Non-cuspidal representations of $\mathrm{GL}_2(F)$

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Notation

- F non-archimedean local field with ring of integers $\mathfrak{o}$ and residue field k
- $G = \mathrm{GL}_2(F) \supset B = N \rtimes T$ where B is the standard Borel, T the standard torus and N the unipotent radical of B

Recall

If $\sigma \in \mathsf{Rep}(T)$ then we can inflate it to a representation of B along the quotient map $B \to B/N = T$ which we still call σ. Since $B\backslash G$ is compact, the Duality Theorem (see Talk 3, Thm. 5.6) tells us that

$$(\mathrm{Ind}_B^G \sigma)^\vee \cong \mathrm{Ind}_B^G(\delta_B^{-1} \otimes \sigma^\vee)$$

where $\delta_B(\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}) = ||a^{-1}b||$ is the modular character of B.

The Jacquet module

Construction. Let $(\pi, V) \in \mathsf{Rep}(G)$. Define $V(N) \subseteq V$ to be the $\mathbf{C}$-subvector space spanned by the vectors $v - \pi(n)v$, $n \in N$, $v \in V$. Set $V_N := V/V(N)$. This is the unique maximal quotient of V on which N acts trivially. It admits an action of $B/N = T$. The resulting representation (π_N, V_N) is called the *Jacquet module* of (π, V). We obtain a functor

$$\begin{aligned} \mathsf{Rep}(G) &\to \mathsf{Rep}(T) \\ (\pi, V) &\mapsto (\pi_N, V_N) \end{aligned}$$

which is exact and additive.

Proposition 1. *Let $(\pi, V) \in \mathsf{Rep}(G)$ irreducible. The following are equivalent:*

(i) The Jacquet module of (π, V) is non-zero,

(ii) π is equivalent to a subrepresentation of $\mathrm{Ind}_B^G \chi$ for some character χ of T.

Proof. Let χ be a character of T, viewed as a representation of B trivial on N. By Frobenius reciprocity,

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_B^G \chi) \cong \mathrm{Hom}_B(\pi, \chi).$$

Since χ is trivial on N, any B-homomorphism $\pi \to \chi$ factors through the map $\pi \to \pi_N$ (as a map of T-representations), hence

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_B^G \chi) \cong \mathrm{Hom}_T(\pi_N, \chi).$$

Thus, if π embeds into $\mathrm{Ind}_B^G \chi$ then clearly π_N is non-zero, showing that (ii) implies (i).

For the converse, suppose that $V_N \neq 0$. If we can show that it admits an irreducible T-quotient then we are done. Indeed, such a representation is necessarily onedimensional (cf. Talk 2, Cor. 21), i.e. a character, say χ. The quotient map $\pi_N \to \chi$ corresponds to a nontrivial map $\pi \to \mathrm{Ind}_B^G \chi$ by Frobenius reciprocity, which is an embedding since π is irreducible.
It remains to construct such a quotient. Let $0 \neq v \in V$. Since V is irreducible over G, its translates $\pi(g)v$, $g \in G$, span V over $\mathbf{C}$. On the other hand, since π is smooth, v is fixed by some compact open subgroup K' of $K_0 := \mathrm{GL}_2(\mathfrak{o})$. As K_0/K' is finite, there are only finitely many distinct elements $\pi(k)v$, $k \in K_0$, say $v_1, \dots, v_r$. By the Iwasawa decomposition $G = BK_0$ (see Talk 3, Prop. 2.1) these vectors generate V over B, hence their images generate V_N over T, so that V_N is finitely generated as a T-representation. Let $\{u_1, \dots, u_t\}$, $t \geq 1$, be a minimal generating set. By Zorn's lemma there exists a T-subspace $U \subseteq V_N$ containing $u_1, \dots, u_{t-1}$ which is maximal for the property that $u_t \notin U$. Therefore, $U \subseteq V_N$ is a maximal T-subspace, so that V_N/U is irreducible. □

Definition. An irreducible smooth representation (π, V) of G is called *cuspidal* if V_N is zero. Otherwise π is called *non-cuspidal* or to be in the *principal series.*

Proposition 2. *Any non-cuspidal representation of G is admissible.*

Proof. Passing to subrepresentations preserves admissibility, hence by Prop. 1 it suffices to show that if χ is a character of T then $\mathrm{Ind}_B^G \chi$ is admissible.

Write $\mathrm{Ind}_B^G \chi = (\Sigma, X)$ (cf. Talk 2, Construction on p. 8f.) and let $K \subseteq K_0 = \mathrm{GL}_2(\mathfrak{o})$ be a compact open subgroup. The space X^K of K-fixed points in X consists of functions $f : G \to \mathbf{C}$ such that

$$f(bgk) = \chi(b)f(g) \qquad \forall b \in B, g \in G, k \in K. \tag{$*$}$$

By the Iwasawa decomposition, $B\backslash G/K$ is finite, and on each double coset BgK there is at most one function (up to scalar) satisfying $(*)$. It follows that X^K is finite-dimensional. □

More Notation

- Let $(\pi, V) \in \mathsf{Rep}(G)$ and ϕ a character of $F^\times$. The *twist* $(\phi\pi, V) \in \mathsf{Rep}(G)$ of π by ϕ is defined via
$$\phi\pi(g) := \phi(\det g)\pi(g), \quad g \in G.$$
- Let $\chi = \chi_1 \otimes \chi_2$ be a character of T and ϕ a character of $F^\times$. The *twist* $\phi \cdot \chi$ of χ by ϕ is the character of T defined by
$$\phi \cdot \chi := \phi\chi_1 \otimes \phi\chi_2.$$
This is compatible with twists of G-representations in the sense that there is a canonical isomorphism
$$\mathrm{Ind}_B^G(\phi \cdot \chi) \cong \phi \,\mathrm{Ind}_B^G \chi.$$
- Let $\sigma \in \mathsf{Rep}(T)$. We define
$$\iota_B^G \sigma := \mathrm{Ind}_B^G(\delta_B^{-1/2} \otimes \sigma).$$
This defines a functor $\mathsf{Rep}(T) \to \mathsf{Rep}(G)$ called *normalized smooth induction.* The Duality Theorem then reads
$$(\iota_B^G \sigma)^\vee \cong \iota_B^G(\sigma^\vee).$$

The following result explains the structure of the Jacquet module of an induced representation.

Lemma 3 (Restriction-Induction)**.** *Let $\sigma \in \mathsf{Rep}(T)$. There is a short exact sequence of T-representations*
$$0 \to \sigma^w \otimes \delta_B^{-1} \to (\mathrm{Ind}_B^G \sigma)_N \to \sigma \to 0$$
where $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the permutation matrix and $\sigma^w(t) := \sigma(wtw^{-1})$, $t \in T$.

The main result needed for the classification is

Theorem 4 (Irreducibility Criterion)**.** *Let $\chi = \chi_1 \otimes \chi_2$ be a character of T. Then*

(i) $\mathrm{Ind}_B^G \chi$ *is reducible iff $\chi = \phi \cdot \mathbb{1}_T$ or $\phi \cdot \delta_B^{-1}$ for some character ϕ of $F^\times$.*

(i)' $\iota_B^G \chi$ *is reducible iff $\chi = \phi \cdot \delta_B^{\pm 1/2}$ for some character ϕ of $F^\times$.*

(ii) Suppose that $\mathrm{Ind}_B^G \chi$ *is reducible. Then*

  (a) its G-composition length is 2*,*

  (b) one composition factor is one-dimensional, the other is infinite-dimensional,

  (c) it admits a one-dimensional G-subrepresentation iff $\chi = \phi \cdot \mathbb{1}_T$ for some character ϕ of $F^\times$,

(d) *it admits a one-dimensional G-quotient iff $\chi = \phi \cdot \delta_B^{-1}$ for some character ϕ of $F^\times$.*

Remark. A smooth representation (π, V) of G has a *composition series* if there is a chain of G-subspaces

$$V = V_0 \supset V_1 \supset \cdots \supset V_l = 0$$

such that V_j/V_{j+1} is irreducible for each j. The subquotients V_j/V_{j+1} are called the *composition factors*, and the *composition length* of π is the number of factors. It is independent of the composition series.

We need the following result on homomorphisms between induced representations:

Proposition 5. *Let χ, ξ be characters of T. Then*

$$\dim_{\mathbf{C}} \operatorname{Hom}_G(\operatorname{Ind}_B^G \chi, \operatorname{Ind}_B^G \xi) = \begin{cases} 1 & \text{if } \xi = \chi \text{ or } \chi^w \delta_B^{-1}, \\ 0 & \text{else.} \end{cases}$$

Proof. By Frobenius reciprocity,

$$\operatorname{Hom}_G(\operatorname{Ind}_B^G \chi, \operatorname{Ind}_B^G \xi) \cong \operatorname{Hom}_T((\operatorname{Ind}_B^G \chi)_N, \xi).$$

By Restriction-Induction, there is a short exact sequence of T-representations

$$0 \to \chi^w \delta_B^{-1} \to (\operatorname{Ind}_B^G \chi)_N \to \chi \to 0.$$

If $\chi \neq \chi^w \delta_B^{-1}$ then this sequence splits and we are done. On the other hand, if $\chi = \chi^w \delta_B^{-1}$ then $\operatorname{Ind}_B^G \chi$ is irreducible by Thm. 4 and we are also done. $\square$

Remark. Prop. 5 gives a counter-example to the converse of Schur's Lemma for representations of locally profinite groups: $\operatorname{End}_G(\operatorname{Ind}_B^G \mathbb{1}_T)$ is one-dimensional, but $\operatorname{Ind}_B^G \mathbb{1}_T$ is not irreducible: It admits the trivial G-representation $\mathbb{1}_G$ as a one-dimensional subrepresentation with embedding $\mathbb{1}_G \to \operatorname{Ind}_B^G \mathbb{1}_T$ given by the constant functions. This leads us to

The Steinberg representation

The irreducible G-quotient of $\operatorname{Ind}_B^G \mathbb{1}_T$ is called the *Steinberg representation* of G, denoted St_G, i.e. it is defined by the short exact sequence

$$0 \to \mathbb{1}_G \to \operatorname{Ind}_B^G \mathbb{1}_T \to \operatorname{St}_G \to 0. \tag{$*$}$$

By twisting with a character ϕ of $F^\times$ we obtain the *special representations* $\phi \cdot \operatorname{St}_G$ of G:

$$0 \to \phi \cdot \mathbb{1}_G \to \operatorname{Ind}_B^G(\phi \cdot \mathbb{1}_T) \to \phi \cdot \operatorname{St}_G \to 0.$$

Taking the smooth dual of $(*)$ we get

$$0 \to \operatorname{St}_G^\vee \to \operatorname{Ind}_B^G \delta_B^{-1} \to \mathbb{1}_G \to 0.$$

Prop. 4 implies $\mathrm{St}_G \cong \mathrm{St}_G^\vee$. Indeed, there is a nontrivial map $\mathrm{Ind}_B^G \mathbb{1}_T \to \mathrm{Ind}_B^G \delta_B^{-1}$. It must contain $\mathbb{1}_G$ in its kernel because otherwise $\mathrm{Ind}_B^G \delta_B^{-1}$ would admit a one-dimensional subrepresentation which by the Irreducibility Criterion is not the case. Thus we get an induced map $\mathrm{St}_G \to \mathrm{Ind}_B^G \delta_B^{-1}$. Its image is irreducible, hence contained in $\mathrm{St}_G^\vee$, giving a nontrivial map $\mathrm{St}_G \to \mathrm{St}_G^\vee$ which is an isomorphism since $\mathrm{St}_G^\vee$ is irreducible.

Theorem 6 (Classification Theorem)**.** *The following is a complete list of the isomorphism classes of irreducible non-cuspidal representations of G:*

(i) the irreducible induced representations $\iota_B^G\chi$, where $\chi \neq \phi \cdot \delta_B^{\pm 1/2}$ for any character ϕ of $F^\times$,

(ii) the one-dimensional representations $\phi \circ \det$, where ϕ is a character of $F^\times$,

(iii) the special representations $\phi \cdot \mathrm{St}_G$, where ϕ is a character of $F^\times$.

The classes in this list are all distinct except that, in (i), we have $\iota_B^G\chi \cong \iota_B^G\chi^w$.