

K-theory and Number theory (I)

Let K be a number field, $\mathcal{O}_K$ the ring of integers

~~def~~ $\zeta_K(s) = \prod_{\mathfrak{p} \in \text{Spec } \mathcal{O}_K} \frac{1}{1 - N(\mathfrak{p})^{-s}}$

$$\zeta_K(s) = \prod_{\mathfrak{p} \in \text{Spec } \mathcal{O}_K} \left(1 - \frac{1}{N(\mathfrak{p})^s}\right)^{-1}$$

Example $K = \mathbb{Q}$, $\zeta_{\mathbb{Q}}(s)$ is the Riemann Zeta function

Prims 0) Motivic invariant, all are automorphic ~~functions~~ (hard analysis)
1) $\zeta_K(s)$ is a meromorphic function ~~with a pole at s=1~~

• Only pole is at $s=1$, it is a simple pole $\rightarrow$ Langlands-type properties

2) $\zeta_K(s)$ has a functional equation

• [We don't know precisely the zeros] $\rightarrow$ Riemann hypothesis

• Theorem Analytic class formula $\rightarrow$ only hard thm

$$\lim_{s \rightarrow 1} (s-1) \zeta_K(s) = \frac{2^{r_1} (2\pi)^{r_2}}{w \sqrt{|D|}} \quad R_h = \frac{2^h R_0^{h/2}}{\sqrt{|D|}} \quad \begin{matrix} * K_0(\mathcal{O}_K)_{\text{tors}} \\ * \mathcal{O}_K^\times(\mathcal{O}_K)_{\text{tors}} \end{matrix}$$

Where

$r_1 = \text{real emb of } K$

$R = \text{regulator of } K$

$r_2 = \text{Complex emb of } K / 2$

$h = \text{class number of } K =$

$w = \# \text{ roots of unity in } K = \# K_1(\mathcal{O}_K)_{\text{tors}} = \# \mathcal{O}_K^\times(\mathcal{O}_K)_{\text{tors}}$

$|D| = \text{discriminant of } K$

Rank 5/Examples

1) $K = \mathbb{Q}_v$: $v_1 = 1$ $v_2 = 0$ hence $\lim_{s \rightarrow 1} (s-1) \zeta_K(s) = 1$
 $W = 2$ $R = 1$
 $h = 1$ ~~$D = 1$~~ $D = 1$

2) $K = \mathbb{Q}(i)$: $v_1 = 0$ $v_2 = 2$ $\lim_{s \rightarrow 1} (s-1) \zeta_K(s) = \frac{\pi}{4}$
 $W = 4$ $R = 1$
 $D = -4$ $h = 1$ PID

Observation

- Prediction formula to compute the K_0

For K global, $\zeta_K(s) = \prod_p L(p, \chi_s)$
 $\downarrow$ general $\downarrow$ $\begin{matrix} p \text{ Gal } K \rightarrow \mathbb{C} \\ \text{trivial} \\ \text{non trivial} \end{matrix}$ $\downarrow$ const. isomorphism
 Subseq. No pole at $s=1$ $\Rightarrow$ ~~not~~

For $\mathbb{Q}$: $\text{Gal } \mathbb{Q}(i) \rightarrow \mathbb{Q}_2$
 No trivial

We get by 2) $L(p, 1) = \frac{\pi}{4} =$

$= 1 - \frac{1}{2} + \frac{1}{5} - \frac{1}{7}$

equivalent to

$\rightarrow \mathbb{Z}[i]$ being a PID

Example

(2)

Q1) $K=Q$

$$r_1=1 \quad r_2=0$$

$$\lim_{s \rightarrow 1} (s-1) \zeta(s) = 1$$

$$W=2 \quad R=1$$

$$h=1 \quad D=1$$

2) $K=Q(i)$

$$r_1=0 \quad r_2=1$$

$$\lim_{s \rightarrow 1} (s-1) \zeta_K(s) = \frac{\pi}{4}$$

$$W=4 \quad R=1$$

$$D=4 \quad g=1 \quad \text{PID}$$

This is extraordinary!

This actually implies

Theorem

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \text{a PID is equivalent to } 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$$

This is because only nontrivial character of $\mathbb{Z}/2\mathbb{Z}$

$$\zeta_K(s) = \zeta(s) L(X, s)$$

Artin L function

Multiplying by $(s-1)$ evaluates

to 1 by the above

$$\Rightarrow \lim_{s \rightarrow 1} (s-1) L(X, s) = \frac{\pi}{4}$$

giving the alternating sign sum.

Quillen question: Can we generalize this to higher k -groups?

yes! One way:

Theorem

$$\sum_{n=1}^{\infty} \frac{(1-2K) (-1)^K}{2^n} = \frac{|K_{4K+2}(\mathbb{Z})|}{|K_{4K+1}(\mathbb{Z})|}$$

no Analogous but does Not become precise Formula

For $\mathbb{F}$ totally Real

$$\sum_{n \in \mathbb{F}} \frac{(1-2K) (-1)^K}{2^{r_1}} = \frac{|K_{4K+2}(\mathbb{O}_{\mathbb{F}})|}{|K_{4K+1}(\mathbb{O}_{\mathbb{F}})|}$$

This gives also the values of Quadratic Artin L-functions hence there the quotients of L-functions

$K \in \mathbb{N}$ Rmk for F not totally real, ζ_F has a pole of order r_2 at $1-2K$.

• For $\zeta(1-2K) = 0$ $K \geq 1 \rightarrow$ Also for general fields by functional equation this gives also $\zeta(2K)$ for $K \in \mathbb{N}_{\geq 1}$

• The functional equation also gives $\zeta(0) = -\frac{1}{2}$ (analytic class formula (for general field often this is zero too)

• Values at odd integers $\rightarrow$ hard!!! $\rightarrow$ Ask Richi

• ~~even integers~~ ~~odd integers~~

order $r_1 + r_2$

Borel Results (After Lule)

Let us prove something ~~Regulator~~

Theorem ~~Dirichlet~~ (Borel or Quillen)

$K_m(\mathbb{Z})$ is finitely generated for all $m > 0$

(Recall $m=1,2$ the theorems of Minkowski and Dirichlet)

Proof

1) Recall the definition $K_m(A) = \pi_m(BGL(A)^+)$

Where $BGL(A)^+ = BGL(A) + \text{some cells in dim } 2,3, \dots$, $GL(A) = \varinjlim_{n \geq 1} GL_n(A)$

We need 1.A) That $BGL(A)^+$ is an (homotopy) top group

1.B) That $BGL(A) \rightarrow BGL(\mathbb{Q}^*)^+$ gives an isomorphism on cohomology with locally constant coefficients

2) Topology fact (can be made effective)

For X connected CW complex The following are equivalent

2.a) For $\forall m \geq 1$ $\pi_m(X)$ is fin. gen.

2.b) For $\forall m \geq 1$ $\pi_1(X)$ is fin. gen. and

~~the~~ $H_m(X, \mathbb{Z})$ is fin. gen. $\forall m \geq 2$

Hence we are reduced to show $H_m(G_L(\mathbb{Z}), \mathbb{Z})$ is finitely generated for all $m \geq 1$

$$\pi_1(BGL(\mathbb{Z})^+) = \pi_1(BGL(\mathbb{Z})) = H_1(BGL(\mathbb{Z}), \mathbb{Z}) = H_1(BGL(\mathbb{Z}), \mathbb{Z})$$

$\downarrow$
only ~~represent~~ H -group

$$\left(\begin{array}{l} \text{Since by 1.A } \pi_1(BGL(\mathbb{Z})^+) = H_1(BGL(\mathbb{Z})^+, \mathbb{Z}) \\ = H_1(BGL(\mathbb{Z}), \mathbb{Z}) \\ \downarrow \\ \text{1.b} \end{array} \right)$$

Oh No But This $G_L(\mathbb{Z})$ is as big as shit, No worries! be happy

3) Theorem (Quillen stability)

Let A be a Noetherian comm. ring with finite Krull dim. Then, for fixed m , the natural map

$H_m(G_L(N), \mathbb{Z}) \rightarrow H_m(G_L(N), \mathbb{Z})$ is π_m for N high enough

Born is explicit bound: $N \geq \max(2m+1, m + \dim(A)-1)$

in our case $N \geq 2m+1$

We are reduced to:

$\forall m \geq 0, N \geq 1$ the group $H_m(G_{L_N}(\mathbb{Z}), \mathbb{Z})$ is finitely generated

4) If we could find a top space S such that:

A) S contractible

B) S has free $G = GL_n(\mathbb{Z})$ action

C) S compact

or

TCS

$\downarrow$ retract

$T/GL_n(\mathbb{Z})$

compact

Then we would be done b:

by A+B $S/\Gamma \xrightarrow{\text{Hopf,}} BGL_n(\mathbb{Z})$

by C chosen for generated

~~Problem~~ No such S ($GL_n(\mathbb{Z})$ has nontrivial torsion)

~~Solution~~ ~~First~~ ~~idea~~ ~~work~~ with smaller group

~~$\Gamma = \mathbb{Z} + 3\mathbb{M}$, $\text{Aut } M_n(\mathbb{Z}) \rightarrow$ Torsion free~~

~~Lemma~~ if ~~Γ~~ $\exists S$ that satisfies

~~A, B, C~~ for ~~Γ~~ we are done ~~immediately~~

~~Proof~~

~~$GL_n(\mathbb{Z})/\Gamma = GL_n(\mathbb{F}_3) \rightarrow$ Finite group~~

Does anybody have any idea where to find such an S ?

(6)

5) We pick X the space of ~~positive~~ real quadratic forms on $\mathbb{R}^N$

$$X = \{ M \in M_N(\mathbb{R}) \mid v^T M v \geq 0 \ \forall v \in \mathbb{R}^N \}$$

5.a) Natural $\mathbb{R}_{>0}^{**}$ action (multiplication) on X

5.b) " $GL_N(\mathbb{Z})$ Similitude action on X

Lemma

$S = X / \mathbb{R}_{>0}^{**}$ is contractible $\Rightarrow$ A is satisfied

Proof

Exercise in Linear algebra $\square$

Lemma

if $g \in GL_N(\mathbb{Z})$ is such that $\exists s \in S$ with

$g \cdot s = s$ Then g is torsion

Proof

Exercise in topology $\square$

6) Problem $GL_N(\mathbb{Z})$ has No Trivial Torsion!

Solution $\Gamma = 1 + 3M_N(\mathbb{Z}) \subseteq GL_N(\mathbb{Z})$ is

Torsion Free and $GL_N(\mathbb{Z}) / \Gamma = GL_N(\mathbb{F}_3)$

so we get

Lemma

A) We are reduced to find S desc 4 with respect to only 1 Γ action

B) The S we found in 5 satisfies 4.A and 4.B w.r.t Γ

Proof

A. by hochschild-Serre ~~is~~ $H_p(G, H_q^*(\Gamma, \mathbb{Z})) \Rightarrow H_p(G, \mathbb{Z})$
 B. Obv from what I have seen already $\square$

7) Now we only need $T \subset S$ st. T/Γ is $\textcircled{7}$

compact but $T \subset S$ is a retract.

Idea for any $M \in X$ a matrix $v \in \mathbb{R}_+^n$
the set $\{v \in \mathbb{Z}^n \mid v^T M v \leq K\}$ is finite

In particular we define $M_{\min}(M) = \{v \in \mathbb{Z}^n \setminus \{0\} \mid v^T M v = \min\}$

$\tilde{T} \subseteq X$ as $\{M \in X \mid M_{\min}(M) \text{ spans } \mathbb{R}^n\}$

Lemma (Serre)

1) we have an effective $GL_n(\mathbb{Z})$ equivalent
retraction $\tilde{T} \subset X$

2) $\tilde{T}/\Gamma$ is compact

so now we have won! $\square$

~~Example for~~

Example of the ~~same~~ series Lemma for $SL_2(\mathbb{Z})$

Here X is the upper half plane, $SL_2(\mathbb{Z})$ -action

restricts us to fundamental domain

$\tilde{T}$ is given by those in the

Picture

