ccc} A & \xrightarrow{\varphi_A} & A^{\text{perf}} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^{\text{perf}}} \\ B & \xrightarrow{\varphi_B} & B^{\text{perf}} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g^{\text{perf}}} \\ C & \xrightarrow{\varphi_C} & C^{\text{perf}} \end{array}$$

The two small squares commute by construction of f^{perf} and g^{perf}. Therefore, also the outer square commutes. Thus, uniqueness gives $g^{\text{perf}} \circ f^{\text{perf}} = (g \circ f)^{\text{perf}}$.

Altogether, we have a functor $(\cdot)^{\text{perf}}$

$$\mathbb{F}_p\text{-alg} \ni A \mapsto A^{\text{perf}} \in \mathbb{F}_p\text{-perf}$$

$$\text{Hom}_{\mathbb{F}_p\text{-alg}}(A, B) \ni f \mapsto f^{\text{perf}} \in \text{Hom}_{\mathbb{F}_p\text{-perf}}(A^{\text{perf}}, B^{\text{perf}}).$$

Let us now show that $(\cdot)^{\text{perf}}$ is left adjoint to ι, for this let $A \in \mathbb{F}_p$-alg and let $B \in \mathbb{F}_p$-perf. Consider

$$\begin{aligned} \Phi &= (f \mapsto \tilde{f}) : \text{Hom}_{\mathbb{F}_p\text{-alg}}(A, B) \to \text{Hom}_{\mathbb{F}_p\text{-perf}}(A^{\text{perf}}, B) \text{ and} \\ \Psi &= (g \mapsto g \circ \varphi_A) : \text{Hom}_{\mathbb{F}_p\text{-perf}}(A^{\text{perf}}, B) \to \text{Hom}_{\mathbb{F}_p\text{-alg}}(A, B), \end{aligned}$$

we see that for $f \in \text{Hom}_{\mathbb{F}_p\text{-alg}}(A, B)$ we have

$$(\Psi \circ \Phi)(f) = \Psi\left(\tilde{f}\right) = \tilde{f} \circ \varphi_A = f.$$

For the other direction note that both $\widetilde{\varphi_A}$ and $id_{A^{\text{perf}}}$ make the diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi_A} & A^{\text{perf}} \\ & {\scriptstyle \varphi_A}\searrow & \downarrow \\ & & A^{\text{perf}} \end{array}$$

commutative, i.e. $\widetilde{\varphi_A} = id_{A^{\text{perf}}}$ by uniqueness. Furthermore, both g and $\widetilde{g \circ \varphi_A}$ make the diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi_A} & A^{\text{perf}} \\ & {\scriptstyle g\circ\varphi_A}\searrow & \downarrow \\ & & B \end{array}$$

commutative, i.e. $g = \widetilde{g \circ \varphi_A}$ by uniqueness. Now we calculate

$$(\Phi \circ \Psi)(g) = \Phi(g \circ \varphi_A) = \widetilde{g \circ \varphi_A} = g \circ id_{A^{\text{perf}}} = g$$

and thus $\text{Hom}_{\mathbb{F}_p\text{-alg}}(A, B) \cong \text{Hom}_{\mathbb{F}_p\text{-perf}}(A^{\text{perf}}, B)$, i.e. $(\cdot)^{\text{perf}}$ is left adjoint to ι.

Let us now focus on $(\cdot)_{\text{perf}}$, for this let $f \in \text{Hom}_{\mathbb{F}_p\text{-alg}}(A, B)$ and define $f \circ \psi_A : A_{\text{perf}} \to B$, by the universal property of B_{perf} we get the commutative diagram:

$$\begin{array}{ccc} A_{\text{perf}} & \xrightarrow{\exists! f_{\text{perf}} := \widetilde{f \circ \psi_A}} & B_{\text{perf}} \\ & \underset{f\circ\psi_A}{\searrow} & \downarrow{\scriptstyle \psi_B} \\ & & B \end{array}$$

Thus, $\psi_B \circ f_{\text{perf}} = f \circ \psi_A$.

Now consider the diagram

$$\begin{array}{ccc} A_{\text{perf}} & \longrightarrow & A_{\text{perf}} \\ & \underset{id_A\circ\psi_A}{\searrow} & \downarrow{\scriptstyle \psi_A} \\ & & A \end{array}$$

and notice that both $id_{A_{\text{perf}}}$ and $(id_A)_{\text{perf}}$ make the diagram commutative and therefore by uniquenuess $id_{A_{\text{perf}}} = (id_A)_{\text{perf}}$.

Now let $f \in \text{Hom}_{\mathbb{F}_p\text{-alg}}(A, B)$ and let $g \in \text{Hom}_{\mathbb{F}_p\text{-alg}}(B, C)$ and consider $((g \circ f) \circ \psi_A) : A_{\text{perf}} \to C$. We notice that both $(g \circ f)_{\text{perf}}$ and $g_{\text{perf}} \circ f_{\text{perf}}$ make the diagram

$$\begin{array}{ccc} A_{\text{perf}} & \longrightarrow & C_{\text{perf}} \\ & \underset{(g\circ f)\circ\psi_A}{\searrow} & \downarrow{\scriptstyle \psi_C} \\ & & C \end{array}$$

commutative. To see this, consider the diagram:

$$\begin{array}{ccccc} A_{\text{perf}} & \xrightarrow{f_{\text{perf}}} & B_{\text{perf}} & \xrightarrow{g_{\text{perf}}} & C_{\text{perf}} \\ {\scriptstyle \psi_A}\downarrow & & \downarrow{\scriptstyle \psi_B} & & \downarrow{\scriptstyle \psi_C} \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \end{array}$$

The two small squares commute by construction of f_{perf} and g_{perf}. Therefore, also the outer square commutes. Thus, by uniqueness $(g \circ f)_{\text{perf}} = g_{\text{perf}} \circ f_{\text{perf}}$.

Altogether we have a functor $(\cdot)_{\text{perf}}$

$$\mathbb{F}_p\text{-alg} \ni A \mapsto A_{\text{perf}} \in \mathbb{F}_p\text{-perf}$$

$$\text{Hom}_{\mathbb{F}_p\text{-alg}}(A, B) \ni f \mapsto f_{\text{perf}} \in \text{Hom}_{\mathbb{F}_p\text{-perf}}(A^{\text{perf}}, B^{\text{perf}}).$$

Let us now show that $(\cdot)_{\text{perf}}$ is right adjoint to ι, for this let $A \in \mathbb{F}_p$-alg and let $B \in \mathbb{F}_p$-perf. Consider

$$\begin{aligned}\Phi = (f \mapsto \tilde{f}) &: \mathrm{Hom}_{\mathbb{F}_p\text{-alg}}(B, A) \to \mathrm{Hom}_{\mathbb{F}_p\text{-perf}}(B, A_{\text{perf}}) \ \text{ and}\\ \Psi = (g \mapsto \psi_A \circ g) &: \mathrm{Hom}_{\mathbb{F}_p\text{-perf}}(B, A_{\text{perf}}) \to \mathrm{Hom}_{\mathbb{F}_p\text{-alg}}(B, A),\end{aligned}$$

we see that for $f \in \mathrm{Hom}_{\mathbb{F}_p\text{-alg}}(B, A)$ we have

$$(\Psi \circ \Phi)(f) = \Psi\left(\tilde{f}\right) = \psi_A \circ \tilde{f} = f.$$

For the other direction note that both $id_{A_{\text{perf}}}$ and $\widetilde{\psi_A}$ make the diagram

$$\begin{array}{ccc} A_{\text{perf}} & \longrightarrow & A_{\text{perf}} \\ & {\scriptstyle \psi_A} \searrow & \downarrow {\scriptstyle \psi_A} \\ & & A \end{array}$$

commutative, i.e. $id_{A_{\text{perf}}} = \widetilde{\psi_A}$ by uniqueness. Furthermore, both $id_{A_{\text{perf}}} \circ g$ and $\widetilde{\psi_A \circ g}$ make the diagram

$$\begin{array}{ccc} B & \longrightarrow & A_{\text{perf}} \\ & {\scriptstyle \psi_A \circ g} \searrow & \downarrow {\scriptstyle \psi_A} \\ & & A \end{array}$$

commutative, and by uniqueness we see once more that $id_{A_{\text{perf}}} \circ g = \widetilde{\psi_A \circ g}$. This gives us for $g \in \mathrm{Hom}_{\mathbb{F}_p\text{-perf}}(B, A_{\text{perf}})$

$$(\Phi \circ \Psi)(g) = \Phi(\psi_A \circ g) = \widetilde{\psi_A \circ g} = id_{A_{\text{perf}}} \circ g = g.$$

Thus, $\mathrm{Hom}_{\mathbb{F}_p\text{-alg}}(B, A) \cong \mathrm{Hom}_{\mathbb{F}_p\text{-perf}}(B, A_{\text{perf}})$, i.e. $(\cdot)_{\text{perf}}$ is right adjoint to ι.

□

Remark 1.19
Note that we can give an explicit description of f^{perf} by considering the mapping $g := ([a] \mapsto [f(a)]) : A^{\text{perf}} \to B^{\text{perf}}$. Let us first note that if $[a] = [b]$ in A^{perf} where $a \in A_n$ and $b \in A_m$, there is $k \in \mathbb{N}$ with $n \leq k$ and $m \leq k$ such that by definition of the inductive limit $F_A^{k-n}(a) = F_A^{k-m}(b)$ and applying f gives us $F_B^{k-n}(f(a)) = F_B^{k-m}(f(b))$, i.e. $[f(a)] = [f(b)]$ in B^{perf}. Of course g is a ring homomorphism and for $a \in A$ we have

$$(g \circ \varphi_A)(a) = g([a]) = [f(a)] = \varphi_B(f(a)) = (\varphi_B \circ f)(a).$$

And therefore $f^{\text{perf}} = g$ by uniqueness since both g and f^{perf} make the following diagram commutative:

$$\begin{array}{ccc} A & \xrightarrow{\varphi_A} & A^{\text{perf}} \\ & {\scriptstyle \varphi_B \circ f} \searrow & \downarrow \\ & & B^{\text{perf}} \end{array}$$

Likewise we can give an explicit description of f_{perf} as well. For this consider the mapping $g := ((a_n)_n \mapsto (f(a_n))_n) : A_{\text{perf}} \to B_{\text{perf}}$. Note that this defines a well defined ring homomorphism since $F_B(f(a_{n+1})) = f(F_A(a_{n+1})) = f(a_n)$, i.e. $(f(a_n))_n \in B_{\text{perf}}$. Now let $(a_n)_n \in A_{\text{perf}}$ and calculate

$$(\psi_B \circ g)((a_n)_n) = \psi_B((f(a_n))_n) = f(a_0) = f(\psi_A((a_n)_n)) = (f \circ \psi_A)((a_n)_n).$$

By uniqueness $f_{\text{perf}} = g$ since both make the following diagram commutative:

$$\begin{array}{ccc} A_{\text{perf}} & \longrightarrow & B_{\text{perf}} \\ & \searrow^{f\circ\psi_A} & \downarrow{\scriptstyle \psi_B} \\ & & B \end{array}$$

Lemma 1.20
Localizations of perfect $\mathbb{F}_p$-algebras are perfect.

Proof
Let $A \in \mathbb{F}_p$-perf and $S \subseteq A$ multiplicatively closed. We need to check that $F_{AS^{-1}}$ is an isomorphism. For that let $\frac{a}{s} \in AS^{-1}$ such that $\left(\frac{a}{s}\right)^p = 0$ in AS^{-1}. By definition of AS^{-1} there exists an $u \in S$ such that $a^p u = 0$ in A. But then

$$F_A(au) = (au)^p = a^p u^p = (a^p u)\, u^{p-1} = 0$$

and this means $au = 0$ in A since A is perfect, by definition of AS^{-1} we see that $\frac{a}{s} = 0$ in AS^{-1}, i.e. $F_{AS^{-1}}$ is injective. Now let $\frac{a}{s} \in AS^{-1}$, then $\frac{a}{s} = \frac{as^{p-1}}{s^p}$ with $as^{p-1} \in A$, by surjectivity of F_A there exists $b \in A$ such that $b^p = as^{p-1}$, it follows

$$F_{AS^{-1}}\left(\frac{b}{s}\right) = \left(\frac{b}{s}\right)^p = \frac{b^p}{s^p} = \frac{as^{p-1}}{s^p} = \frac{a}{s}.$$

Thus, $F_{AS^{-1}}$ is an isomorphism. □

Proposition 1.21
$(\cdot)^{\text{perf}}$ commutes with localizations in the following sense, let $A \in \mathbb{F}_p$-alg and let $S \subseteq A$ multiplicatively closed. Then

$$\varphi_A(S) \subseteq \varphi_A(S)^{\text{perf}} := \left\{a \in A^{\text{perf}} : \exists n \geq 0 : a^{p^n} \in \varphi_A(S)\right\} \subseteq A^{\text{perf}}$$

are both multiplicatively closed and we have

$$A^{\text{perf}}\varphi_A(S)^{-1} \cong A^{\text{perf}}((\varphi_A(S))^{\text{perf}})^{-1} \cong (AS^{-1})^{\text{perf}}.$$

Proof
It is clear that the two sets in question are multiplicatively closed. Let us first show that $A^{\text{perf}}\varphi_A(S)^{-1} \cong (AS^{-1})^{\text{perf}}$. For this consider the canonical ring homomorphisms

$$f_{AS^{-1}} : A \to AS^{-1} \quad \text{and} \quad f_{A^{\text{perf}}\varphi_A(S)^{-1}} : A^{\text{perf}} \to A^{\text{perf}}\varphi_A(S)^{-1}.$$

Define $\varphi := f_{A^{\text{perf}}\varphi_A(S)^{-1}} \circ \varphi_A : A \to A^{\text{perf}}\varphi_A(S)^{-1}$ and note that $\varphi(s) \in (A^{\text{perf}}\varphi_A(S)^{-1})^\times$ for all $s \in S$.

By the universal property of localization we obtain a unique ring homomorphism $\widetilde{\varphi}$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{f_{AS^{-1}}} & AS^{-1} \\ & \searrow^{\varphi} & \Big\downarrow{\scriptstyle \exists! \widetilde{\varphi}} \\ & & A^{\mathrm{perf}}\varphi_A(S)^{-1} \end{array}$$

We now claim that $(A^{\mathrm{perf}}\varphi_A(S)^{-1}, \widetilde{\varphi})$ has the universal property of $(AS^{-1})^{\mathrm{perf}}$, i.e. $A^{\mathrm{perf}}\varphi_A(S)^{-1}$ is a perfect closure of AS^{-1}. For this note that $A^{\mathrm{perf}}\varphi_A(S)^{-1}$ is perfect by the previous lemma. Let $C \in \mathbb{F}_p$-perf and let $g \in \mathrm{Hom}_{\mathbb{F}_p\text{-alg}}(AS^{-1}, C)$. We apply the universal property of A^{perf} to $(g \circ f_{AS^{-1}}) : A \to C$ and we obtain a unique ring homomorphism $(g \circ f_{AS^{-1}})^{\sim}$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\varphi_A} & A^{\mathrm{perf}} \\ & \searrow^{g \circ f_{AS^{-1}}} & \Big\downarrow{\scriptstyle \exists! (g \circ f_{AS^{-1}})^{\sim}} \\ & & C \end{array}$$

It follows that for all $u \in \varphi_A(S)$ we have $(g \circ f_{AS^{-1}})^{\sim}(u) \in C^{\times}$ and thus, by the universal property of $A^{\mathrm{perf}}\varphi_A(S)^{-1}$ we obtain a unique ring homomorphism $\tilde{g}$ such that the following diagram commutes:

$$\begin{array}{ccc} A^{\mathrm{perf}} & \xrightarrow{f_{A^{\mathrm{perf}}\varphi_A(S)^{-1}}} & A^{\mathrm{perf}}\varphi_A(S)^{-1} \\ & \searrow^{(g \circ f_{AS^{-1}})^{\sim}} & \Big\downarrow{\scriptstyle \exists! \tilde{g}} \\ & & C \end{array}$$

Note that $\tilde{g}$ makes the diagram

$$\begin{array}{ccc} AS^{-1} & \xrightarrow{\tilde{\varphi}} & A^{\mathrm{perf}}\varphi_A(S)^{-1} \\ & \searrow^{g} & \Big\downarrow{\scriptstyle \tilde{g}} \\ & & C \end{array}$$

commutative, because if $\frac{a}{s} \in AS^{-1}$, then

$$\begin{aligned} &(\tilde{g} \circ \tilde{\varphi}) \left(\frac{a}{s}\right) = \tilde{g} \left(\tilde{\varphi} \left(\frac{a}{s}\right)\right) \\ &= \tilde{g} \left(\varphi(a)\varphi(s)^{-1}\right) = \tilde{g}(\varphi(a))(\tilde{g}(\varphi(s)))^{-1} \\ &= (\tilde{g} \circ f_{A^{\mathrm{perf}}\varphi_A(S)^{-1}} \circ \varphi_A)(a)((\tilde{g} \circ f_{A^{\mathrm{perf}}\varphi_A(S)^{-1}} \circ \varphi_A)(s))^{-1} \\ &= ((g \circ f_{AS^{-1}})^{\sim} \circ \varphi_A)(a)(((g \circ f_{AS^{-1}})^{\sim} \circ \varphi_A)(s))^{-1} \\ &= (g \circ f_{AS^{-1}})(a)((g \circ f_{AS^{-1}})(s))^{-1} \\ &= g(f_{AS^{-1}}(a)\,(f_{AS^{-1}}(s))^{-1}) = g\left(\frac{a}{s}\right). \end{aligned}$$

Assume $h : A^{\text{perf}}\varphi_A(S)^{-1} \to C$ makes the diagram commutative as well, i.e. $h \circ \tilde{\varphi} = g$. Then we have

$$h \circ f_{A^{\text{perf}}\varphi_A(S)^{-1}} \circ \varphi_A = h \circ \varphi = h \circ \tilde{\varphi} \circ f_{AS^{-1}} = g \circ f_{AS^{-1}}$$

and hence, by the uniqueness in the universal property of A^{perf} it follows that $h \circ f_{A^{\text{perf}}\varphi_A(S)^{-1}} = (g \circ f_{AS^{-1}})^{\sim}$, i.e. $h = \tilde{g}$ by the uniqueness in the universal property of the localization $A^{\text{perf}}\varphi_A(S)^{-1}$. Thus, $A^{\text{perf}}\varphi_A(S)^{-1}$ is a perfect closure of AS^{-1} and by corollary 1.5 $A^{\text{perf}}\varphi_A(S)^{-1} \cong (AS^{-1})^{\text{perf}}$ as claimed.

It is clear from the universal property that $\varphi_A(S)$ and $\varphi_A(S)^{\text{perf}}$ lead to isomorphic localizations.

□

Remark 1.22
The analogous statement for A_{perf} is wrong which is the main reason why we cannot glue $(\cdot)_{\text{perf}}$ to a functor on $\mathbb{F}_p$-schemes.

Example 1.23
Let $A := \mathbb{F}_p[t]/(t^p)$ and $S := \{t^n : n \in \mathbb{N}\}$. It is easy to check that S is multiplicatively closed. Obviously $0 \in S$ and therefore we have $AS^{-1} = 0$ and thus $(AS^{-1})_{\text{perf}} = 0$. Moreover it is easy to check that the ring homomorphism $(\alpha \mapsto (\alpha, \alpha, \alpha, ...)) : \mathbb{F}_p \to A_{\text{perf}}$ is an isomorphism of $\mathbb{F}_p$-algebras. But then $\psi_A^{-1}(S) = \{1\}$ and we see

$$A_{\text{perf}}\psi_A^{-1}(S) \cong A_{\text{perf}} \cong \mathbb{F}_p \neq 0 = AS^{-1}.$$

Lemma 1.24
If $f \in \text{Hom}_{\mathbb{F}_p\text{-alg}}(A, B)$ is injective, so are f^{perf} and f_{perf}. If $f \in \text{Hom}_{\mathbb{F}_p\text{-alg}}(A, B)$ is surjective, so is f^{perf}.

Proof
Assume $f : A \to B$ is injective. Say $[a] \in A^{\text{perf}}$ represented by $a \in A_n = A$, such that $0 = f^{\text{perf}}([a]) = [f(a)]$ where $[f(a)]$ is represented by $f(a) \in B_n = B$. Then there exists $m \geq n$ such that $0 = F_B^{m-n}(f(a)) = f(a^{p^{m-n}})$ and by injectivity of f we see that $a^{p^{m-n}} = 0$, i.e. $F_A^{m-n}(a) = 0$ and thus $[a] = 0$, i.e. f^{perf} is injective.
If $(a_n)_n \in A_{\text{perf}}$ such that $0 = f_{\text{perf}}((a_n)_n) = (f(a_n))_n$, then $f(a_n) = 0$ for all $n \in \mathbb{N}$. Hence $a_n = 0$ for all $n \in \mathbb{N}$ by injectivity of f and thus f_{perf} is injective.

Let us now assume that $f : A \to B$ is surjective and let $[b] \in B^{\text{perf}}$ be represented by $b \in B_n = B$. Since f is surjective, there is $a \in A_n = A$ such that $f(a) = b$, but then $f^{\text{perf}}([a]) = [f(a)] = [b]$, i.e. f^{perf} is surjective. □

Proposition 1.25
If $f \in \text{Hom}_{\mathbb{F}_p\text{-alg}}(A, B)$, then $A^{\text{perf}}/\ker(f^{\text{perf}}) \cong (A/\ker(f))^{\text{perf}}$.

Proof
Let $\pi : A \to A/\ker(f)$ be the canonical map. By the previous lemma we see that $\pi^{\text{perf}} : A^{\text{perf}} \to (A/\ker(f))^{\text{perf}}$ is surjective. Thus, we claim $\ker(f^{\text{perf}}) = \ker(\pi^{\text{perf}})$. Let $[a] \in \ker(\pi^{\text{perf}})$ be representated by $a \in A_n = A$. Then we have $0 = \pi^{\text{perf}}([a]) = [\pi(a)]$ in $(A/\ker(f))^{\text{perf}}$, i.e. there exists $m \geq n$ such that $0 = F_{A/\ker(f)}^{m-n}(\pi(a)) = \pi(a^{p^{m-n}})$. This means that $a^{p^{m-n}} \in \ker(f)$, i.e. $0 = f(a^{p^{m-n}}) = F_B^{m-n}(f(a))$, but then $0 = [f(a)] = f^{\text{perf}}([a])$, i.e. $[a] \in \ker(f^{\text{perf}})$. Reversing these steps shows that $\ker(f^{\text{perf}}) \subseteq \ker(\pi^{\text{perf}})$.
□

Proposition 1.26
If $A_{\text{red}} := A/\operatorname{Rad}_A(0)$ then the canonical map $\pi : A \to A_{\text{red}}$ induces an isomorphism $\pi^{\text{perf}} : A^{\text{perf}} \to (A_{\text{red}})^{\text{perf}}$.

Proof
By the previous proposition we have $(A_{\text{red}})^{\text{perf}} \cong A^{\text{perf}}/\ker(\pi^{\text{perf}})$ and therefore it is enough to see that $\ker(\pi^{\text{perf}}) = 0$. For this let $[a] \in \ker(\pi^{\text{perf}})$ be represented by $a \in A_n = A$. Then $0 = \pi^{\text{perf}}([a]) = [\pi(a)]$ in $(A_{\text{red}})^{\text{perf}}$ and thus, there exists $m \geq n$ such that $0 = F^{m-n}_{A_{\text{red}}}(\pi(a)) = \pi(a^{p^{m-n}})$, i.e. $a^{p^{m-n}} \in \ker(\pi) = \operatorname{Rad}_A(0) = \ker(\varphi_A)$ by Lemma 1.7. This means $0 = \varphi_A(a^{p^{m-n}}) = F^{m-n}_{A^{\text{perf}}}(\varphi_A(a))$, i.e. $\varphi_A(a) = 0$. Recall that by definition of A^{perf} we therefore have that $0 = \varphi_A(a) = [a]^{p^n}$, i.e. $[a] = 0$. □

Example 1.27
The analogous statement for A_{perf} is wrong, i.e. in general $(A_{\text{red}})_{\text{perf}} \neq (A_{\text{perf}})$. Let $B := \mathbb{F}_p[t]^{\text{perf}} = \bigcup_{n\geq 0} \mathbb{F}_p[t^{1/p^n}]$ and $A := B/(t)$.
Then we have $\operatorname{Rad}_A(0) = (t^{1/p^n} + (t) : n \in \mathbb{N})$ and therefore $A_{\text{red}} \cong \mathbb{F}_p$, this means $(A_{\text{red}})_{\text{perf}} \cong \mathbb{F}_p$. But the canonical ring homomorphism $\pi : B \to A$ induces a ring isomomorphism

$$\begin{aligned} \pi_{\text{perf}} :\ & B \cong B_{\text{perf}} \to A_{\text{perf}} \\ & b \mapsto \left(b^{1/p^n}\right)_n \mapsto \left(b^{1/p^n} + (t)\right)_n \end{aligned}$$

and thus,

$$(A_{\text{perf}})_{\text{red}} \cong B_{\text{red}} = B \neq \mathbb{F}_p = (A_{\text{red}})_{\text{perf}}.$$

Proposition 1.28
Let $A \in \mathbb{F}_p$-alg and let B and C be A-algebras. Then the canonical map

$$B \otimes_A C \to B^{\text{perf}} \otimes_{A^{\text{perf}}} C^{\text{perf}}$$

induces an isomorphism of perfect $\mathbb{F}_p$-algebras

$$(B \otimes_A C)^{\text{perf}} \to B^{\text{perf}} \otimes_{A^{\text{perf}}} C^{\text{perf}}.$$

Proof
The strategy of the proof is to use the Yoneda lemma after showing that for arbitrary $D \in \mathbb{F}_p$-perf we have $\operatorname{Hom}_{\mathbb{F}_p\text{-alg}}((B \otimes_A C)^{\text{perf}}, D) \cong \operatorname{Hom}_{\mathbb{F}_p\text{-alg}}(B^{\text{perf}} \otimes_{A^{\text{perf}}} C^{\text{perf}}, D)$.
For more details see Lemma 4.10 in [1]. □

Proposition 1.29
Let $f \in \operatorname{Hom}_{\mathbb{F}_p\text{-alg}}(A, B)$. If f is a flat ring homomorphism so is the ring homomorphism $f^{\text{perf}} \in \operatorname{Hom}_{\mathbb{F}_p\text{-alg}}(A^{\text{perf}}, B^{\text{perf}})$.

Proof
Since f is a flat ring homomorphism, we see that B_n is flat over A_n for all $n \in \mathbb{N}$. Thus, for all $n \in \mathbb{N}$ we get that $B_n \otimes_{A_n} A^{\text{perf}}$ is flat over A^{perf} and since direct limits preserve flatness, we see that $\varinjlim_{n\in\mathbb{N}} B_n \otimes_{A_n} A^{\text{perf}}$ is flat over A^{perf}. Finally, we note that $\varinjlim_{n\in\mathbb{N}} B_n \otimes_{A_n} A^{\text{perf}} \cong B^{\text{perf}}$ and thus, B^{perf} is flat over A^{perf} via f^{perf}. The isomorphism is due to Bourbaki in [2], Chapter 2, § 6, Corollary 2 after Proposition 7. □

Proposition 1.30
If $A \in \mathbb{F}_p$-alg then $\mathrm{Spec}(\varphi_A) : \mathrm{Spec}(A^{\mathrm{perf}}) \to \mathrm{Spec}(A)$ is a homeomorphism.

Proof
Let $\mathfrak{p} \in \mathrm{Spec}(A)$ and $\pi : A \to A/\mathfrak{p}$ be the canonical map. Note that together with $A/\mathfrak{p}$ also its perfect closure $(A/\mathfrak{p})^{\mathrm{perf}}$ is an integral domain. Thus, 0 is a prime ideal of $(A/\mathfrak{p})^{\mathrm{perf}}$ and $f(\mathfrak{p}) := \ker(\pi^{\mathrm{perf}} : A^{\mathrm{perf}} \to (A/\mathfrak{p})^{\mathrm{perf}}) = (\pi^{\mathrm{perf}})^{-1}(0)$ is a prime ideal of A^{perf}.

We can now define $f := (\mathfrak{p} \mapsto \ker(\pi^{\mathrm{perf}} : A^{\mathrm{perf}} \to (A/\mathfrak{p})^{\mathrm{perf}})) : \mathrm{Spec}(A) \to \mathrm{Spec}(A^{\mathrm{perf}})$ and we claim that f is the inverse of $\mathrm{Spec}(\varphi_A)$:
Let $\mathfrak{p} \in \mathrm{Spec}(A)$. Then we have

$$\begin{aligned}
(\mathrm{Spec}(\varphi_A) \circ f)(\mathfrak{p}) &= \varphi_A^{-1}\left(\ker\left(\pi^{\mathrm{perf}}\right)\right) \\
&= \left\{a \in A : \varphi_A(a) \in \ker(\pi^{\mathrm{perf}})\right\} \\
&= \left\{a \in A : 0 = \pi^{\mathrm{perf}}(\varphi_A(a)) = [\pi(a)]\right\} \\
&= \left\{a \in A : \exists m \geq 0 : 0 = F_{A/\mathfrak{p}}^m(\pi(a)) = \pi(a^{p^m})\right\} \\
&= \left\{a \in A : \exists m \geq 0 : a^{p^m} \in \ker(\pi) = \mathfrak{p}\right\} \\
&= \{a \in A : a \in \mathfrak{p}\} \\
&= \mathfrak{p}.
\end{aligned}$$

And for $\mathfrak{p} \in \mathrm{Spec}(A^{\mathrm{perf}})$ we have

$$\begin{aligned}
(f \circ \mathrm{Spec}(\varphi_A))(\mathfrak{p}) &= f\left(\varphi_A^{-1}(\mathfrak{p})\right) \\
&= \ker\left(A^{\mathrm{perf}} \to (A/\varphi_A^{-1}(\mathfrak{p}))^{\mathrm{perf}}\right) \\
&= \left\{a \in A^{\mathrm{perf}} : \exists n \geq 0 : a^{p^n} \in \varphi_A(\varphi_A^{-1}(\mathfrak{p}))\right\} \\
&= \left\{a \in A^{\mathrm{perf}} : \exists n \geq 0 : a^{p^n} \in \mathfrak{p}\right\} \\
&= \mathfrak{p}
\end{aligned}$$

where we used that if $x \in \mathfrak{p}$, there exists $n \in \mathbb{N}$ such that $x^{p^n} \in \varphi_A(A)$, i.e. $x^{p^n} = \varphi_A(a)$ with $a \in \varphi_A^{-1}(\mathfrak{p})$. Therefore f is the inverse of $\mathrm{Spec}(\varphi_A)$ as claimed.

It is a well-known fact that $\mathrm{Spec}(\varphi_A)$ is continuous and therefore we just need to show that f is continuous as well. For this let $I \subseteq A^{\mathrm{perf}}$ be an ideal, then we have the equality $f^{-1}(V(I)) = V(\varphi_A^{-1}(I))$. In order to see this let $\mathfrak{q} \in f^{-1}(V(I))$, then $f(\mathfrak{q}) \in V(I)$, i.e. $I \subseteq f(\mathfrak{q})$, but then

$$\varphi_A^{-1}(I) \subseteq \varphi_A^{-1}(f(\mathfrak{q})) = (\mathrm{Spec}(\varphi_A) \circ f)(\mathfrak{q}) = \mathfrak{q},$$

i.e. $\mathfrak{q} \in V(\varphi_A^{-1}(I))$.

Conversely let $\mathfrak{q} \in V(\varphi_A^{-1}(I))$, then $\varphi_A^{-1}(I) \subseteq \mathfrak{q}$. Now let $x \in I$, then there exists $n \geq 0$ such that $x^{p^n} \in \varphi_A(A)$, say $x^{p^n} = \varphi_A(a)$ with $a \in \varphi_A^{-1}(I) \subseteq \mathfrak{q}$, then

$$\pi^{\mathrm{perf}}\left(x^{p^n}\right) = \pi^{\mathrm{perf}}\left(\varphi_A(a)\right) = [\pi(a)] = 0.$$

Hence, $x^{p^n} \in f(\mathfrak{q})$, i.e. $x \in f(\mathfrak{q})$ and thus $I \subseteq f(\mathfrak{q})$. But this means that $f(\mathfrak{q}) \in V(I)$, i.e. $\mathfrak{q} \in f^{-1}(V(I))$. Consequently, $\mathrm{Spec}(\varphi_A)$ is a homeomorphism. $\square$

2 Perfect closure of $\mathbb{F}_p$-schemes

Lemma 2.1
If $A \in \mathbb{F}_p$-alg then the following statements are equivalent:

(i) A is perfect.

(ii) For all $\mathfrak{p} \in \mathrm{Spec}(A)$ it holds that $A_\mathfrak{p}$ perfect.

Proof
(i) $\Rightarrow$ (ii): Follows from Lemma 1.20.

(ii) $\Rightarrow$ (i): Let B be the scalar restriction of A with respect to F_A, i.e. $B := F_{A*}A$ where $ab := F_A(a)b$ for all $a, b \in A$. Now we can view $F_A : A \to B$ as an A-linear map and thus, F_A is bijective if and only if for all $\mathfrak{p} \in \mathrm{Spec}(A)$

$$(F_A)_\mathfrak{p} := id_{A_\mathfrak{p}} \otimes F_A : A_\mathfrak{p} \otimes_A A \cong A_\mathfrak{p} \to B_\mathfrak{p} := A_\mathfrak{p} \otimes_A B$$

is bijective.

Let $\mathfrak{p} \in \mathrm{Spec}(A)$ and let $\varphi = ((x, a) \mapsto x^p a) : A_\mathfrak{p} \times B \to A_\mathfrak{p}$. We note that φ is A-balanced and thus, by the universal property of the tensor product we obtain a unique ring homomorphism $\tilde{\varphi} = (x \otimes a \mapsto x^p a) : A_\mathfrak{p} \otimes_A B \to A_\mathfrak{p}$. Now we claim that $\tilde{\varphi}$ is an isomorphism. If $\frac{a}{s} \in A_\mathfrak{p}$ then $\tilde{\varphi}\left(\frac{1}{s} \otimes as^{p-1}\right) = \frac{a}{s}$, i.e. $\tilde{\varphi}$ is surjective. Now let $x \in \ker(\tilde{\varphi})$, then $x = \frac{1}{s} \otimes a$ since every element in $A_\mathfrak{p} \otimes_A B$ is a simple tensor. By definition of localization $0 = \tilde{\varphi}(x) = \tilde{\varphi}\left(\frac{1}{s} \otimes a\right) = \frac{a}{s^p}$ gives us the existence of $u \in A \setminus \mathfrak{p}$ such that $au = 0$ in A. Thus, we see that $x = \frac{1}{s} \otimes a = \frac{u}{su} \otimes a = \frac{1}{su} \otimes au^p = 0$, i.e. $\tilde{\varphi}$ is injective. By construction the diagram

$$\begin{array}{ccc}
A_\mathfrak{p} & \xrightarrow{F_{A_\mathfrak{p}}} & A_\mathfrak{p} \\
\cong \downarrow & & \uparrow \tilde{\varphi} \\
A_\mathfrak{p} \otimes_A A & \xrightarrow[(F_A)_\mathfrak{p}]{} & A_\mathfrak{p} \otimes_A B
\end{array}$$

is commutative. Since $F_{A_\mathfrak{p}}$ is bijective we see that $(F_A)_\mathfrak{p}$ is bijective. Altogether, A is perfect. □

Proposition 2.2
For an $\mathbb{F}_p$-scheme X the following statements are equivalent:

(i) X admits an affine open covering $X = \bigcup_{i \in I} X_i$ such that for all $i \in I$ the ring $\mathcal{O}_X(X_i)$ is perfect.

(ii) For all $x \in X$ the ring $\mathcal{O}_{X,x}$ is perfect.

(iii) For all $U \subseteq X$ affine open the ring $\mathcal{O}_X(U)$ is perfect.

(iv) For all $U \subseteq X$ open the ring $\mathcal{O}_X(U)$ is perfect.

Proof
(i)$\Rightarrow$(ii): Let $x \in X$ and choose $i \in I$ such that $x \in X_i$. Write $X_i = \mathrm{Spec}(A_i)$ and write $x = \mathfrak{p}$, viewed as a prime ideal of A_i. Then $\mathcal{O}_{X,x} = \mathcal{O}_{X_i,x} \cong (A_i)_\mathfrak{p}$ and $(A_i)_\mathfrak{p}$ is perfect by Lemma 2.1 since $A_i \cong \mathcal{O}_X(X_i)$ is perfect.

(ii)⇒(i): Let $(X_i)_{i\in I}$ be any affine open covering of X and write $X_i = \mathrm{Spec}(A_i)$. Then we have $(A_i)_\mathfrak{p} \cong \mathcal{O}_{X,\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Spec}(A_i)$, i.e. $(A_i)_\mathfrak{p}$ is perfect for all $\mathfrak{p} \in \mathrm{Spec}(A_i)$, but then $\mathcal{O}_X(X_i) \cong A_i$ is perfect by Lemma 2.1.

(ii)⇒(iii): Let $U \subseteq X$ be affine open, write $U = \mathrm{Spec}(A)$ and let $\mathfrak{p} \in U$. Then $A_\mathfrak{p} \cong \mathcal{O}_{X,\mathfrak{p}}$ and thus, $A_\mathfrak{p}$ is perfect for all $\mathfrak{p} \in \mathrm{Spec}(A)$. By Lemma 2.1 we see that $\mathcal{O}_X(U) \cong A$ is perfect.

(iii)⇒(iv): Let $U \subseteq X$ be open, then $(U, \mathcal{O}_U)$ is a scheme and thus, U admits an affine open covering $(U_i)_{i\in I}$. By assumption for all $i \in I$ the ring $\mathcal{O}_U(U_i)$ is perfect. By (i) and (ii) we see that we have a ring homomorphism $\mathcal{O}_X(U) \hookrightarrow \prod_{x\in U} \mathcal{O}_{U,x}$ where all the $\mathcal{O}_{U,x}$ are perfect. In particular, all $\mathcal{O}_{U,x}$ are reduced and thus, $\mathcal{O}_X(U)$ is reduced. This shows that $F_{\mathcal{O}_X(U)}$ is injective. For the surjectivity let $g \in \mathcal{O}_X(U)$. Since all the $\mathcal{O}_U(U_i)$ are perfect, we find a preimage f_i for $g|_{U_i}$, i.e. $f_i^p = g|_{U_i}$. By the injectivity of the Frobenius on $\mathcal{O}_U(U_i \cap U_j)$ we can glue the f_i to $f \in \mathcal{O}_X(U)$, such that by construction $f^p = g$, i.e. $F_{\mathcal{O}_X(U)}$ is surjective. Altogether, $\mathcal{O}_X(U)$ is perfect.

(iv)⇒(i): Clear.

□

Definition 2.3
Let X be an $\mathbb{F}_p$-scheme, we call X perfect if X satisfies the equivalent conditions in Proposition 2.2.

Recall that the absolute Frobenius morphism $F_X : X \to X$ of an $\mathbb{F}_p$-scheme X is the identity on the underlying topological space and is the p-power map on $\mathcal{O}_X(U)$ for any open subset $U \subseteq X$. As an immediate consequence of Proposition 2.2 we obtain the following result.

Corollary 2.4
X is perfect if and only if the absolute Frobenius morphism $F_X : X \to X$ is an isomorphism.

Note that the absolute Frobenius is functorial in the sense that if $f : X \to Y$ is a morphism of $\mathbb{F}_p$-schemes then the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{F_X} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[F_Y]{} & Y \end{array} \tag{1}$$

Definition 2.5
Let X be an $\mathbb{F}_p$-scheme. A perfect closure of X is a perfect $\mathbb{F}_p$-scheme X^{perf} together with a morphism $\varphi_X : X^{\mathrm{perf}} \to X$, in short $(X^{\mathrm{perf}}, \varphi_X)$, such that it satisfies the following universal property:
For any morphism of $\mathbb{F}_p$-schemes $f : Z \to X$ with Z a perfect $\mathbb{F}_p$-scheme there exists a unique morphism $\tilde{f} : Z \to X^{\mathrm{perf}}$ satisfying $f = \varphi_X \circ \tilde{f}$, i.e., the following diagram commutes:

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ & {\scriptstyle \exists!\tilde{f}}\searrow & \uparrow{\scriptstyle \varphi_X} \\ & & X^{\mathrm{perf}} \end{array}$$

Proposition 2.6
Every $\mathbb{F}_p$-scheme X has a perfect closure $(X^{\mathrm{perf}}, \varphi_X)$.

Proof
There are several approaches to the construction of X^{perf}. We will start with a construction based on projective limits in the category of $\mathbb{F}_p$-schems. This approach is also carried out in [6], Theorem 6.2.3. We will comment on alternative constructions below, relying on the results on perfect closures of rings from the the previous section.

Let X be an $\mathbb{F}_p$-scheme and consider the projective system (X_n, φ_{mn}) over $\mathbb{N}$ given by $X_n := X$, $\varphi_{mn} := F_X^{n-m} : X_n \to X_m$ for $m \leq n$. By Lemma 32.2.2 in [7], the limit $X^{\mathrm{perf}} := \varprojlim_{n\in\mathbb{N}} X_n$ exists in the category of $\mathbb{F}_p$-schemes. For this note that the powers of $F_X : X \to X$ are affine because they are the identity on the underlying topological space of X. Consider the canonical projections $\pi_n : X^{\mathrm{perf}} \to X_n$ and set $\varphi_X := \pi_0$.
Let $\psi_n : X_{n+1} \to X_n$ be the identity map and consider $\psi_n \circ \pi_{n+1} : X^{\mathrm{perf}} \to X_n$.
Then for $m \leq n$

$$\begin{array}{ccc} X_{n+1} & \xrightarrow{\psi_n} & X_n \\ {\scriptstyle \varphi_{m+1,n+1}}\downarrow & & \downarrow{\scriptstyle \varphi_{mn}} \\ X_{m+1} & \xrightarrow[\psi_m]{} & X_m \end{array}$$

commutes, and thus

$$\begin{array}{ccc} X^{\mathrm{perf}} & \xrightarrow{\psi_n\circ\pi_{n+1}} & X_n \\ {\scriptstyle \psi_m\circ\pi_{m+1}}\downarrow & \swarrow{\scriptstyle \varphi_{mn}} & \\ X_m & & \end{array}$$

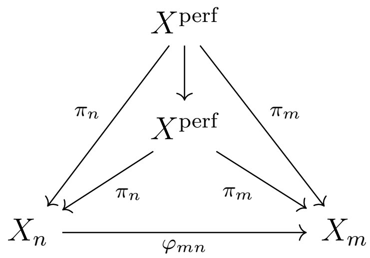
commutes. By the universal property of projective limits there exists a unique morphism $\psi : X^{\mathrm{perf}} \to X^{\mathrm{perf}}$ such that

$$\begin{array}{ccc} X^{\mathrm{perf}} & \xrightarrow{\psi} & X^{\mathrm{perf}} \\ {\scriptstyle \psi_n\circ\pi_{n+1}}\downarrow & \swarrow{\scriptstyle \pi_n} & \\ X_n & & \end{array}$$

commutes. Together with (1) we see that

$$\begin{aligned} \pi_n \circ F_{X^{\mathrm{perf}}} \circ \psi &= F_{X_n} \circ \pi_n \circ \psi \\ &= F_{X_n} \circ \psi_n \circ \pi_{n+1} \\ &= \psi_n \circ F_{X_{n+1}} \circ \pi_{n+1} \\ &= \varphi_{n,n+1} \circ \pi_{n+1} \\ &= \pi_n. \end{aligned}$$

And thus, both $id_{X^{\mathrm{perf}}}$ and $F_{X^{\mathrm{perf}}} \circ \psi$ make



commutative. Hence, by the uniqueness statement in the universal property of projective limits we have $F_{X^{\mathrm{perf}}} \circ \psi = id_{X^{\mathrm{perf}}}$. Analogously one shows that $\psi \circ F_{X^{\mathrm{perf}}} = id_{X^{\mathrm{perf}}}$ and thus, X^{perf} is a perfect $\mathbb{F}_p$-scheme.
Now let Z be a perfect $\mathbb{F}_p$-scheme and let $f : Z \to X$ be a morphism of $\mathbb{F}_p$-schemes. For $n \in \mathbb{N}$ set $f_n := f \circ F_Z^{-n}$ where $F_Z^{-n} := (F_Z^{-1})^n$. Let $n \geq m$. On the level of topological spaces it is clear that $\varphi_{mn} \circ f_n = f_m$ and for $U \subseteq X$ open we have by (1)

$$\begin{aligned}(\varphi_{mn} \circ f_n)_U^{\#} &= \left(\varphi_{mn} \circ f \circ F_Z^{-n}\right)_U^{\#} \\ &= \left(f \circ F_Z^{n-m} \circ F_Z^{-n}\right)_U^{\#} \\ &= \left(f \circ F_Z^{-m}\right)_U^{\#} \\ &= (f_m)_U^{\#}.\end{aligned}$$

And thus, by the universal property of the projective limit there exists a unique morphism $\tilde{f} : Z \to X^{\mathrm{perf}}$ such that $\pi_n \circ \tilde{f} = f_n$ for all $n \in \mathbb{N}$. In particular:

$$\varphi_X \circ \tilde{f} = \pi_0 \circ \tilde{f} = f_0 = f.$$

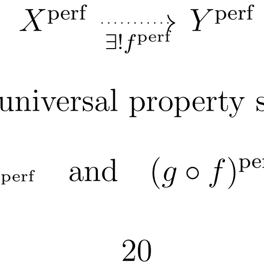
□

The following strong uniqueness property comes directly out of the universal property of $(X^{\mathrm{perf}}, \varphi_X)$.

Corollary 2.7
The perfect closure is unique up to unique isomorphism, i.e., if (Y, ψ_Y) is a perfect closure of X, then there is a unique isomorphism $\widetilde{\varphi_X} : X^{\mathrm{perf}} \to Y$ of $\mathbb{F}_p$-schemes satisfying $\varphi_X = \psi_Y \circ \widetilde{\varphi_X}$.

Proof
This follows immediately from the universal property. □

The universal property shows that $(\cdot)^{\mathrm{perf}}$ is a functor from the category of $\mathbb{F}_p$-schemes into the category of perfect $\mathbb{F}_p$-schemes:



Moreover, the uniqueness in the universal property shows that

$$(id_X)^{\mathrm{perf}} = id_{X^{\mathrm{perf}}} \quad \text{and} \quad (g \circ f)^{\mathrm{perf}} = g^{\mathrm{perf}} \circ f^{\mathrm{perf}}.$$

Corollary 2.8
If $X = \mathrm{Spec}(A)$ is an affine scheme, then $(\mathrm{Spec}(A^{\mathrm{perf}}), \mathrm{Spec}(\varphi_A))$ has the universal property of X^{perf}.

Proof
In fact, this follows directly from the construction of projective limits of affine schemes. Namely, setting $X = \mathrm{Spec}(A)$, the proof of Lemma 32.2.2 in [7], shows that

$$\begin{aligned} X^{\mathrm{perf}} &= \mathrm{Spec}(A)^{\mathrm{perf}} \\ &= \varprojlim_{n \in \mathbb{N}} \mathrm{Spec}(A) \\ &= \mathrm{Spec}(\varinjlim_{n \in \mathbb{N}}(A)) \\ &= \mathrm{Spec}\left(A^{\mathrm{perf}}\right). \end{aligned}$$

One can also argue via the universal property of A^{perf} as follows. Recall that if $(X, \mathcal{O}_X)$ is an $\mathbb{F}_p$-scheme and if $(Y, \mathcal{O}_Y)$ is an affine $\mathbb{F}_p$-scheme, then the map

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Sch}}((X, \mathcal{O}_X), (Y, \mathcal{O}_Y)) &\to \mathrm{Hom}_{\mathbb{F}_p\text{-alg}}(\mathcal{O}_Y(Y), \mathcal{O}_X(X)) \\ \left(f, f^{\#}\right) &\mapsto f^{\#}_Y \end{aligned}$$

is a bijection (cf. [7], Lemma 26.6.4). Now let Z be a perfect $\mathbb{F}_p$-scheme and let $f : Z \to X = \mathrm{Spec}(A)$. Together with Proposition 1.18 we get

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Sch}}(Z, \mathrm{Spec}(A)) &\cong \mathrm{Hom}_{\mathbb{F}_p\text{-alg}}(A, \mathcal{O}_Z(Z)) \\ &\cong \mathrm{Hom}_{\mathbb{F}_p\text{-alg}}(A^{\mathrm{perf}}, \mathcal{O}_Z(Z)) \\ &\cong \mathrm{Hom}_{\mathrm{Sch}}\left(Z, \mathrm{Spec}\left(A^{\mathrm{perf}}\right)\right) \end{aligned}$$

and therefore we obtain a morphism $\tilde{f} : Z \to \mathrm{Spec}\left(A^{\mathrm{perf}}\right)$. By construction it is the unique morphism such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Spec}(A^{\mathrm{perf}}) & \xrightarrow{\mathrm{Spec}(\varphi_A)} & \mathrm{Spec}(A) = X \\ & \nwarrow^{\tilde{f}} & \uparrow f \\ & & Z \end{array}$$

□

Remark 2.9

(i) If $X = \bigcup_{i \in I} X_i$ is an affine open covering then $X^{\mathrm{perf}} = \bigcup_{i \in I} X_i^{\mathrm{perf}}$ is an affine open covering. This follows from the construction of projective limits of schemes in the proof of Lemma 32.2.2 in [7].

(ii) Together with Proposition 1.28 and Corollary 2.8 this shows that the fiber product commutes with the perfect closure, i.e. $(X \times_Z Y)^{\mathrm{perf}} \cong X^{\mathrm{perf}} \times_{Z^{\mathrm{perf}}} Y^{\mathrm{perf}}$ because in the affine case we have for $X = \mathrm{Spec}(A), Y = \mathrm{Spec}(B), Z = \mathrm{Spec}(C)$:

$$\begin{aligned} (X \times_Z Y)^{\mathrm{perf}} &\cong \mathrm{Spec}(A \otimes_C B)^{\mathrm{perf}} \\ &\cong \mathrm{Spec}\left((A \otimes_C B)^{\mathrm{perf}}\right) \\ &\cong \mathrm{Spec}\left(A^{\mathrm{perf}} \otimes_{C^{\mathrm{perf}}} B^{\mathrm{perf}}\right) \\ &\cong X^{\mathrm{perf}} \times_{Z^{\mathrm{perf}}} Y^{\mathrm{perf}}. \end{aligned}$$

For the general case let $f : X \to Z$ and $g : Y \to Z$. The construction of the fiber product of schemes shows that if $(Z_i)_{i\in I}$ is an affine open covering of Z and if for all $i \in I$, $(X_{ij})_{j\in J(i)}$ and $(Y_{ik})_{k\in K(i)}$ are affine open coverings of $X_i := f^{-1}(S_i), Y_i := g^{-1}(S_i)$, then $X \times_Z Y = \bigcup_{i\in I} \bigcup_{j\in J(i)} \bigcup_{k\in K(i)} X_{ij} \times_{Z_i} Y_{ik}$ is an affine open covering of $X \times_Z Y$ and thus:

$$\begin{aligned}(X \times_Z Y)^{\mathrm{perf}} &= \bigcup_{i\in I} \bigcup_{j\in J(i)} \bigcup_{k\in K(i)} (X_{ij} \times_{Z_i} Y_{ik})^{\mathrm{perf}} \\ &\cong \bigcup_{i\in I} \bigcup_{j\in J(i)} \bigcup_{k\in K(i)} X_{ij}^{\mathrm{perf}} \times_{Z_i^{\mathrm{perf}}} Y_{ik}^{\mathrm{perf}} \\ &= X^{\mathrm{perf}} \times_{Z^{\mathrm{perf}}} Y^{\mathrm{perf}}.\end{aligned}$$

Remark 2.10
One can give a more explicit construction of X^{perf}. Namely, we can realize $(X^{\mathrm{perf}}, \mathcal{O}_{X^{\mathrm{perf}}}) = (X, \mathcal{O}_X^{\mathrm{perf}})$ where $\mathcal{O}_{X^{\mathrm{perf}}}(U) := \mathcal{O}_X^{\mathrm{perf}}(U) := \mathcal{O}_X(U)^{\mathrm{perf}}$ for all $U \subseteq X$ open and restriction maps $\mathcal{O}_X(V)^{\mathrm{perf}} \to \mathcal{O}_X(U)^{\mathrm{perf}}$ beeing the perfect closure of the restriction maps $\mathcal{O}_X(V) \to \mathcal{O}_X(U)$ for all $U \subseteq V$ open. The morphism φ_X is the identity on the level of topological spaces and $\varphi_{X,U}^{\#} := \varphi_{\mathcal{O}_X(U)} : \mathcal{O}_X(U) \to \mathcal{O}_X(U)^{\mathrm{perf}}$. That this really gives a sheaf of rings with local stalks can be checked using the results on perfect closures of rings. For a morphism $f : Z \to X$, $\tilde{f}$ is just f on the level of topological spaces and $\tilde{f}_U^{\#} := (f_U^{\#})^{\mathrm{perf}} : \mathcal{O}_X(U)^{\mathrm{perf}} \to \mathcal{O}_Z(f^{-1}(U))^{\mathrm{perf}} \cong \mathcal{O}_Z(f^{-1}(U))$ since Z is perfect.

Alternatively, the perfect closure can be constructed in the affine case using the proof of Corollary 2.8 and glueing. That the perfect closure preserves open immersions can be deduced from Proposition 1.21. More conceptually, it follows from the following fact.

Proposition 2.11
$\varphi_X : X^{\mathrm{perf}} \to X$ is a universal homeomorphism, i.e., for every morphism of $\mathbb{F}_p$-schemes $f : Z \to X$, the morphism $\varphi_X \times id_Z : X^{\mathrm{perf}} \times_X Z \to X \times_X Z \cong Z$ is a homeomorphism on the level of topological spaces.

Proof
By Remark 2.9 (i), an affine open covering $X = \bigcup_{i\in I} X_i$ gives rise to an affine open covering $X^{\mathrm{perf}} = \bigcup_{i\in I} X_i^{\mathrm{perf}}$. Let us first check that it is enough to show that $\varphi_X^{-1}(X_m) \to X_m$ is a universal homeomorphism. In fact, by Remark 2.10 $\varphi_X^{-1}(X_m) = X_m^{\mathrm{perf}}$. Let us now assume that $X_m^{\mathrm{perf}} \to X_m$ is a universally homeomorphism and show that this is sufficient for the claim of the proposition. By Lemma 29.45.5 in [7] a morphism of schemes is a universal homeomorphism if and only if it is surjective, integral and universally injective.
It is clear that the surjectivity of $X_m^{\mathrm{perf}} \to X_m$ implies the surjectivity of φ_X and by Lemma 29.44.2 (3) in [7] φ_X is integral. If $X_m^{\mathrm{perf}} \to X_m$ is universally injective, using Lemma 29.10.2 in [7] we see that the diagonal morphism $X_m^{\mathrm{perf}} \to X_m$ is surjective. Since $X^{\mathrm{perf}} \times_X X^{\mathrm{perf}} = \bigcup_{i\in I} X_i^{\mathrm{perf}} \times_{X_i} X_i^{\mathrm{perf}}$ is an affine open covering, it follows that $X^{\mathrm{perf}} \to X^{\mathrm{perf}} \times_X X^{\mathrm{perf}}$ is surjective. Hence, φ_X is universally injective by Lemma 29.10.2 in [7].

Altogether, by Remark 2.10 we can reduce to the affine case $X = \mathrm{Spec}(A)$ with $\varphi_X = \mathrm{Spec}(\varphi_A)$. As seen above, it is sufficient to show that $\mathrm{Spec}(\varphi_A)$ is universally injective, surjective and integral. Let us first show that $\mathrm{Spec}(\varphi_A)$ is universally injective. By Lemma 29.10.2 in [7] we may show that $\mathrm{Spec}(\varphi_A)$ is radicial. By Proposition 1.30 it is clear that $\mathrm{Spec}(\varphi_A)$ is injective. Let $\mathfrak{p} \in \mathrm{Spec}(A^{\mathrm{perf}})$ and let $x \in \kappa(\mathfrak{p}) = A_{\mathfrak{p}}^{\mathrm{perf}}/\mathfrak{p}A_{\mathfrak{p}}^{\mathrm{perf}}$. Let $x = [a]$ be represented by $a \in A_{\mathfrak{p}}^{\mathrm{perf}}$. Note that by Proposition 1.21 $A_{\mathfrak{p}}^{\mathrm{perf}} \cong (A_{\varphi_A^{-1}(\mathfrak{p})})^{\mathrm{perf}}$ where we used that $A^{\mathrm{perf}}\backslash\mathfrak{p} = \varphi_A(A\backslash\varphi_A^{-1}(\mathfrak{p}))^{\mathrm{perf}}$. But then we get a well-defined ring homomorphism

$$\varphi := \overline{\varphi_{A_{\varphi_A^{-1}(\mathfrak{p})}}} : \kappa(\mathrm{Spec}(\varphi_A(\mathfrak{p}))) = A_{\varphi_A^{-1}(\mathfrak{p})}/\varphi_A^{-1}(\mathfrak{p})A_{\varphi_A^{-1}(\mathfrak{p})} \to A_{\mathfrak{p}}^{\mathrm{perf}}/\mathfrak{p}A_{\mathfrak{p}}^{\mathrm{perf}}.$$

Interpreting a as element of $(A_{\varphi_A^{-1}(\mathfrak{p})})^{\mathrm{perf}}$, we find $b \in A_{\varphi_A^{-1}(\mathfrak{p})}$ such that $a^{p^n} = \varphi_{A_{\varphi_A^{-1}(\mathfrak{p})}}(b)$ for some $n \geq 0$. By construction we have

$$x^{p^n} = \left[a^{p^{p^n}}\right] = [\varphi_{A_{\varphi_A^{-1}(\mathfrak{p})}}(b)] = \varphi([b]) \in \kappa(\mathrm{Spec}(\varphi_A((p)))),$$

where we identify $\kappa(\mathrm{Spec}(\varphi_A(\mathfrak{p})))$ with the image of φ. This shows that the extension $\kappa(\mathfrak{p})/\kappa(\mathrm{Spec}(\varphi_A)(\mathfrak{p}))$ is purely inseparable, i.e., $\mathrm{Spec}(\varphi_A)$ is radicial. It is clear by Proposition 1.30 that $\mathrm{Spec}(\varphi_A)$ is surjective. By Lemma 29.44.2 (2) in [7], $\mathrm{Spec}(\varphi_A)$ is integral if and only if φ_A is integral. Let $a \in A^{\mathrm{perf}}$, then we find $n \geq 0$ and $a_0 \in A$ such that $a^{p^n} = \varphi_A(a_0)$, hence φ_A is integral. □

The following results are also contained in [5], Lemma 3.4.

Proposition 2.12
Let $f : X \to Y$ be a morphism of $\mathbb{F}_p$-schemes. The following properties hold for f if and only if they hold for f^{perf}.

(i) quasi-compact

(ii) quasi-separated

(iii) affine

(iv) separated

(v) integral

(vi) universally closed

(vii) a universal homeomorphism

If f has one of the following properties, then so does f^{perf}.

(viii) a closed immersion

(ix) an open immersion

(x) flat

Proof
(i) and (iii): Since f and f^{perf} agree on the level of topological spaces this is clear.

(ii): By definition $f : X \to Y$ is quasi-separated if and only if the diagonal morphism $\triangle_{X/Y}\colon X \to X \times_Y X$ is quasi-compact. Since $(X \times_Y X)^{\mathrm{perf}} \cong X^{\mathrm{perf}} \times_{Y^{\mathrm{perf}}} X^{\mathrm{perf}}$ we get $\triangle_{X/Y}^{\mathrm{perf}} = \triangle_{X^{\mathrm{perf}}/Y^{\mathrm{perf}}}$. Therefore, the statement follows from (i).

(iv): If f is separated then the diagonal morphism $\triangle_{X/Y}$ is a closed immersion. By (viii) this implies that so is $\triangle_{X/Y}^{\mathrm{perf}} = \triangle_{X^{\mathrm{perf}}/Y^{\mathrm{perf}}}$ whence f^{perf} is separated by definition. Conversely, if f^{perf} is separated then $\triangle_{X/Y}^{\mathrm{perf}}$ is a closed immersion. But since the diagonal morphism commutes with base change it is in fact universally closed. By (vi) so is $\triangle_{X/Y}$. In particular, $\triangle_{X/Y}$ is a closed immersion whence f is separated by definition.

(v): By definition we may assume that $f = \mathrm{Spec}(g) : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is affine. Moreover, by Corollary 2.8 $f = \mathrm{Spec}(g^{\mathrm{perf}})^{\mathrm{perf}} : \mathrm{Spec}(B^{\mathrm{perf}}) \to \mathrm{Spec}(A^{\mathrm{perf}})$. This means it is enough to show that a ring homomorphism $g : A \to B$ is integral if and only if $g^{\mathrm{perf}} : A^{\mathrm{perf}} \to B^{\mathrm{perf}}$ is integral. Now assume that $g : A \to B$ is integral and let $x \in B^{\mathrm{perf}}$, then there exists $n \geq 0$ such that $x^{p^n} \in \varphi_B(B)$, say $x^{p^n} = \varphi_B(b)$ for some $b \in B$. Since g is integral there is an equation

$$0 = b^m + g(a_{m-1})b^{m-1} + ... + g(a_0).$$

By applying φ_B we obtain the equation

$$0 = x^{mp^n} + g^{\mathrm{perf}}(\varphi_A(a_{m-1}))x^{(m-1)p^n} + ... + g^{\mathrm{perf}}(\varphi_A(a_0)).$$

For this recall that by the construction of g^{perf} the diagram

$$\begin{array}{ccc} A & \xrightarrow{\;g\;} & B \\ {\scriptstyle\varphi_A}\downarrow & & \downarrow{\scriptstyle\varphi_B} \\ A^{\mathrm{perf}} & \xrightarrow[g^{\mathrm{perf}}]{} & B^{\mathrm{perf}} \end{array}$$

commutes. Hence, g^{perf} is integral.
For the converse let $b \in B$, since $\varphi_B(b) \in B^{\mathrm{perf}}$ we find an equation

$$0 = \varphi_B(b)^n + g^{\mathrm{perf}}(a_{n-1})\varphi_B(b)^{n-1} + ... + g^{\mathrm{perf}}(a_0).$$

Since $a_j \in A^{\mathrm{perf}}$ we find $m_j \geq 0$ such that $a_j^{p^{m_j}} = \varphi_A(\tilde{a_j})$ for some $\tilde{a_j} \in A$. Let $m := \max\{m_0, ..., m_{n-1}\}$ then using the commutativity of the above diagram

$$\begin{aligned} 0 &= \varphi_B(b)^n + g^{\mathrm{perf}}(a_{n-1})\varphi_B(b)^{n-1} + ... + g^{\mathrm{perf}}(a_0) \\ &= (\varphi_B(b)^n + g^{\mathrm{perf}}(a_{n-1})\varphi_B(b)^{n-1} + ... + g^{\mathrm{perf}}(a_0))^{p^m} \\ &= \varphi_B(b)^{np^m} + g^{\mathrm{perf}}\left(a_{n-1}^{p^m}\right)\varphi_B(b)^{(n-1)p^m} + ... + g^{\mathrm{perf}}(a_0^{p^m}) \\ &= \varphi_B(b)^{np^m} + g^{\mathrm{perf}}\left(a_{n-1}^{p^{m_{n-1}}}\right)^{p^m - p^{m_{n-1}}} \varphi_B(b)^{(n-1)p^m} + ... + g^{\mathrm{perf}}\left(a_0^{p^{m_0}}\right)^{p^m - p^{m_0}} \\ &= \varphi_B(b)^{np^m} + g^{\mathrm{perf}}(\varphi_A(\widetilde{a_{n-1}}))^{p^m - p^{m_{n-1}}}\varphi_B(b)^{(n-1)p^m} + ... + g^{\mathrm{perf}}(\varphi_A(\tilde{a_0}))^{p^m - p^{m_0}} \\ &= \varphi_B(b)^{np^m} + \varphi_B(g(\widetilde{a_{n-1}}))^{p^m - p^{m_{n-1}}}\varphi_B(b)^{(n-1)p^m} + ... + \varphi_B(g(\tilde{a_0}))^{p^m - p^{m_0}} \\ &= \varphi_B\left(b^{np^m} + g(\widetilde{a_{n-1}}^{p^m - p^{m_{n-1}}})b^{(n-1)p^m} + ... + g(\tilde{a_0}^{p^m - p^{m_0}})\right). \end{aligned}$$

And thus, $b^{np^m} + g\left(\tilde{a_{n-1}}^{p^m - p^{m_{n-1}}}\right) b^{(n-1)p^m} + ... + g\left(\tilde{a_0}^{p^m - p^{m_0}}\right) \in \ker(\varphi_B) = \mathrm{Rad}_B(0)$ by Lemma 1.7. Taking this element to a high power we get an integral equation for b over $g(A)$. This implies that g is integral.

(vi): If $Z \to Y$ is any Y-scheme, Remark 2.9 (ii) implies that $(f \times id_Z)^{\text{perf}}$ can be identified with $f^{\text{perf}} \times id_{Z^{\text{perf}}}$. Since a morphism and its perfect closure coincide on the level of topological spaces and since closed immersions are stable under base change, the claim follows.

(vii): f is a universal homeomorphism if and only if f is integral, surjective and universally injective. Since f and f^{perf} agree on the level of topological spaces, it is enough by (v) to show that f is universally injective if and only if f^{perf} is universally injective. But that is clear by Lemma 29.10.2 in [7] and $\triangle_{X^{\text{perf}}/Y^{\text{perf}}} = \triangle_{X/Y}^{\text{perf}}$.

(viii) and (ix): This follows from the construction in Remark 2.10 and Lemma 1.24.

(x): This follows from Lemma 29.25.3 (3) in [7] and Proposition 1.29.

□

Remark 2.13
If $f : X \to \mathrm{Spec}(\mathbb{F}_p)$ is the structure morphism of X and f is of finite type, then $f^{\text{perf}} : X^{\text{perf}} \to \mathrm{Spec}(\mathbb{F}_p)^{\text{perf}} = \mathrm{Spec}(\mathbb{F}_p)$ is the structure morphism of X^{perf}. But in general f^{perf} is not of finite type. For example $\mathbb{F}_p[t]^{\text{perf}} = \bigcup_{n \in \mathbb{N}} \mathbb{F}_p[t^{1/p^n}]$ is no longer of finite type over $\mathbb{F}_p$.

Definition 2.14
A ring homomorphism $f : R \to S$ is called étale if and only if it satisfies the following properties:

(i) $f : R \to S$ is of finite presentation, i.e., f makes S a finitely generated R-algebra.

(ii) $f : R \to S$ is flat.

(iii) $f : R \to S$ is unramified, i.e., if $\mathfrak{q}$ is a prime ideal of S and if $\mathfrak{p} = \mathfrak{q} \cap R$, then $\mathfrak{p}S_{\mathfrak{q}} = \mathfrak{q}S_{\mathfrak{q}}$ and the field extension

$$\kappa(\mathfrak{q}) := S_{\mathfrak{q}}/\mathfrak{q}S_{\mathfrak{q}} \mid R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} =: \kappa(\mathfrak{p})$$

is finite separable.

Definition 2.15
Let $f : X \to Y$ be a morphism of schemes.

(i) We say that f is étale at $x \in X$ if there exists an affine open neighbourhood $\mathrm{Spec}(A) = U \subseteq X$ of x and affine open $\mathrm{Spec}(B) = V \subseteq Y$ with $f(U) \subseteq V$ such that the induced ring map $B \to A$ is étale.

(ii) We say that f is étale if f is étale at every point of X.

Remark 2.16
If K is a field then a ring homomorphism $K \to A$ is étale if and only if $A \cong \prod_{i=1}^{r} K_i$ is a finite direct product of finite separable field extensions K_i/K. For more details see Lemma 10.143.4 in [7].

Definition 2.17
The small étale site of a scheme X is the category $X_{\text{ét}}$ where the objects are all X-schemes $f : Y \to X$ such that the structure morphism f is étale. The morphisms are all morphisms of X-schemes and the coverings of $Y \to X$ are all jointly surjective families $(f_i : Y_i \to Y)_{i \in I}$ of morphisms of étale X-schemes.

If $f : Y \to X$ is étale and $g : Z \to X$ is arbitrary, then $f \times g : Y \times_X Z \to Z$ is étale (cf. [7], Lemma 29.36.4). This way one gets a functor

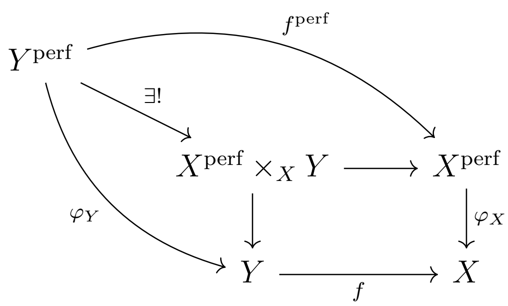
$$X_{\text{ét}} \to Z_{\text{ét}}, (Y \to X) \mapsto (Y \times_X Z \to Z),$$

from the small étale site of X to the small étale site of Z.

Remark 2.18
If $f : Y \to X$ is a morphism of $\mathbb{F}_p$-schemes then

$$\begin{array}{ccc} Y^{\text{perf}} & \xrightarrow{f^{\text{perf}}} & X^{\text{perf}} \\ {\scriptstyle \varphi_Y}\downarrow & & \downarrow{\scriptstyle \varphi_X} \\ Y & \xrightarrow[f]{} & X \end{array}$$

commutes and by the universal property of the fiber product there exists a unique morphism $Y^{\text{perf}} \to X^{\text{perf}} \times_X Y$ such that the following diagram commutes:



Proposition 2.19
Let $f : Y \to X$ be an étale morphism of $\mathbb{F}_p$-schemes, then:

(i) The canonical morphism $Y^{\text{perf}} \to X^{\text{perf}} \times_X Y$ is an isomorphism.

(ii) The functor $X_{\text{ét}} \to X_{\text{ét}}^{\text{perf}}, Y \mapsto Y^{\text{perf}}$, is an equivalence of categories.

Proof
(i): We may assume $Y = \text{Spec}(B), X = \text{Spec}(A)$ affine and that f corresponds to a ring homomorphism $\chi : A \to B$. By the universal property of A^{perf} the ring homomorphism $A \xrightarrow{\chi} B \xrightarrow{\varphi_B} B^{\text{perf}}$ gives rise to a ring homomorphism $\tilde{\chi} : A^{\text{perf}} \to B^{\text{perf}}$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\varphi_A} & A^{\text{perf}} \\ {\scriptstyle \chi}\downarrow & & \downarrow{\scriptstyle \tilde{\chi}} \\ B & \xrightarrow{\varphi_B} & B^{\text{perf}} \end{array}$$

This implies that the mapping $B \times A^{\text{perf}} \to B^{\text{perf}}, (b,a) \mapsto \varphi_B(b)\tilde{\chi}(a)$, where we view A^{perf} and B as A-modules via φ_A respectively χ, is A-balanced and induces a ring homomorphism $B \otimes_A A^{\text{perf}} \to B^{\text{perf}}$. By Corollary 2.8 and by the interrelation of tensor product and fiber product for affine schemes this corresponds on the level of spectra to the morphism $Y^{\text{perf}} \to X^{\text{perf}} \times_X Y$, i.e.

$$\begin{array}{ccc} \operatorname{Spec}(B^{\text{perf}}) & \xrightarrow{\cong} & \operatorname{Spec}(B)^{\text{perf}} = Y^{\text{perf}} \\ \downarrow & & \downarrow \\ \operatorname{Spec}(B \otimes_A A^{\text{perf}}) & \xrightarrow{\cong} & X^{\text{perf}} \times_X Y \end{array}$$

commutes. Thus, we need to see that $B \otimes_A A^{\text{perf}} \to B^{\text{perf}}$ is an isomorphism.
Now as an A-module, $A^{\text{perf}} = \varinjlim_n F^n_{A*}(A)$ is the inductive limit of the A-modules $F^n_{A*}(A)$ obtained from A via scalar restriction along $F^n_A : A \to A$. Since tensor products commute with inductive limits, we have $B \otimes_A A^{\text{perf}} \cong \varinjlim_n (B \otimes_A F^n_{A*}(A))$ with transition maps $id_B \otimes F_A : B \otimes_A F^n_{A*}(A) \to B \otimes_A F^{n+1}_{A*}(A)$. The resulting map

$$\varinjlim_n (B \otimes_A F^n_{A*}(A)) \cong B \otimes_A A^{\text{perf}} \to B^{\text{perf}} = \varinjlim_n B$$

is the inductive limit of the ring homomorphisms:

$$F^n_{B|A} : B \otimes_A F^n_{A*}(A) \to B, b \otimes a \mapsto b^{p^n}\chi(a).$$

This is the so-called n-th relative Frobenius homomorphism of B over A. It is a standard fact that if $\chi : A \to B$ is étale then all n-th relative Frobenius homomorphisms $B \otimes_A F^n_{A*}(A) \to B$ are isomorphisms by Lemma 41.14.3 in [7] and hence so is the ring homomorphism $B \otimes_A A^{\text{perf}} \to B^{\text{perf}}$. Let us briefly sketch the argument.

Set $Y^{(p^n)} := \operatorname{Spec}(B \otimes_A F^n_{A*}(A)) = Y \times_X X$ where the fiber product is formed with respect to $f : Y \to X$ and $F^n_X : X \to X$. Since F^n_X is a universal homeomorphism, so is its fiber product $pr_Y : Y^{(p^n)} = Y \times_X X \xrightarrow{id_Y \times F^n_X} Y \times_X X \cong Y$. Setting $F^n_{Y|X} := \operatorname{Spec}(F^n_{B|A})$ the composition $Y \xrightarrow{F^n_{Y|X}} Y^{(p^n)} \xrightarrow{pr_Y} Y$ is equal to F^n_Y, as is easily checked on the level of rings. Since F^n_Y and pr_Y are universal homeomorphisms, so is $F^n_{Y|X}$.

On the other hand, $\chi : A \to B$ is étale and so is the A-algebra $B \otimes_A F^n_{A*}(A) = B \otimes_{A,F^n_A} A$ obtained via scalar extension. Now any homomorphism of A-algebras between étale A-algebras is an étale ring homomorphism by Lemma 10.143.8 in [7]. In particular, $F^n_{B|A}$ is étale. Since $F^n_{Y|X} = \operatorname{Spec}(F^n_{B|A})$ is surjective the ring homomorphism $F^n_{B|A}$ is faithfully flat. Moreover, $F^n_{B|A} : C := B \otimes_A F^n_{A*}(A) \to B$ is an epimorphism of rings. In fact, by Lemma 10.107.1 in [7], we need to see that the multiplication map $B \otimes_C B \to B, b \otimes b' \mapsto bb'$, is an isomorphism. On the level of spectra this is the diagonal morphism which is an open immersion because $F^n_{B|A}$ is unramified (cf [7], Lemma 29.35.13). Moreover, it is surjective because $F^n_{Y|X}$ is universally injective by Lemma 29.10.2 in [7]. Thus, $B \otimes_C B \to B$ is an isomorphism and $F^n_{B|A}$ is an epimorphism of rings as claimed. Finally, by Lemma 10.107.7 in [7], any faithfully flat epimorphism of rings is an isomorphism.

(ii) This is a formal consequence of (i) and a more general result about universal homeomorphisms. For more details see Theorem 59.45.2 in [7]. □

Example 2.20
Let $X := \mathrm{Spec}(K)$ where K is a field of characteristic p. By Corollary 2.8 we have $X^{\mathrm{perf}} = \mathrm{Spec}(K^{\mathrm{perf}})$ where K^{perf} is again a field of characteristic p. In particular, by Example 1.6 we have

$$K^{\mathrm{perf}} = \left\{x \in K^{\mathrm{alg}} : \exists n \geq 0 : x^{p^n} \in K\right\}.$$

By Proposition 2.19 (ii) we have an equivalence of categories

$$F : X_{\text{ét}} \cong X^{\mathrm{perf}}_{\text{ét}}, Y \mapsto Y^{\mathrm{perf}}, f \mapsto f^{\mathrm{perf}}.$$

Moreover, by Lemma 29.36.7 (2) in [7], étale K-schemes are of the form $\coprod_{i\in I} \mathrm{Spec}(K_i)$ where K_i/K is a finite separable field extension of K. Since by Corollary 2.8 and Proposition 1.30 the functor $(\cdot)^{\mathrm{perf}}$ does not change the underlying topological space, the equivalence F restricts to an equivalence of the connected étale schemes. It is clear that $\coprod_{i\in I} \mathrm{Spec}(K_i)$ is connected if and only if it is a singleton, i.e. $\coprod_{i\in I} \mathrm{Spec}(K_i) = \mathrm{Spec}(L)$ for some finite separable extension of K. Moving to global sections we get an equivalence of categories

$$\begin{aligned}\{L/K : L/K \text{ finite and separable}\} &\to \left\{L'/K^{\mathrm{perf}} : L'/K^{\mathrm{perf}} \text{ finite}\right\}\\ L &\mapsto L^{\mathrm{perf}}\\ \sigma : L_1 \to L_2 &\mapsto \sigma^{\mathrm{perf}} : L_1^{\mathrm{perf}} \to L_2^{\mathrm{perf}}.\end{aligned}$$

Note that any finite extension of K^{perf} is automatically separable because K^{perf} is perfect. Furthermore, by Proposition 2.19 (i) we know that for L/K finite separable the canonical ring homomorphism $K^{\mathrm{perf}} \otimes_K L \to L^{\mathrm{perf}}$ is an isomorphism. In particular

$$\dim_{K^{\mathrm{perf}}}(L^{\mathrm{perf}}) = \dim_{K^{\mathrm{perf}}}(K^{\mathrm{perf}} \otimes_K L) = \dim_K(L).$$

More precisely, since $L^{\mathrm{alg}} = K^{\mathrm{alg}}$, by Example 1.6 we can realize

$$L^{\mathrm{perf}} = \left\{x \in K^{\mathrm{alg}} : \exists n \geq 0 : x^{p^n} \in L\right\}.$$

Then it is immediately clear that $L \hookrightarrow L^{\mathrm{perf}} \hookleftarrow K^{\mathrm{perf}}$. Therefore, the multiplication map $K \times L^{\mathrm{perf}} \to L^{\mathrm{perf}}, (x, y) \mapsto xy$, induces a well-defined ring homomorphism

$$K^{\mathrm{perf}} \otimes_K L \to L^{\mathrm{perf}}.$$

This is the above ring isomorphism. Let us check directly that it is bijective. We have $\mathrm{im}(K^{\mathrm{perf}} \otimes_K L \to L^{\mathrm{perf}}) = LK^{\mathrm{perf}}$, where LK^{perf} denotes the compositum. Furthermore, we have $L \cap K^{\mathrm{perf}} = K$ since $(L \cap K^{\mathrm{perf}})/K$ is separable and purely inseparable at the same time. But then

$$[LK^{\mathrm{perf}} : K] = [L : K][K^{\mathrm{perf}} : K] = \dim_K(K^{\mathrm{perf}} \otimes_K L)$$

and thus, $K^{\mathrm{perf}} \otimes_K L$ maps bijectively to $LK^{\mathrm{perf}} \subseteq L^{\mathrm{perf}}$.
It remains to show that $LK^{\mathrm{perf}} = L^{\mathrm{perf}}$. For this write $L = K[\alpha]$ with some primitive element $\alpha \in L$. It is sufficient to show that $L^{\mathrm{perf}} = K^{\mathrm{perf}}[\alpha]$. First note that $K^{\mathrm{perf}}[\alpha]$ is perfect because K^{perf} is. In fact, any finite extension of $K^{\mathrm{perf}}[\alpha]$ is also one of K^{perf}, hence is separable over K^{perf}, hence over $K^{\mathrm{perf}}[\alpha]$. In particular, $K^{\mathrm{perf}}[\alpha]$ contains all p^n-th roots of α. Now if $x \in L^{\mathrm{perf}}$ then $x^{p^n} \in L = K[\alpha]$ for some $n \geq 0$. If we write $x^{p^n} = \sum_i a_i \alpha^i$ with elements $a_i \in K$ then $x = \sum_i a_i^{1/p^n} \alpha^{i/p^n} \in K^{\mathrm{perf}}[\alpha]$.

This shows $L^{\text{perf}} \subseteq K^{\text{perf}}[\alpha]$. The reverse inclusion is obvious.

We also note that the inverse functor is given by $L' \mapsto L' \cap K^{\text{sep}}$. By the above we may write $L' = LK^{\text{perf}} = L^{\text{perf}}$ for some finite separable extension L/K. Then we have

$$L' \cap K^{\text{sep}} = L^{\text{perf}} \cap K^{\text{sep}} = L,$$

since $(L^{\text{perf}} \cap K^{\text{sep}})/L$ is both separable and purely inseparable at the same time.

It is a formal consequence of the equivalence of categories that L/K is finite Galois if and only if $L^{\text{perf}}/K^{\text{perf}}$ is finite Galois. As a consequence, the restriction homomorphism

$$\begin{aligned} \operatorname{Gal}(K^{\text{alg}}/^{\text{perf}}) &\mapsto \operatorname{Gal}(K^{\text{sep}}/K) \\ \sigma &\mapsto \sigma|_{K^{\text{sep}}}, \end{aligned}$$

is a topological isomorphism. Indeed, we have

$$\begin{aligned} \operatorname{Gal}(K^{\text{alg}}/K^{\text{perf}}) &\cong \varprojlim_{L'/K^{\text{perf}} \text{ finite Galois}} \operatorname{Gal}\left(L'/K^{\text{perf}}\right) \\ &\cong \varprojlim_{L/K \text{ finite Galois}} \operatorname{Gal}\left(LK^{\text{perf}}/K^{\text{perf}}\right) \\ &\cong \varprojlim_{L/K \text{ finite Galois}} \operatorname{Gal}\left(L/L \cap K^{\text{perf}}\right) \\ &= \varprojlim_{L/K \text{ finite Galois}} \operatorname{Gal}(L/K) \\ &\cong \operatorname{Gal}(K^{\text{sep}}/K). \end{aligned}$$

Remark 2.21
$\operatorname{Gal}(K^{\text{sep}}/K)$ (resp. $\operatorname{Gal}(K^{\text{alg}}/K^{\text{perf}})$) can be interpreted as the so-called étale fundamental group of $X = \operatorname{Spec}(K)$ (resp. $X^{\text{perf}} = \operatorname{Spec}(K^{\text{perf}})$). Although we will not define this terminology in detail, it is generally true that the equivalence of categories in Proposition 2.19 (ii) induces an isomorphism between the étale fundamental groups of X and X^{perf}.

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