The local Langlands correspondence for GL_2

ESAGA Babyseminar SS 2023

Organizer: Luca Marannino

Time and place: Wednesdays 2-4 pm in WSC S-U-3.02

Introduction

Providing a general introduction to the Langlands program is definitely not an easy task. The original formulation - proposed by Robert Langlands in the late 60's and building up on pre-existing ideas - can be described as a vast generalization of class field theory. So let us say a few words about class field theory.

Let F denote a local field (i.e. a finite extension of $\mathbb{Q}_p$ or the ring of Laurent power series with coefficients in a finite field), with residue field $\boldsymbol{k}$ of cardinality $q = p^n$ for some $n \geq 1$ and some prime number p. Let Ω_F denote its absolute Galois group (i.e. $\Omega_F = \mathrm{Gal}(F^{sep}/F)$ with F^{sep} a separable closure of F) and let F^{unr} denote the maximal unramified extension of F inside F^{sep}. Galois theory gives a fundamental short exact sequence of profinite groups

$$1 \to \mathrm{Gal}(F^{sep}/F^{unr}) \longrightarrow \Omega_F \longrightarrow \mathrm{Gal}(F^{unr}/F) \cong \mathrm{Gal}(\boldsymbol{k}^{ac}/\boldsymbol{k}) \to 1.$$

Usually people call $I_F := \mathrm{Gal}(F^{sep}/F^{unr})$ the inertia subgroup of Ω_F. On the other hand $\mathrm{Gal}(\boldsymbol{k}^{ac}/\boldsymbol{k})$ is easily seen to be isomorphic to the profinite completion of $\mathbb{Z}$ and it is topologically generated by the so-called Frobenius automorphism (i.e. the map $x \mapsto x^q$). We let $\mathcal{W}_F$ denote the so-called Weil group of F. It can be defined as the preimage of the copy of $\mathbb{Z}$ generated by the Frobenius inside $\mathrm{Gal}(F^{unr}/F)$ under the surjection $\Omega_F \twoheadrightarrow \mathrm{Gal}(F^{unr}/F)$. The topology on $\mathcal{W}_F$ will be the one induced by discrete topology on $\mathbb{Z}$ and the profinite topology on I_F.

Local class field theory for F provides a unique injective group homomorphism with dense image

$$\mathrm{rec_F} : F^\times \longrightarrow \Omega_F^{ab} \tag{1}$$

such that for every uniformizer π of F it holds that $\mathrm{rec}_F(\pi)$, restricted to the maximal unramified extension of F, is the so-called geometric Frobenius (i.e. it induces the *inverse* of Frobenius at the level of residue fields). The image of the reciprocity map $\mathrm{rec_F}$ is precisely the abelianization $\mathcal{W}_F^{ab}$ of the Weil group $\mathcal{W}_F$, i.e. we have a topological isomorphism

$$\mathrm{rec_F} : F^\times \cong \mathcal{W}_F^{ab}$$

Thus one immediately obtains a bijection between the following sets

$$\{\text{continuous characters } \xi : \mathcal{W}_F \to \mathbb{C}^\times\} \overset{1:1}{\longleftrightarrow} \{\text{continuous characters } \chi : F^\times \to \mathbb{C}^\times\} \tag{2}$$

In the global setting, for K a global field (i.e. K a number field or a global function field) one has a global reciprocity map

$$\mathrm{rec_K} : \mathbb{A}_K^\times / K^\times \longrightarrow \Omega_K \tag{3}$$

where $\mathbb{A}_K^\times$ is the group of idéles of K and Ω_K is the absolute Galois group of K. This reciprocity map is compatible with the local versions (1) in a precise way. One can also define a good (but less apparent) notion of Weil group $\mathcal{W}_K$, to obtain the analogue of the bijection (2) in the global setting.

Note that $\mathbb{C}^\times = \mathrm{GL}_1(\mathbb{C})$, $F^\times = \mathrm{GL}_1(F)$ and similarly $\mathbb{A}_K^\times = \mathrm{GL}_1(\mathbb{A}_K)$. The idea of Langlands was to propose a generalization of a correspondence as in (2) to the case of GL_n for general $n \geq 1$ and then further to the case of any reductive group.

Such correspondences are supposed to satisfy a certain number of requirements, such as:

- compatibility of L-functions and ε-factors on both sides
- local-global compatibility (as it happens with class field theory)
- functoriality, i.e. they should be in some sense *natural* with respect to morphisms $G \to H$ of reductive algebraic groups.

For a more precise discussion on Langlands conjectures and Langlands functoriality in wider generality, we refer to [GH22, chapter 12].

Confining ourselves to the local situation and to the case $G = \mathrm{GL}_n$, the analogues of two sides of the correspondence (2) are then given by:

- isomorphism classes of Weil-Deligne representations of dimension n on the Galois side
- isomorphism classes of irreducible smooth admissible representations of $\mathrm{GL}_n(F)$ on the automorphic side.

The local Langlands correspondence for GL_n is now a theorem due to the efforts of several mathematicians. In the function field case, the proof for general n is due to Laumon-Rapoport-Stuhler [LRS93], building up on previous work of Drinfeld. For p-adic fields the general case is due to Harris-Taylor [HT01] and Henniart [Hen00]. More recently Scholze [Sch13] gave a new proof, revisiting and simplyfying some of the arguments in [HT01].

The aim of this seminar is to understand and to prove the local Langlands conjecture for $\mathrm{GL}_2(F)$, for F (*almost*) any local field. Our guiding reference will be the book [BH06], which is meant to be an accessible source for math PhD students with a bit of number-theoretic background.

A more detailed descrition of the talks will be given in the next section. Here we describe the structure of the seminar.

- Talks 2-3: These two talks discuss the basics regarding representation theory and measure theory for locally profinite groups, introducing and discussing the examples which are more relevant for the rest of the seminar.
- Talks 4-5-6-7: These talks are the main body of the seminar. The aim is to give a classification of irreducible smooth representations of $\mathrm{GL}_2(F)$ (i.e. to understand the automorphic side). In particular, the classification of cuspidal representations is probably the most technical and deep part of the seminar.
- Talks 8-9: These talks discuss automorphic L-functions and local constants for automorphic representations, starting from the case of GL_1 and then moving to the case of GL_2. If you have never studied Tate's thesis, you should repent and give talk 8. Talk 9 is slightly more advanced.
- Talks 10-11-12: These talks describe the Galois side of the story and then give the construction of the local Langlands correspondence for GL_2. Talk 10 is rather elementary and could be a good way to become more familiar with infinite Galois theory and local class field theory. Talk 11 contains the main results concerning the Galois side and talk 12 puts everything together to prove the local Langlands conjecture for GL_2.

- TALKS 13-14: These talks do not have a precise program at the moment. The idea is to give to a survey of a different/more advanced aspect of the Langlands program (see below). **People who are more expert in the area and want to learn more advanced topics/techniques should volunteer for these talks and agree with the organizer on a feasible plan.**

WARNING! As it is an established tradition for the Babyseminar (and not only the Babyseminar), the program of some of the talks might contain an excessive amount of material. As usual, the organizer is happy to discuss with the speakers about their talks, in particular to decide what can be skipped in case of lack of time.

Overview of the talks

Talk 1 (April 5th): Introduction and distribution of the talks

Talk 2 (April 12th): Smooth representations of locally profinite groups

The aim of this talk is to give an introduction to the theory of smooth representations of locally profinite groups, recovering analogues of (some of) the well-known results in the representation theory of finite groups.

- Give the definition of locally profinite group as in [BH06, §1.1] and introduce the relevant notation for non-archimedean local fields as in [BH06, §1.2] and give the first examples of locally profinite groups as in [BH06, §1.3, §1.4].
- Define characters of a locally profinite group G and their first properties as in [BH06, §1.6] and discuss the cases $G = (F, +)$ and $G = (F^\times, \cdot)$ as in [BH06, §1.7, §1.8]. You are invited/allowed to be sketchy here.
- Give the definition of smooth/admissible representation of a locally profinite group G as in [BH06, §2.1]. State the proposition and the lemma in [BH06, §2.2] briefly discussing how the smoothness assumption allows to reduce the statements to the analogous ones for finite dimensional representations of finite groups.
- State corollary 1 in [BH06, §2.3] and discuss the functor $(-)^\infty$ (namely the right adjoint to the forgetful functor from smooth representations to abstract representations).
- Define induced representations, state and prove *Frobenius reciprocity* and the following proposition in [BH06, §2.4]. Define compact induction as in [BH06, §2.5] and discuss its properties without proof.
- State and prove *Schur's lemma* for irreducible smooth representations under the hypothesis in [BH06, §2.6], discuss its first corollaries.
- Follow [BH06, §2.7] to discuss how semisimplicity behaves with respect to induction.

Talk 3 (April 19th): Measures and duality

This talk develops more tools in the study of smooth representations of locally profinite groups and applies them to the case of interest for us, i.e. $G = \mathrm{GL}_2(F)$.

- Give a crash course on Haar measures on locally profinite groups following [BH06, §3.1-3.4], including as explicit examples $G = (F, +)$, $G = (F^\times, \cdot)$, $G = \mathrm{GL}_2(F)$ (see [BH06, §7.4, 7.5]).

- Define the subgroups B, N, T, Z of $G = \mathrm{GL}_2(F)$ as in [BH06, §7.1] (they should look extremely familiar after last term's Babyseminar). Following [BH06, §7.6], define and study the corresponding Haar measures on these subgroups.
- Discuss briefly the Iwasawa decomposition and the Cartan decomposition ([BH06, §7.2]). Define the standard Iwahori subgroup I and state the Iwahori decomposition ([BH06, §7.3]). If time permits, discuss the corresponding Haar measures (as in the exercise at the end of [BH06, §7.6]).
- Discuss the notion of duality for smooth representations of locally profinite groups and its first properties as in [BH06, §2.8, 2.9].
- State the *duality theorem* in [BH06, §3.5], sketch its proof and deduce the corresponding result in [BH06, §7.7]

Talk 4 (April 26th): Irreducible non-cuspidal representations

This talk inaugurates the study of irreducible smooth representations of $G = \mathrm{GL}_2(F)$ (for F a non-archimedean local field), achieving the classification of the so-called *non-cuspidal* ones.

- Introduce the mirabolic subgroup M of G and explain the results concerning its representations contained in section 8 of [BH06].
- Introduce the Jacquet module attached to a smooth representation of G and its properties ([BH06, §9.1]), obtaining the definition of cuspidal representation. Use the proposition in [BH06, §9.1] to prove that non-cuspidal irreducible smooth representations of G are admissible ([BH06, §9.4]).
- State the *irreducibility criterion* for representations of the form $\mathrm{Ind}_B^G(\chi_1 \otimes \chi_2)$ in [BH06, §9.6] and sketch its proof ([BH06, §9.7-9.9]).
- State and prove the *classification theorem* for irreducible, non-cuspidal, smooth representations of $\mathrm{GL}_2(F)$ following [BH06, §9.10-9.11].

Talk 5 (May 3rd): Irreducible cuspidal representations I

In this talk we restrict our attention to cuspidal representation of $\mathrm{GL}_2(F)$ and give the first results concerning their classification. We assume that the residue characteristic of F is $p \neq 2$.

- Define the Hecke algebra $\mathcal{H}(G)$ of a unimodular locally profinite group G and the idempotents attached to compact open subgroups of G. Sketch the equivalence between the category of smooth representations of G and the category of smooth (left) $\mathcal{H}(G)$-modules in [BH06, §4.2]. State the proposition in [BH06, §4.3] (you can omit the proof).
- Following [BH06, §10.1-10.2], introduce the matrix coefficients of smooth representation of $G = \mathrm{GL}_2(F)$ and sketch the equivalence between cuspidality and γ-cuspidality. Deduce that every irreducible smooth cuspidal representation is admissible. If time permits and you miss category theory in this seminar, mention the *projectivity theorem* of appendix 10a in [BH06].
- Discuss intertwining (in particular proposition 2 of [BH06, §11.1]) and introduce the spherical Hecke algebra $\mathcal{H}(G, \rho)$ as in [BH06, §11.2].
- State and prove the theorem in [BH06, §11.4]. Sketch the sample construction in [BH06, §11.5].

Talk 6 (May 10th): Irreducible cuspidal representations II

This talk continues the survey concerning the classification of irreducible cuspidal representations of $\mathrm{GL}_2(F)$. Assume always $p \neq 2$ to simplify things.

- Give a quick overview of the theory of chain orders and fundamental strata in sections 12 and 13 of [BH06], with the aim of stating the theorem and corollary in [BH06, §13.3] and the proposition in [BH06, §13.4].
- State the *exhaustion theorem* in [BH06, §14.5] and give an idea of its proof. Here you might want to make a short detour on representations with Iwahori-fixed vector, which are treated in section 17 of [BH06].
- Define cuspidal types and explain the classification of cuspidal representations in terms of (conjugacy classes of) cuspidal types via the *induction theorem* and the following corollary in [BH06, §15.5].

Talk 7 (May 17th): Irreducible cuspidal representations III (& some Fourier analysis)

This talk concludes the classification of irreducible cuspidal representations of $\mathrm{GL}_2(F)$ and introduces some analytic tools which will be crucial in the following talks.

- Define and study admissible pairs and the cuspidal representations attached to them as in [BH06, sections 18-19].
- State the *parametrization theorem* and sketch its proof following [BH06, sections 20-21], always ignoring the case of residue characteristic $p = 2$.
- Introduce the Fourier transform of a Schwarz function on F (resp. on $\mathrm{M}_2(F)$) and sketch the proof of the Fourier inversion formula ([BH06, §23.1 and §24.1]).

Talk 8 (May 24th): Automorphic L-functions I

This talk is devoted to a review of the functional equation for smooth representations of $\mathrm{GL}_1(F)$ (i.e. the local version of Tate's thesis) and to the corresponding study for non-cuspidal representations of $\mathrm{GL}_2(F)$.

- Introduce the local zeta integrals for $\mathrm{GL}_1(F)$ and prove the corresponding functional equation as in the theorem in [BH06, §23.2]. Cover the rest of the material in section 23 till [BH06, §23.4].
- State theorems 1 and 2 in [BH06, §24.2] and deduce the functional equation for the local constants.
- Following [BH06, section 26], compute the L-functions and the ε-factors for non-cuspidal irreducible representations of $\mathrm{GL}_2(F)$ and prove theorems 1 and 2 in [BH06, §24.2] in the non-cuspidal case. For the case of one-dimensional and special representations (i.e. the discussion in [BH06, §26.5-26.7]) you can just give the statements and omit the proofs.
- State the *converse theorem* in [BH06, §27.1]. Show that it can be proven separately for cuspidal and non-cuspidal representations and sketch the proof in the non-cuspidal case ([BH06, §27.2-27.3]).

Talk 9 (May 31th): Automorphic L-functions II

In this talk we study the functional equation for the L-functions attached to cuspidal representations and we introduce Whittaker and Kirillov models.

- Sketch the proof of the theorems 1 and 2 in [BH06, §24.2] in the cuspidal case ([BH06, §24.4-24.6]).
- Introduce the Whittaker and Kirillov models of irreducible smooth representations of $\mathrm{GL}_2(F)$ of infinite dimension and study their first properties ([BH06, section 36]). Feel free to omit (parts of) the proof of the theorem in [BH06, §36.1].
- Discuss the description of the local constant $\varepsilon(\pi, s, \psi)$ in terms of the Kirillov model of π, for π an irreducible cuspidal representation of $\mathrm{GL}_2(F)$. Deduce the *converse theorem* in the cuspidal case from the theorem in [BH06, §37.3].

Talk 10 (June 7th): Representations of the Weil group I

This talk addresses the Galois side of the Langlands correspondence, namely representations of the Weil group $\mathcal{W}_F$. We study their first properties and define the local Artin L-functions associated with them.

- Recall the facts about Galois theory contained in [BH06, §28.1-28.3].
- Define the Weil group $\mathcal{W}_F$ and discuss its first properties ([BH06, §28.4-28.5]).
- Give the first results concerning the $\mathbb{C}$-valued representations of $\mathcal{W}_F$ contained in [BH06, §28.6-28.7], including the proofs if time permits.
- Recall quickly the statements of local class field theory ([BH06, §29.1]).
- Following [BH06, §29.2-29.4], define the local Artin L-function associated to finite-dimensional, smooth, semisimple representations of $\mathcal{W}_F$ and state the theorem in [BH06, §29.4] concerning the local constants $\varepsilon(\rho, s, \psi)$.
- Define Deligne representations as in [BH06, §31.1] and give the example of $\mathrm{Sp}(n)$. Extend the notion of direct sum, tensor product and duals to the case of Deligne representations as in [BH06, §32.2]. Define L-functions and local constants for Deligne representations as in [BH06, §32.3]. If time permits, try to motivate the definition of Deligne representation, taking inspiration from [BH06, section 32] (in particular the theorem in [BH06, §32.7] and the proposition in [BH06, §32.7]).

Talk 11 (June 14th): Representations of the Weil group II

This talk is devoted to the study of the local constants for semisimple smooth representations of $\mathcal{W}_F$. Moreover, it gives a parametrization of irreducible two-dimensional smooth representations of $\mathcal{W}_F$, almost realizing the cuspidal Langlands correspondence.

- Introduce the Grothendieck group of a profinite group G and induction constants on G and explain the results in [BH06, §30.1].
- Restate the theorem in [BH06, §29.4] and the proposition following it.
- State the theorem in [BH06, §30.2] and use it to sketch the proof of the results of [BH06, §29.4], following the discussion in [BH06, §30.3-30.6].

- Mention that the proof of the theorem in [BH06, §30.2] relies on global methods. If time permits and you are a bit familiar with the adélic language, give an idea of the main ingredients of this proof ([BH06, §30.7-30.9]).
- Realize the correspondence between admissible pairs and irreducible two-dimensional smooth representations of $\mathcal{W}_F$ ([BH06, §34.1]).

Talk 12 (June 21th): The Langlands correspondence

In this talk we prove the main result of the seminar, namely the local Langlands correspondence for $\mathrm{GL}_2(F)$ for F a local field (of residue characteristic $p \neq 2$).

- State the Langlands correspondence for $\mathrm{GL}_2(F)$ and realize the correspondence between non-cuspidal irreducible representations of $\mathrm{GL}_2(F)$ and two-dimensional reducible Deligne representations of $\mathcal{W}_F$ ([BH06, §33.1-33.3]).
- Discuss briefly how to obtain the cuspidal Langlands correspondence from the bijection realized in [BH06, §34.2], omitting all the technical details.
- Define the Weil representation associated to a couple $(E/F, \Theta)$ (E/F being a quadratic separable extension of F and Θ being a *regular* character of $E^\times$) following [BH06, section 39]. You can use as a black box the results of [BH06, section 38] (in particular the theorem in [BH06, §38.6]).
- Recover the Langlands correspondence for $\mathrm{GL}_2(F)$ using Weil representations as in [BH06, §40.1].
- Explain in a sentence why the case of residue characteristic 2 is more complicated and needs more refined techniques (key word = non-dihedral representations).
- If time permits, give an overview of the ℓ-adic Langlands correspondence, following [BH06, section 35].

Talk 13 (June 28th): Trace formulae and applications I

In this talk (and the following one) we introduce some versions of the Arthur-Selberg trace formula and an application to the proof of the so-called Jacquet-Langlands correspondence for GL_2. We always work in the number field setting and, whenever it can be useful to achieve more clarity, we assume $G = \mathrm{GL}_n$ or even $G = \mathrm{GL}_2$.

- Introduce the global Jacquet-Langlands correspondence for GL_2 (cf. [Gel96, lecture VI] and/or [KR97, section 1]).
- Explain the material in [GH22, section 16.1] concerning the automorphic kernel function, i.e. the *analytic side* of the trace formula. If time permits, mention the generalization to relative traces ([GH22, section 16.2]).
- Discuss functions with cuspidal image, following [Gel96, section 16.4] (cf. also pagg. 476-477 in section 18.2 of loc. cit.) and state the simple trace formula as in [Gel96, theorem 3.1 page 50] (or more generally [GH22, corollary 18.4.1]). Refer to the next talk for the description of the *geometric side* of the formula.
- State the trace formula of proposition [Gel96, proposition 2.1, page 49], emphasizing the presence of extra terms and trying to motivate their appearance.

Talk 14 (July 5th): Trace formulae and applications II

This talk completes the discussion concerning the simple trace formula and sketches the proof of the Jacquet-Langlands correspondence.

- Discuss local (relative) orbital integrals as in [GH22, section 17.5].
- Give an overview on global orbital integrals, with the aim of stating [GH22, thoerem 17.8.4].
- Sketch the proof of the (relative) simple trace formula, following [GH22, sections 18.2 and 18.4].
- Sketch the proof of the global Jacquet-Langlands correspondence for GL_2, following [Gel96, lecture VI, sections 1-2] and/or [KR97, section 1].

Other possible more advanced topics

- Survey on the proof of the local Langlands correspondence for GL_n over p-adic fields. The notes [Wed08] explain the ingredients in the proof given by Harris and Taylor in [HT01] (which contains a nice and readable introduction where the main ideas are sketched). One could also try to describe the similar proof given by Scholze in [Sch13] (see also the survey article [Sch20]).
- Insights on what the global situation for GL_2 looks like. The book [GH11] is probably the most friendly introduction to automorphic forms on $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. For a survey of what global Langlands should be, chapter 12 of [GH22] could be a good reference.
- Introduction to the p-adic Langlands program (see for instance [Eme14] or some parts of the scary preprint [EGH22]).

References

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