Master Seminar

Topics in Category Theory

University of Duisburg-Essen, summer term 2023

Organizer: Jan Kohlhaase
Day and Time: Tuesday 14:15 – 15:45
Place: WSC-S-U-3.02
Credits: 9 ECTS (two talks) or 6 ECTS (one talk)

Contents: Categorical constructions and arguments can be found everywhere in algebra, geometry and topology. The seminar will provide an introduction to a few particularly useful concepts. Among other things this concerns abelian categories, embedding theorems, Morita theory, localizations, categories of complexes, homotopy categories, triangulated categories, t-structures, derived categories, derived functors, adjoint functor theorems and model categories.

Prerequisites: You should know what categories, functors and natural transformations are (cf. [6], 1.1, [7], 1.1–1.3, [11], I.2, or [13], Chapter 1). In case of need, many other basic concepts can be looked up in [1], [3] or [9].

1. Representability and the Yoneda Lemma [April 4]: [1], 8.1–8.4, [7], 1.4, [11], I.3, [13], 2.1–2.2; the category of set-valued functors, representable functors, the Yoneda embedding, examples of representable functors, obstructions to representability, examples of non-representable functors

2. Limits, colimits and adjoint functors [April 11]: [1], Chapter 5, 9.1–9.3 and 9.6, [7], 1.5, 2.1–2.2, [11], I.4–I.5, [12], 2.5–2.7, [13], 3.1–3.5, 4.1–4.2 and 4.5; (co)limits, examples, (co)complete categories, (co)completeness and the existence of (co)products and (co)equalizers, pullbacks and pushouts, adjoint functors, the (co)unit of an adjunction, uniqueness of adjoints, examples, right (resp. left) adjoints commute with limits (resp. with colimits), on a small category any set-valued functor is the colimit of representable functors

3. Abelian Categories [April 18]: [6], 1.2, [7], 8.2–8.3, [9], Chapter VIII, [11], II.1–II.2.3, [12], 4.1–4.4; (pre)additive categories, (co)kernels, (co)images, abelian categories, (non)examples, exact sequences, additive and left/right exact functors, sketch a proof of your favorite *metatheorem* in an abelian category (isomorphism theorems, snake lemma, etc.) using only universal properties

4. The Freyd-Mitchell Embedding Theorem [April 25]: [2], [11], II.2.4-II.3.2, [12], 4.13–4.14; injective and projective objects and (co)generators, Morita's theorem, Grothendieck abelian categories, existence of injective cogenerators, the category of left exact functors, the Freyd-Mitchell embedding theorem for small abelian categories

5. Morita Theory [May 2]: [8], 17.B and 18.A–E; Morita equivalent rings, the example of matrix rings, Morita contexts, Morita's Main Theorems I&II, the consequences in [8], (18.33/36/42)

6. Localizations of Categories [May 9]: [6], 1.6, [7], 7.1–7.3, [11], V.2; localizations of categories and their universal property, (saturated) multiplicative systems of morphisms, construction of localizations, (co)kernels/(co)products/(co)limits in localizations, localizations of subcategories, localization of functors

7. Categories of complexes [May 16]: [6], 1.3–1.4, [7], 11.2–11.3 and 12.1–12.3, [11], IV.1.1–IV.1.5; (co)chain complexes in additive and abelian categories, the translation functors $[n]$, mapping cones, bounded and acyclic complexes, the homotopy category, the (co)homology functors H^n and H_n

8. Triangulated categories [June 13]: [6], 1.5 and Proposition 1.4.4, [7], Chapter 10, [11], V.1 and V.3; translations, exact triangles, triangulated categories, exact and cohomological functors, the homotopy category of complexes is triangulated, H^0 and Hom as cohomological functors, null systems and localization of triangulated categories, localization of triangulated functors

9.+10. Derived categories and derived functors [May 16+23]: [6], 1.7–1.8, [7], 13.1–13.2, [11], IV.3.1–IV.3.3 and V.4.–V.4.5; quasi-isomorphisms of complexes, two views on the derived category (complexes localized at quasi-isomorphisms, homotopy category modulo acyclic complexes), truncation functors and boundedness conditions, the fully faithful functor $\mathcal{A} \to D(\mathcal{A})$, injective and projective resolutions, left and right derived functors, the examples of Ext and Tor, compositions of derived functors, the isomorphisms $\mathrm{Ext}^n_{\mathcal{A}}(X,Y) \cong \mathrm{H}^n(\mathrm{RHom}_{\mathcal{A}}(X,Y)) \cong \mathrm{Hom}_{D(\mathcal{A})}(X,Y[n])$

11. t-Structures [June 20]: [6], 10.1; t-structures on a triangulated category, the standard t-structure on a derived category, the truncation functors, the cohomology functors H^n, the heart of a t-structure as an abelian category

12. Adjoint Functor Theorems [June 27]: [9], V.6–V.9, [12], 2.9–2.11, [13], 4.6; left adjoints and initial objects in comma categories, the general adjoint functor theorem, (co)separating sets, the special adjoint functor theorem, Freyd's representability theorem, existence of adjoints between locally presentable categories, examples

13.+14. Model Categories I+II [July 4+11]: [5], Chapters 1–2, [11], Chapter VI; retracts, the definition of a model category, (co)fibrant objects, the homotopy category, left/right lifting properties, (co)fibrations in (co)cartesian squares, Ken Brown's lemma, examples, cylinder and path objects, left and right homotopies, weak equivalences between fibrant-cofibrant objects, (co)fibrant replacements, the equivalences $\mathcal{C}[\mathcal{W}]^{-1} \cong \mathrm{Ho}(\mathcal{C}) \cong \mathrm{Ho}(\mathcal{C}_{cf})$, Quillen adjunctions and Quillen equivalences, the small object argument, cofibrantly generated model categories, construction of model categories

References

[1] S. Awodey: *Category theory*, Oxford Logic Guides **49**, Oxford University Press, 2006

[2] J. Bailie: Abelian Categories and Mitchell's Embedding Theorem, *preprint*, 2017, available at `https://jameshbailie.github.io`

[3] M. Brandenburg: *Einführung in die Kategorientheorie*, Springer Spektrum, 2016

[4] P. Freyd: *Abelian Categories*, Harper & Row, 1964

[5] M. Hovey: *Model Categories*, *Math. Surveys Monographs* **63**, AMS, 1999

[6] M. Kashiwara, P. Shapira: *Sheaves on Manifolds*, Grundlehren Math. Wiss. **292**, Springer, 1990

[7] M. Kashiwara, P. Shapira: *Categories and Sheaves*, Grundlehren Math. Wiss. **332**, Springer, 2006

[8] T.Y. Lam: *Lectures on Modules and Rings*, Graduate Texts in Mathematics **189**, Springer, 1999

[9] S. Mac Lane: *Categories for the Working Mathematician*, Second Edition, Springer 1998

[10] B. Mitchell: *Theory of Categories*, Academic Press, 1965

[11] S. Morel: *Homological Algebra*, Lecture Notes, available at `https://web.math.princeton.edu/~smorel/notes540.pdf`

[12] B. Pareigis: *Categories and Functors*, Academic Press, 1970

[13] E. Riehl: *Category Theory in Context*, Dover Publications, 2016