Master seminar

Brauer Groups

University of Duisburg-Essen, winter term 2021/22

Organizers: Jan Kohlhaase, Marc Kohlhaw
Day and time: Tuesday 14:15 – 15:45
Place: WSC-S-U-3.02 or online
Credits: 9 ECTS for two talks

Contents: The Brauer group classifies the finite dimensional central simple algebras over a field up to equivalence. In many examples, its computation constitutes a non-trivial problem with important applications to number theory and algebraic geometry. The seminar covers the algebraic theory of central simple algebras, the cohomological description of the Brauer group, the Brauer group of local and global fields, Brauer-Severi varieties, and Brauer groups of schemes.

Prerequisites: For 1. and 2. it is sufficient to have a solid background in algebra. For 3. and 4. some prior knowledge of number theory is helpful. Parts 5. and 6. require some prior knowledge of algebraic geometry.

1. Central simple algebras [26.10. + 02.11.]: [6], §§1–6; [7], Chapters II and III.1; [9], Chapters 12–13; central simple algebras over fields; Wedderburn's structure theorem; tensor products of central simple algebras; the Brauer group of a field; dimension, index and degree; the theorem of Skolem-Noether; splitting fields and maximal commutative subfields; relative Brauer groups; examples: the Brauer group of algebraically closed fields, of finite fields, and of the real numbers

2. Cohomological description of the Brauer group [09.11. + 16.11.]: [2], §§3–4; [6], §§7–10; [7], III.2–6; [9], Chapter 14–15; cohomology groups for G-modules; H^0 and H^1 for non-abelian G-groups; the long exact cohomology sequence; the partial exact sequence [2], Proposition 4.4.1, in the non-commutative setting; inflation, restriction and corestriction; state [2], Lemma 4.3.3, and indicate how everything generalizes to continuous cohomology; the isomorphism $\mathrm{Br}(L|K) \cong H^2(\mathrm{Gal}(L|K), L^\times)$ and passage to the direct limit (cf. [6], §8.3–4; [7], Theorem 3.14; [9], Theorem 14.2 and Theorem 14.6); the Brauer group is torsion; the exponent divides the index; the isomorphism $\mathrm{Br}(L|K) \cong L^\times/N_{L|K}(K^\times)$ in the cyclic case (cf. [6], Satz 10.6; [7], Corollary 3.34; [9], Proposition 15.1.b); true/false/open statements on cyclic algebras (cf. [6], §10.10; [9], §15.7)

3. The Brauer group of a local field [23.11. + 30.11.]: [6], §§11–13; [4], §§7–8; [9], Chapter 17; discrete valuations on fields and central simple algebras; completions; extensions of valuations and the fundamental equation $ef = n$; unramified extensions; local fields; norm groups of unramified extensions; existence of unramified splitting fields; the invariant map $\mathrm{Br}(K) \xrightarrow{\cong} \mathbb{Q}/\mathbb{Z}$ for a local field K; the exponent is equal to the index

4. The Brauer group of a global field [07.12. + 14.12.]: [4], §§12–14; [9], Chapter 18; valuations on number fields and their extensions; the product formula; the ring of adèles and the idèle group; state the Hasse norm principle without proof (cf. [4], Corollary 13.24; [9], Theorem 18.4); the Brauer-Hasse-Noether theorem; the invariant map for global fields; construct the sequence

$1 \to \mathrm{Br}(K) \to \bigoplus_v \mathrm{Br}(K_v) \to \mathbb{Q}/\mathbb{Z} \to 1$ and prove as much as possible of its exactness; use the Grundwald-Wang theorem, Artin's global reciprocity law and the Tchebotarev density theorem without proof

5. Brauer-Severi varieties [11.01. + 18.01.]: [1], §§1.1 and 6.1; [2], §§1 and 5; [3], §8.13; [5]; [10]; quaternion algebras and conics; $\mathrm{Br}(L|K)$ and $H^1(\mathrm{Gal}(L|K), \mathrm{PGL}_n(L))$ (cf. [2], §2.4; [5], Theorem 3.6; [10], Satz 2.3.13); reminder on $\mathbb{P}^n_K$ as a scheme; forms of projective space; Brauer-Severi varieties; Châtelet's theorem (cf. [2], Theorem 5.1.3; [10], Satz 2.4.6); the relation between isomorphism classes of Brauer-Severi varieties and $H^1(\mathrm{Gal}(L|K), \mathrm{PGL}_n(L))$ (cf. [2], Theorem 5.2.1; [10], Korollar 2.2.3); state Amitsur's theorem and prove as much of it as you can (cf. [2], §5.4); the Brauer-Severi variety of a central simple algebra as a functor (cf. [3], §8.13)

6. Brauer groups of schemes [25.01. + 01.02.]: [1], §§2.1–2 and §3; [8], Chapter IV; Azumaya algebras over local rings; the Brauer group of a local ring; the generalized Skolem-Noether theorem; Brauer groups of (strictly) henselian local rings; maximal étale subalgebras; Azumaya algebras over schemes; the Brauer-Azumaya group of a scheme; the Skolem-Noether theorem over schemes; outline the definition and basic properties of the étale cohomology of schemes; the Brauer-Grothendieck group $H^2_{\text{ét}}(X, \mathbb{G}_m)$ of a scheme X; the injection of the Brauer-Azumaya group into the Brauer-Grothendieck group (cf. [1], Theorem 3.3.1; [8], Theorem IV.2.5); torsion properties of the Brauer-Azumaya group (cf. [8], Proposition IV.2.7); comparison results for spectra of local henselian rings (cf. [8], Corollary IV.2.12); Gabber's comparison theorem (cf. [1], Theorem 3.3.2)

References

[1] J-L. Colliot-Thélène, A. Skorobogatov: *The Brauer-Grothendieck Group*, Ergebnisse der Mathematik und ihrer Grenzgebiete **71**, Springer, 2021.

[2] P. Gille, T. Szamuely: *Central Simple Algebras and Galois Cohomology*, Cambridge Studies in Advanced Mathematics **101**, Cambridge University Press, 2006.

[3] U. Görtz, T. Wedhorn: *Algebraic Geometry I*, Vieweg+Teubner, 2010.

[4] D. Harari: *Galois Cohomology and Class Field Theory*, Universitext, Springer, 2020.

[5] J. Jahnel: The Brauer-Severi variety associated with a central simple algebra: A survey, *preprint*, 2000, available at `https://www.math.uni-bielefeld.de/LAG/`

[6] I. Kersten: *Brauergruppen*, Universitätsdrucke, Universitätsverlag Göttingen, 2007.

[7] A. Knapp: *Advanced Algebra*, Cornerstones, Birkhäuser, 2007.

[8] J. Milne: *Étale Cohomology*, Princeton Mathematical Series, Princeton University Press, 1980.

[9] R. Pierce: *Associative Algebras*, Graduate Texts in Mathematics **88**, Springer, 1982.

[10] F. Wittbold: *Galoisabstieg für Varietäten*, Bachelor thesis, 2018, available at `https://www.esaga.uni-due.de/jan.kohlhaase/students/`