Irreducible cuspidal representations III (and some Fourier analysis)

As usual, we let F be a non-Archimedean local field with valuation ring $\mathcal{O}_F$ and maximal ideal $\mathfrak{p}$. We set $k = \mathcal{O}_F/\mathfrak{p}$ and we write $p = \operatorname{char} k$, $q = \# k$.
Moreover, $U_F := \mathcal{O}_F^\times$ and $U_F^n = 1+\mathfrak{p}^n$ for $n \geq 1$. We also set $G = GL_2(F)$

Let us first recall some results from the previous talk.
We have defined what is a "stratum" of G, namely a triple $(\mathfrak{A}, n, \alpha)$ where $\mathfrak{A}$ is a chain order in $A = \operatorname{End}_F(F^2)$, $n \in \mathbb{Z}$, $\alpha \in \mathfrak{P}^{-n}$ for $\mathfrak{P} = \operatorname{rad} \mathfrak{A}$.
We have classified cuspidal representations of G according to the strata they contain.
By the exhaustion theorem, a representation ϖ of G such that $\ell(\varpi) \leq \ell(\varpi\chi)$ for all $\chi \in \hat{F}^\times$ is cuspidal if and only if

1) $\ell(\varpi) = 0$ and ϖ contains a representation of $U_{\mathfrak{M}} \cong GL_2(\mathcal{O}_F)$ inflated from an irreducible cuspidal representation of $GL_2(k)$, or
2) $\ell(\varpi) > 0$ and ϖ contains a simple stratum

Using this, we have built up a bijection

$$\left\{\begin{matrix}\text{Conjugacy classes of}\\ \text{cuspidal types}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{Isomorphism classes of}\\ \text{irreducible cuspidal repr.}\end{matrix}\right\}$$

$$(\mathfrak{A}, J, \Lambda) \longmapsto c\text{-}\operatorname{Ind}_J^G \Lambda$$

Moreover, we have also seen that for a simple stratum $(\mathfrak{A}, n, \alpha)$ the element α is minimal over F, and thus gives a quadratic extenscon $E = F[\alpha]$ of F.

<u>Goal:</u> We wish to extend the map above to a bijection

$$\left\{\begin{matrix}\text{pairs } (E/F, \chi) \text{ where}\\ \cdot)\ E/F \text{ quadratic ext.}\\ \cdot)\ \chi \text{ character of } E^\times\\ \text{such that} \ldots\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{conjugacy classes}\\ \text{of cuspidal types}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{Iso classes of irred.}\\ \text{cusp. repr. of } G\end{matrix}\right\}$$

First, we need to explain what pairs $(E/F, \chi)$ should appear.
Second, we need to construct a cuspidal type out of these pairs.

§1. Admissible pairs

We start by recalling some facts about tame extensions.

As in the previous talk, we fix a character ψ of F of level one.

Def: Let E/F be a field extension. E/F is said to be "tamely ramified" if $p \nmid e(E/F)$, "wildly ramified" otherwise.

Fact: E/F is tamely ramified if and only if $Tr_{E/F}(\mathcal{O}_E) = \mathcal{O}_F$.

In particular, if $p \neq 2$ then every quadratic field extension E/F is tamely ramified. This is the case that we will mainly focus on.

We list some known properties of tame extensions.

<u>Lemma 1:</u> Let E/F be a tamely ramified field extension and set $e := e(E/F)$.

1) For $r \in \mathbb{Z}$, $Tr_{E/F}(\mathfrak{p}_E^{1+r}) = \mathfrak{p}^{1+\lfloor \frac{r}{e} \rfloor} = \mathfrak{p}_E^{1+r} \cap F$;
2) For $m \geq 1$, the norm map $N_{E/F}$ induces an isomorphism $U_E^{em}/U_E^{em+1} \xrightarrow{\sim} U_F^m/U_F^{m+1}$ which satisfies $N_{E/F}(1+x) \equiv 1 + Tr_{E/F}(x) \pmod{\mathfrak{p}^{m+1}}$ for all $x \in \mathfrak{p}_E^{em}$.
3) The norm $N_{E/F}$ induces a map $U_E/U_E^1 \to U_F/U_F^1$ which is surjective if E/F is unramified, and has kernel and cokernel of order $\gcd(e, q-1)$ if E/F is totally ramified.

For a tamely ramified extension E/F, we set $\psi_E = \psi \circ Tr_{E/F}$.

<u>Proposition 2:</u> 1) The character ψ_E has level one.

2) Let $\chi \in \widehat{F^\times}$ be of level $m \geq 1$ and set $\chi_E = \chi \circ N_{E/F}$. Then χ_E has level em. If $c \in \mathfrak{p}^{-m}$ satisfies $\chi(1+x) = \psi(cx)$ for all $x \in \mathfrak{p}^{\lfloor \frac{m}{2} \rfloor + 1}$, then $\chi_E(1+y) = \psi_E(cy)$ for all $y \in \mathfrak{p}_E^{\lfloor \frac{em}{2} \rfloor + 1}$

Proof: 1) follows from part (1) of Lemma 1;
2) The first and the second assertion follow from (2) and (3) of Lemma 1.

Def: An "admissible pair" is a pair $(E/F, \chi)$ where E/F is a tamely ramified quadratic extension and χ is a character of $E^\times$, such that:

1) χ does not factor through the norm map $N_{E/F}: E^\times \to F^\times$, and
2) if $\chi|_{U_E^1}$ factors through $N_{E/F}$, then E/F is unramified.

Two admissible pairs $(E/F, \chi)$ and $(E'/F, \chi')$ are said to be "F-isomorphic" if there is an F-isomorphism $j: E \to E'$ such that $\chi = \chi' \circ j$.

$\mathbb{P}_2(F) := \{$ isomorphism classes of admissible pairs $\}$.

For an admissible pair $(E/F, \chi)$ and a character φ of $F^\times$, the pair $(E/F, \chi \otimes \varphi_E)$ is also admissible (here, $\varphi_E = \varphi \circ N_{E/F}$).

Def: Let $(E/F, \chi)$ be an admissible pair, and call n the level of χ.

$(E/F, \chi)$ is called "minimal" if $\chi|_{U_E^n}$ does not factor through $N_{E/F}$.

Any admissible pair $(E/F, \chi)$ is isomorphic to one of the form $(E/F, \chi' \otimes \varphi_E)$ where $(E/F, \chi')$ is minimal and φ is a character of $F^\times$.

Notice that, if χ is a character of $E^\times$ of level $m \geq 1$, then there is $\alpha \in \mathfrak{p}_E^{-m}$ such that $\chi(1+x) = \psi_E(\alpha x)$ for all $x \in \mathfrak{p}_E^{m}$. This follows from the characterization of the characters of U_E^n/U_E^{n+1} that we gave in the last talk, together with the fact that χ is trivial on U_E^{m+1}.

<u>Proposition 3:</u> Let E/F be a tamely ramified quadratic field extension, χ a character of $E^\times$ of level $m \geq 1$. Let $\alpha \in \mathfrak{p}_E^{-m}$ be such that $\chi|_{U_E^m} = \alpha\psi_E$.

Then $(E/F, \chi)$ is a minimal (admissible) pair if and only if α is minimal

<u>Recall</u>: $\alpha \in E$ is called "minimal over F" if, writing $n = -v_E(\alpha)$, we have:

1) E/F is totally ramified and n is odd;

2) E/F is unramified and, for a prime element π of F, $\pi^n\alpha + \mathfrak{p}_E$ generates k_E/k.

<u>Proof</u>: If $\chi|_{U_E^m}$ factors through $N_{E/F}$, then by Proposition 2 $\chi|_{U_E^m} = c\psi_E|_{U_E^m}$ for some $c \in F$. Since $(U_E^m/U_E^{m+1})^\wedge \cong \mathfrak{p}_E^{-m}/\mathfrak{p}_E^{1-m}$, it follows that $c \equiv \alpha \pmod{\mathfrak{p}_E^{1-m}}$. Conversely, if there exists $c \in F$ such that $c \equiv \alpha \pmod{\mathfrak{p}_E^{1-m}}$, then $c\psi_E|_{U_E^m} = \alpha\psi_E|_{U_E^m} = \chi|_{U_E^m}$, so $\chi|_{U_E^m}$ factors through $N_{E/F}$.

Now, α is not minimal $\iff$ there is $c \in F$ such that $\alpha \equiv c \pmod{\mathfrak{p}_E^{1-m}}$.

$\Leftarrow$) If E/F is totally ramified, then $v_F = \frac{1}{2}v_E|_F$, so $v_E(\alpha) = v_E(c) = 2v_F(c) \in 2\mathbb{Z}$.

If E/F is unramified, then $\pi^m\alpha \equiv \pi^m c \pmod{\mathfrak{p}_E}$, so $\pi^m\alpha + \mathfrak{p}_E = \pi^m c + \mathfrak{p} \in k$.

$\Rightarrow$) If E/F is totally ramified and m is even, $\pi^{\frac{m}{2}}\alpha \equiv d \pmod{\mathfrak{p}_E}$ for some $d \in \mathcal{O}_F$, so $\alpha \equiv \pi^{-\frac{m}{2}}d \pmod{\mathfrak{p}_E^{1-m}}$, and $\pi^{-\frac{m}{2}}d \in F$.

If E/F is unramified and $\pi^n\alpha + \mathfrak{p}_E$ does not generate k_E/k, since $[k_E:k] = 2$ we must have $\pi^n\alpha + \mathfrak{p}_E \in k$, so $\pi^n\alpha \equiv \pi^n c \pmod{\mathfrak{p}_E}$ for some $c \in F$.

$\square$

§2. Construction of cuspidal representations (level 0)

Goal: we wish to associate an irreducible cuspidal representation to an admissible pair $(E/F, \chi)$

We start with admissible pairs of the form $(E/F,\chi)$ with χ of level 0.
In particular, $\chi|_{U_E^1}$ is trivial, so by definition of admissible pair E/F is unramified.

Lemma 4: Let E/F be an unramified quadratic extension, $\chi \in \widehat{E^\times}$ of level 0.
Let $\sigma \in Gal(E/F)$, $\sigma \neq 1$. The following are equivalent:

1) $(E/F,\chi)$ is an admissible pair;
2) $\chi \neq \chi^\sigma$;
3) $\chi|_{U_E} \neq \chi^\sigma|_{U_E}$

Proof: Since E/F is unramified, we have $E^\times = F^\times U_E$, so $(2) \Leftrightarrow (3)$.
E/F is cyclic, so by Hilbert's Theorem 90 the kernel of $N_{E/F}$ consists of all elements of the form $\frac{x^\sigma}{x}$ for $x \in E^\times$. It follows that χ factors through $N_{E/F}$ if and only if for all $x \in E^\times$ $\chi(\frac{x^\sigma}{x}) = 1$, that is, $\chi^\sigma(x) = \chi(x^\sigma) = \chi(x)$. □

Let $(E/F,\chi)$ be an admissible pair with χ of level 0. As in the previous talk, let $A := End_F(F^2) = M_2(F)$, and fix an F-embedding $E \hookrightarrow A$ (for example, we can map a primitive element of E/F to the companion matrix of its minimal polynomial over F). By a result at the beginning of the previous talk, there is a unique chain order $\mathfrak{A} \subseteq A$ such that $E^\times \subseteq K_{\mathfrak{A}}$ (recall that $K_{\mathfrak{A}} = \{g \in G \mid g^{-1}\mathfrak{A} g = \mathfrak{A}\}$.
By conjugating with an element of G, we may take $\mathfrak{A} = \mathfrak{M} = M_2(\mathcal{O}_F)$, so that we have embeddings $\mathcal{O}_E \hookrightarrow \mathfrak{M}$, $k_E \hookrightarrow M_2(k)$.
Since χ has level 0, it is trivial on U_E^1, so $\chi|_{U_E}$ is inflated from a character of $k_E^\times$, say $\tilde\chi$. By Lemma 4, we have $\chi|_{U_E} \neq \chi^\sigma|_{U_E}$, so $\tilde\chi^q \neq \tilde\chi$, since σ reduces modulo $\mathfrak{p}_E$ to $x \mapsto x^q$ in $Gal(k_E/k)$. This condition allows one to construct an irreducible cuspidal representation $\tilde\lambda$ of $GL_2(k)$.
A sketch of the construction of $\tilde\lambda$ is as follows:
Let ψ be any non-trivial character of $N_k = \{\left(\begin{smallmatrix}1&*\\0&1\end{smallmatrix}\right)\} \subseteq GL_2(k)$. If Z_k is the center of $GL_2(k)$, we define the character of $Z_k N_k$: $\chi_\psi : Z_k N_k \to \mathbb{C}^\times$, $\left(\begin{smallmatrix}a&0\\0&a\end{smallmatrix}\right)\cdot u \mapsto \chi(a)\psi(u)$, $a \in k^\times, u \in N_k$.

Then $\tilde{\lambda} = \mathrm{Ind}_{Z_k N_k}^{GL_2(k)} \chi_\psi - \mathrm{Ind}_{k_E^\times}^{GL_2(k)} \chi$ is an irreducible cuspidal representation of $GL_2(k)$ which does not depend on the choice of ψ.
We now inflate $\tilde{\lambda}$ to a representation λ of $U_{\mathfrak{M}} = GL_2(\mathcal{O}_F)$. It can be checked that the restriction of λ to U_F is a multiple of $\chi|_{U_F}$. We may therefore extend λ to a representation Λ of $K_{\mathfrak{M}} = F^\times U_{\mathfrak{M}}$ by imposing that $\Lambda|_{F^\times}$ be a multiple of $\chi|_{F^\times}$.
We have thus constructed a triple $(\mathfrak{M}, K_{\mathfrak{M}}, \Lambda)$ which is a cuspidal type.
We set $\varpi_\chi := c\text{-}\mathrm{Ind}_{K_{\mathfrak{M}}}^{G} \Lambda$: by the classification theorem of last talk, ϖ_χ is an irreducible cuspidal representation of G with $\ell(\varpi) = 0$.
Moreover, ϖ_χ only depends on the isomorphism class of $(E/F, \chi)$.
<u>Notation</u>: $\mathbb{P}_2(F)_0 = \{$ iso classes of admissible pairs $(E/F, \chi)$ with χ of level $0 \}$
$\mathcal{A}_2^\circ(F)_0 = \{$ equivalence classes of irred. cuspidal representations of G of level $0 \}$.

<u>Proposition 5</u>: The map $\mathbb{P}_2(F)_0 \to \mathcal{A}_2^\circ(F)_0$, $(E/F, \chi) \mapsto \varpi_\chi$ is a bijection.
Moreover, if $(E/F, \chi) \in \mathbb{P}_2(F)_0$, then
1) If $\varphi \in \widehat{F^\times}$ has level zero, then $\varpi_{\chi\varphi_E} = \varphi\, \varpi_\chi$;
2) $\omega_{\varpi_\chi} = \chi|_{F^\times}$;
3) the pair $(E/F, \check{\chi})$ is admissible and $\check{\varpi}_\chi = \varpi_{\check{\chi}}$.

<u>Proof</u>: Surjectivity follows from the fact that any cuspidal type of level 0 arises from an admissible pair $(E/F, \chi) \in \mathbb{P}_2(F)_0$ (by the study of cuspidal representations of $GL_2(k)$ and lemma 4), together with the exhaustion theorem.
Injectivity also follows by looking at the corresponding representations of $GL_2(k)$ □

§3. <u>Construction of cuspidal representations</u> (level ≥ 1)

Fix a character ψ of F of level one.
Let $(E/F, \chi)$ be a minimal admissible pair with χ of level $n \geq 1$. We set $\psi_E = \psi \circ \mathrm{Tr}_{E/F}$ and $\psi_A = \psi \circ \mathrm{Tr}_A$. Since $\chi|_{U_E^{n+1}}$ is trivial, χ induces a character in $\left(U_E^{[\frac{n}{2}]+1}/U_E^{n+1}\right)^\wedge \cong \mathfrak{p}_E^{-n}/\mathfrak{p}_E^{-[\frac{n}{2}]}$; we may hence find $\alpha \in \mathfrak{p}_E^{-n}$ such that $\chi(1+x) = \psi_E(\alpha x)$ for all $x \in \mathfrak{p}_E^{[\frac{n}{2}]+1}$.
Fix an F-embedding of E in $A = M_2(F)$ and let $\mathfrak{A}$ be the unique chain order such that $E^\times \subseteq K_{\mathfrak{A}}$. Then $e_{\mathfrak{A}} = e(E/F)$ and the triple $(\mathfrak{A}, n, \alpha)$ is a simple stratum.

(indeed, we have shown that α is minimal in Proposition 3).

Thus, to a minimal admissible pair $(E/F, \chi)$ with χ of level $n \geq 1$ we can attach a simple stratum $(\mathfrak{A}, n, \alpha)$.

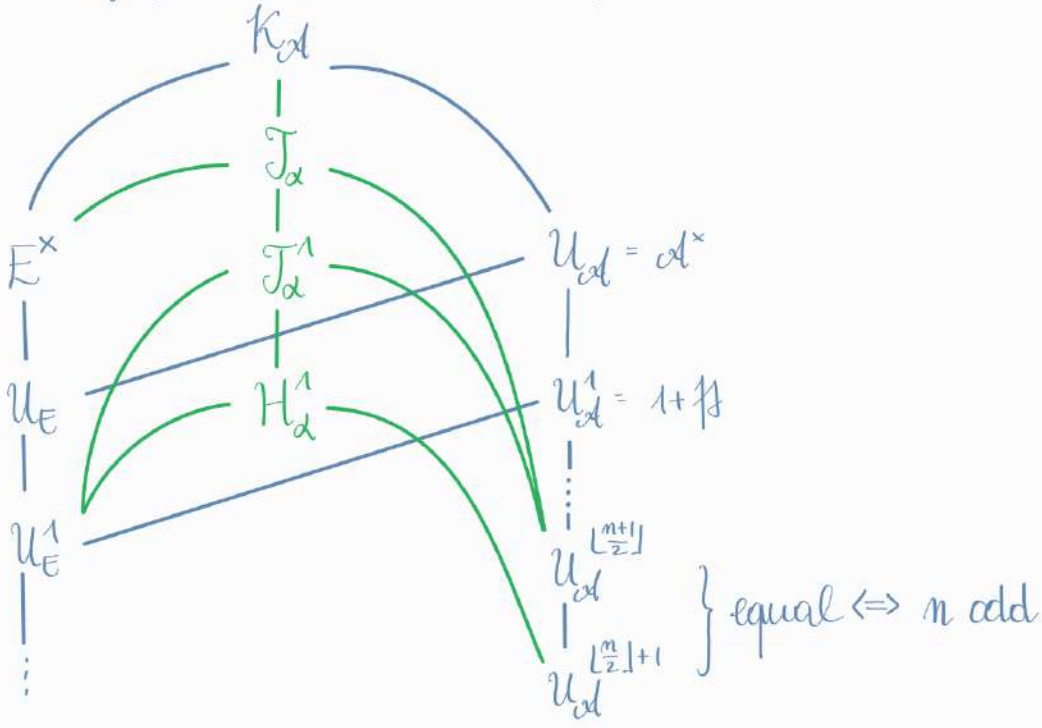
<u>Next step:</u> construct from this a cuspidal type.

(Set $J_\alpha = E^\times U_{\mathfrak{A}}^{\lfloor \frac{n+1}{2} \rfloor}$; what we are missing is a representation $\Lambda \in C(\psi_\alpha, \mathfrak{A})$, i.e. a representation Λ of J_α such that $\Lambda|_{U_{\mathfrak{A}}^{\lfloor \frac{n+1}{2} \rfloor}}$ is a multiple of ψ_α.)

We need some definitions:

$J_\alpha = E^\times U_{\mathfrak{A}}^{\lfloor \frac{n+1}{2} \rfloor}$, $\quad H_\alpha^1 = U_E^1 U_{\mathfrak{A}}^{\lfloor \frac{n}{2} \rfloor + 1}$, $\quad J_\alpha^1 = J_\alpha \cap U_{\mathfrak{A}}^1 = U_E^1 U_{\mathfrak{A}}^{\lfloor \frac{n+1}{2} \rfloor}$

Notice that $H_\alpha^1 = J_\alpha^1$ if and only if n is odd. Here is a picture



We distinguish two cases.

<u>Case 1:</u> n odd, say $n = 2m+1$.

We can define the character Λ of $J_\alpha = E^\times U_\alpha^{m+1}$ by the conditions:

$\Lambda|_{U_{\mathfrak{A}}^{m+1}} = \psi_\alpha$ and $\Lambda|_{E^\times} = \chi$.

These conditions are compatible: $E^\times \cap U_{\mathfrak{A}}^{m+1} = U_E^{m+1}$, and $\mathrm{Tr}_A|_E = \mathrm{Tr}_{E/F}$, so $\chi|_{E^\times \cap U_{\mathfrak{A}}^{m+1}} = \psi_\alpha|_{E^\times \cap U_{\mathfrak{A}}^{m+1}}$.

It follows that the triple $(\mathfrak{A}, J_\alpha, \Lambda)$ is a cuspidal type of G, so $\varpi_\chi = c\text{-}\mathrm{Ind}_{J_\alpha}^{G} \Lambda$ is an irreducible cuspidal representation of G containing the fundamental stratum $(\mathfrak{A}, n, \alpha)$, and

$\ell(\varpi_\chi) = \dfrac{n}{e(E/F)}$, $\quad \omega_{\varpi_\chi} = \chi|_{F^\times}$

Case 2: n even, say $n=2m$. In particular, E/F is unramified.

We can repeat what we have done in Case 1, but this time we only get a character ϑ of H^1_α; we can indeed define ϑ by the conditions

$\vartheta|_{\mathfrak{U}^{m+1}_\alpha} = \psi_\alpha$ and $\vartheta|_{U^1_E} = \chi$,

which are still compatible. The problem is that the fact that $\lfloor \frac{n+1}{2} \rfloor \neq \lfloor \frac{n}{2} \rfloor + 1$ prevents us from defining ϑ directly on J_α.

Lemma 6: There is a unique irreducible representation η_ϑ of J^1_α such that $\eta_\vartheta|_{H^1_\alpha}$ contains ϑ. Moreover

1) $\dim \eta_\vartheta = q$;

2) $\eta_\vartheta|_{H^1_\alpha}$ is a multiple of ϑ;

3) Let μ_E be the group of roots of unity of E of order prime to p. The action of μ_E on J^1_α by conjugation fixes the representation η_ϑ.

Proof: Quite technical.

Let us consider the representation $\eta = \eta_\vartheta$ as in Lemma 6. Since the subgroup μ_F acts trivially on J^1_α, we may consider the group $\mu_E/\mu_F \ltimes J^1_\alpha$, where the action of μ_E/μ_F over J^1_α is by conjugation.

Proposition 7: There is a unique irreducible representation $\tilde{\eta}$ of $\mu_E/\mu_F \ltimes J^1_\alpha$ such that

$\tilde{\eta}|_{J^1_\alpha} \cong \eta$ and $\mathrm{Tr}\, \tilde{\eta}(\zeta u) = -\vartheta(u)$ for all $u \in H^1_\alpha$, $\zeta \in \mu_E/\mu_F$, $\zeta \neq 1$.

Proof: Long.

Corollary 8: There is a unique representation Λ of J_α such that

1) $\Lambda|_{J^1_\alpha} \cong \eta$;

2) $\Lambda|_{F^\times}$ is a multiple of $\chi|_{F^\times}$;

3) for every $\zeta \in \mu_E \setminus \mu_F$ $\mathrm{Tr}\,\Lambda(\zeta) = -\chi(\zeta)$.

Proof: Uniqueness should be clear; we need to construct Λ with said properties. We may identify μ_E/μ_F with $E^\times/F^\times U^1_E$. The representation $\tilde{\eta}$ of Proposition 7 can be regarded as a representation of $\mu_E/\mu_F \ltimes J^1_\alpha/\ker\vartheta \cong E^\times/F^\times U^1_E \ltimes J^1_\alpha/\ker\vartheta$. We may thus inflate it to a representation ν of $E^\times \ltimes J^1_\alpha/\ker\vartheta$. We define the character $\tilde{\chi}$ of $E^\times \ltimes J^1_\alpha/\ker\vartheta$ by the conditions: $\tilde{\chi}|_{E^\times} = \chi$ and $\tilde{\chi}|_{J^1_\alpha/\ker\vartheta}$ is trivial. (Notice that there is no compatibility to check).

Set $\tilde{\Lambda} = \tilde{\chi} \otimes \nu \in \mathrm{Rep}(E^\times \times J_\alpha^1/\ker\vartheta)$. without the "1"

Consider the homomorphism $E^\times \times J_\alpha^1/\ker\vartheta \to J_\alpha/\ker\vartheta$, $(x,j) \mapsto xj$: this map is surjective, and the fact that it is a homomorphism depends on the fact that $E^\times$ acts by conjugation. The kernel of this map is then given by the elements of the form $(x^{-1}, x\ker\vartheta)$ for $x \in U_E^1$; since $\tilde{\Lambda}$ is trivial on this kernel, it must be the inflation of an irreducible representation Λ_1 of $J_\alpha/\ker\vartheta$. The inflation of Λ_1 to J_α is the representation Λ in the statement. □

Finally, the representation Λ of Corollary 8 lies in $C(\psi_\alpha, \mathfrak{A})$, so we may consider $\varpi_\chi := c\text{-}\mathrm{Ind}_{J_\alpha}^G \Lambda$. This is an irreducible cuspidal representation of G with $\ell(\varpi_\chi) = n$ and $\omega_{\varpi_\chi} = \chi|_{F^\times}$.

<u>Summarizing</u>: we start with a minimal admissible pair $(E/F, \chi)$ with χ of level $n \geq 1$. We choose an embedding $E \hookrightarrow A = M_2(F)$, which gives a unique chain order $\mathfrak{A}$ in A such that $E^\times \subseteq K_{\mathfrak{A}}$. We also fix a character ψ of F of level one and $\alpha \in \mathfrak{p}_E^{-n}$ such that $\chi(1+x) = \psi_E(\alpha x)$ for all $x \in \mathfrak{p}_E^{\lfloor \frac{n}{2} \rfloor + 1}$.
We therefore obtain a minimal fundamental stratum $(\mathfrak{A}, n, \alpha)$.
At this point, we have constructed a particular representation $\Lambda \in C(\psi, \mathfrak{A})$ of J_α, both for n odd and n even. This yields a cuspidal type $(\mathfrak{A}, J_\alpha, \Lambda)$ to which we may associate a cuspidal representation $\varpi_\chi = c\text{-}\mathrm{Ind}_{J_\alpha}^G \Lambda$ of G.

We briefly check that ϖ_χ does not depend on the choices that we have made, but only on the isomorphism type of $(E/F, \chi)$

<u>Proposition 9</u>: The representation ϖ_χ does not depend on the choice of ψ, α or the embedding $E \hookrightarrow A$, but only on the isomorphism class of $(E/F, \chi)$.
Moreover, if φ is a character of $F^\times$ such that $(E/F, \chi\varphi_E)$ is also minimal, we have $\varpi_{\chi\varphi_E} = \varphi\varpi_\chi$.

<u>Proof</u>: The choice of ψ and α does not affect J_α nor Λ. The embeddings $E \hookrightarrow A$ are conjugate to one another, so they yield conjugate cuspidal types, hence the same representation ϖ_χ. If $(E_1/F, \chi_1)$ and $(E_2/F, \chi_2)$ are isomorphic pairs

via $\sigma: E_1 \xrightarrow{\sim} E_2$, and if we choose two embeddings $j_i: E_i \hookrightarrow A$, as before $j_1(E_1)$ and $j_2(E_2)$ are conjugate by an element of G, and conjugation by such element also turns χ_1 into χ_2.
For the remaining assertion, the pair $(E/F, \chi\varphi_E)$ is associated with the cuspidal type $(\mathcal{A}, J_\alpha, \varphi \circ \det)$, which gives rise to the representation $\varphi \varpi_\chi$. $\square$

We may now extend this procedure to all admissible pairs (not necessarily minimal). Let $(E/F, \chi)$ be an admissible pair; there is a minimal pair $(E/F, \chi')$ and a character φ of $F^\times$ such that $\chi = \chi' \otimes \varphi_E$.
Following the lead of Proposition 9, we define $\varpi_\chi := \varphi \varpi_{\chi'}$.
This is again a cuspidal representation of G with normalized level the same level of χ. As in Proposition 9, ϖ_χ only depends on the isomorphism class of $(E/F, \chi)$.
We have thus constructed a map
$\mathbb{P}_2(F) \longrightarrow \mathcal{A}_2^\circ(F)$, $(E/F, \chi) \mapsto \varpi_\chi$
where $\mathcal{A}_2^\circ(F)$ is the set of irreducible classes of irreducible cuspidal representations of G.

§4. <u>The parametrization theorem</u>

From now on, we assume $p \neq 2$. Our next goal is to prove the following:
<u>Theorem 10:</u> (Parametrization theorem)

The map $\mathbb{P}_2(F) \to \mathcal{A}_0^2(F)$, $(E/F, \chi) \mapsto \varpi_\chi$ is a bijection, and:

1) if χ has level $\ell(\chi)$, then $\ell(\varpi_\chi) = \ell(\chi)/e(E/F)$;
2) $\omega_{\varpi_\chi} = \chi|_{F^\times}$;
3) $\check{\varpi}_\chi = \varpi_{\check{\chi}}$;
4) if φ is a character of $F^\times$, then $\varpi_{\varphi_E \chi} = \varphi \varpi_\chi$.

Before sketching the proof of this result, we need a preliminary definition.
<u>Def:</u> A representation $\varpi \in \mathcal{A}_0^2(F)$ is called "unramified" if there is an unramified character ϕ of $F^\times$ such that $\phi \varpi \cong \varpi$. Otherwise, ϖ is called "totally ramified".

Remark: If $p=2$, one gets an isomorphism between $\mathbb{P}_2(F)$ and the set of unramified classes in $\mathcal{A}_0^2(F)$. In sketching the proof of Theorem 10, we will ignore the case $p=2$.

We now begin the proof of Theorem 10.

Step 1: Preliminary results.

We first focus on characters ϕ of $F^\times$ for which $\phi\varpi \simeq \varpi$. Notice first that, since $\phi\varpi(g) = \phi(\det g)\cdot\varpi(g)$, the central character of $\phi\varpi$ is $\omega_{\phi\varpi}\begin{pmatrix} a & 0 \\ 0 & a\end{pmatrix} = \phi(a^2)\cdot\omega_\varpi\begin{pmatrix} a & 0 \\ 0 & a\end{pmatrix} = (\phi^2\omega_\varpi)\begin{pmatrix} a & 0 \\ 0 & a\end{pmatrix}$, so the relation $\phi\varpi \simeq \varpi$ implies that $\phi^2 = 1$ by comparing central characters.

We distinguish between ramified and unramified characters.

Proposition 11: Let ϖ be an irreducible cuspidal representation of G corresponding to the cuspidal type $(\mathcal{A}, J, \Lambda)$. Then ϖ is unramified if and only if $\mathcal{A} \cong \mathfrak{M}$.

Proof: Let ϕ be an unramified character of $F^\times$ such that $\phi\varpi \cong \varpi$. Suppose $\mathcal{A} \cong \mathfrak{M}$. For ϕ is unramified, $\phi|_{U_F} = 1$, and since $\phi^2 = 1$ we have $\det K_\mathcal{A} = (F^\times)^2 U_F \subseteq \ker\phi$. But ϕ being trivial on $\det K_\mathcal{A}$ implies that $\Lambda \otimes (\phi\circ\det) \cong \Lambda$. Thus:
$\phi\varpi = c\text{-}\mathrm{Ind}_{K_\mathcal{A}}^G (\Lambda\otimes(\phi\circ\det)) \cong c\text{-}\mathrm{Ind}_{K_\mathcal{A}}^G \Lambda = \varpi$.

Conversely, suppose $\mathcal{A} \cong \mathfrak{I}$. Then ϖ is compactly induced from a representation Λ of J, where J is a subgroup of G on which $\phi\circ\det$ does not act trivially. Since $\phi\varpi \cong \varpi$, this would imply that Λ and $\Lambda\otimes\det\phi$ intertwine in G.

However, it can be shown that two representations of $C(\mathcal{A},\psi)$ intertwine if and only if they are isomorphic (the proof is easy when the stratum $(\mathcal{A}, n, \alpha)$ associated with the cuspidal type $(\mathcal{A}, J, \Lambda)$ has n odd, while it is more involved for n even). $\square$

Proposition 12: Suppose $\varpi \in \mathcal{A}_0^2(F)$ is totally ramified.

1) There exists a unique character ϕ of $F^\times$ such that $\phi\varpi \simeq \varpi$, $\phi \neq 1$. ϕ is ramified, of level 0, and order 2. ($\phi^2 = 1$)

2) Let $(\mathfrak{A}, n, \alpha)$ be a simple stratum, $n \geq 1$, and suppose that $\varpi = \vartheta\varpi_0$ for some character ϑ of $F^\times$ and some representation ϖ_0 containing $\psi_\alpha|U_{\mathfrak{A}}^{[\frac{n}{2}]+1}$. Then the field $E = F[\alpha]$ satisfies $N_{E/F}(E^\times) = \ker\phi$.

Proof: Up to twisting ϖ by a character, we may assume that $\ell(\varpi) \leq \ell(\xi\varpi)$ for all characters ξ of $F^\times$. By the exhaustion theorem, ϖ contains a simple stratum $(\mathfrak{A}, n, \alpha)$. However, the previous Proposition tells us that $\mathfrak{A} \neq \mathfrak{M}$, since ϖ is totally ramified; thus, $(\mathfrak{A}, n, \alpha)$ is ramified and n is odd, say $n = 2m+1$. Put $E = F[\alpha]$, $J_\alpha = E^\times U_{\mathfrak{A}}^{[\frac{n}{2}]+1}$; then ϖ is induced by a cuspidal type $(\mathfrak{A}, J_\alpha, \Lambda)$. Now, tame ramification implies that $\det E^\times = N_{E/F}(E^\times) \supseteq U_F^1$, and we also have $\det U_{\mathfrak{A}}^{m+1} \subseteq \det U_{\mathfrak{A}}^1 = U_F^1$. Hence $\det J_\alpha = N_{E/F}(E^\times)$.

By class field theory, $N_{E/F}(E^\times)$ has index two in $F^\times$; there is therefore a unique character ϕ of $F^\times$ such that $\phi \neq 1$ and ϕ is trivial on $N_{E/F}(E^\times)$. Since $[F^\times : N_{E/F}(E^\times)] = 2$, we have $\phi^2 = 1$; ϕ is also ramified and has level 0.
We have $\Lambda = \Lambda \otimes (\phi \circ \det)$ for ϕ is trivial on $N_{E/F}(E^\times)$, so
$\varpi = c\text{-}\mathrm{Ind}_{J_\alpha}^G \Lambda = c\text{-}\mathrm{Ind}_{J_\alpha}^G \Lambda \otimes (\phi \circ \det) = \phi\varpi$.
This proves the existence of ϕ. Let us turn to uniqueness.
Suppose $\xi\varpi \cong \varpi$ for some character ξ of $F^\times$. Then $\xi^2 = 1$, so $\xi|_{U_F^1} = 1$ and ξ has level 0. It follows that ϖ contains both Λ and $\Lambda \otimes (\xi \circ \det)$ (where Λ is a suitable character of J_α). As in the proof of the previous Proposition, Λ and $\Lambda \otimes (\phi \circ \det)$ must intertwine, and they are therefore isomorphic. This implies that ξ is trivial on $N_{E/F}(E^\times)$, so either $\xi = 1$ or $\xi = \phi$. ◻

Step 2: Injectivity of the map $\mathbb{P}_2(F) \to \mathcal{A}_2^\circ(F)$, $(E/F, \chi) \mapsto \varpi_\chi$.
Suppose that there are admissible pairs $(E_1/F, \chi_1)$ and $(E_2/F, \chi_2)$ such that $\varpi_{\chi_1} \cong \varpi_{\chi_2}$. Set $\varpi = \varpi_{\chi_1} = \varpi_{\chi_2}$ for short.

- We prove that $E_1 \cong E_2$

  Indeed, if ϖ is unramified, then it is attached to a cuspidal datum with chain order $\mathfrak{A} \cong \mathfrak{M}$. But then $e(E_i/F) = e_{\mathfrak{A}} = 1$ for $i = 1,2$, which means that E_1, E_2 are unramified. Since F has a unique unramified extension of fixed degree, up to isomorphism, we conclude that $E_1 \cong E_2$.

On the other hand, if ϖ is totally ramified, by Proposition 12 there is a ramified character $\phi \neq 1$ of $F^\times$ such that $\phi\varpi \cong \varpi$, and $N_{E_i/F}(E_i^\times) = \ker\phi$.
Thus $N_{E_1/F}(E_1^\times) = N_{E_2/F}(E_2^\times)$, which implies $E_1 \cong E_2$ by class field theory (a "simple istance of class field theory", apparently)
We may thus assume that $E_1 = E_2$ and we will denote this field by E.
We also have that the level of χ_i is $n = e(E/F) \cdot \ell(\varpi)$.

·) We may finish the argument for injectivity, up to a lemma whose proof we shall postpone to a later stage.
Fix an embedding of E in $A = M_2(F)$ and let $\mathfrak{A}$ be the unique chain order such that $E^\times \subseteq K_{\mathfrak{A}}$. As usual, set $J = E^\times U_{\mathfrak{A}}^{\lfloor \frac{n+1}{2} \rfloor}$, so $(E_i/F, \chi_i)$ is mapped to the cuspidal type $(\mathfrak{A}, J, \Lambda_i)$ for a suitable Λ_i that we have defined in the previous section. We also know that the restriction of Λ_i to $U_{\mathfrak{A}}^{\lfloor \frac{n}{2} \rfloor + 1}$ is a multiple of ψ_{α_i} for some $\alpha_i \in \mathfrak{p}_E^{-n}$.
Since the representations associated with these cuspidal types are isomorphic, $(\mathfrak{A}, J, \Lambda_1)$ and $(\mathfrak{A}, J, \Lambda_2)$ must be conjugate, and in particular ψ_{α_1} and ψ_{α_2} intertwine in G. It can be shown that this implies that ψ_{α_1} is $U_{\mathfrak{A}}$-conjugate to ψ_{α_2}; this follows from the so-called "Conjugacy Theorem", which is a result that we have not stated, but is needed to define the representation Λ arising from an admissible pair $(E/F, \chi)$ as done in the previous section.
Assume for now the following Lemma, which we will prove later:

<u>Lemma 13</u>: There exists $u \in U_{\mathfrak{A}}$ such that $u\alpha_2 u^{-1} \in E$ and $u\alpha_2 u^{-1} \equiv \alpha_1 \pmod{\mathfrak{p}_E^{-\lfloor \frac{n}{2} \rfloor}}$.

By this lemma, conjugating the cuspidal type $(\mathfrak{A}, J, \Lambda_2)$ by u yields a cuspidal type $(\mathfrak{A}, J, \Lambda_2')$ such that $\Lambda_2'|_{U_{\mathfrak{A}}^{\lfloor \frac{n}{2} \rfloor + 1}}$ is a multiple of $\psi_{u\alpha_2 u^{-1}}|_{U_{\mathfrak{A}}^{\lfloor \frac{n}{2} \rfloor + 1}}$, and $u\alpha_2 u^{-1} \in E$.
Applying the lemma again (with α_2 replaced by α_1 and α_1 by $u\alpha_2 u^{-1}$), we may find $v \in G$ such that $v\alpha_1 v^{-1} \in E$ and $v\alpha_1 v^{-1} \equiv u\alpha_2 u^{-1} \pmod{\mathfrak{p}_E^{-\lfloor \frac{n}{2} \rfloor}}$.
As a result, $(\mathfrak{A}, J, \Lambda_1)$ is conjugate to a cuspidal type $(\mathfrak{A}, J, \Lambda_1')$ such that $\Lambda_1'|_{U_{\mathfrak{A}}^{\lfloor \frac{n}{2} \rfloor + 1}}$ is a multiple of $\psi_{v\alpha_1 v^{-1}}|_{U_{\mathfrak{A}}^{\lfloor \frac{n}{2} \rfloor + 1}}$, and $v\alpha_1 v^{-1} \in E$.
The congruence $v\alpha_1 v^{-1} \equiv u\alpha_2 u^{-1} \pmod{\mathfrak{p}_E^{-\lfloor \frac{n}{2} \rfloor}}$ implies that $\psi_{v\alpha_1 v^{-1}} = \psi_{u\alpha_2 u^{-1}}$.

This proves that Λ_1 and Λ_2 intertwine in G. It can be shown that $\Lambda_1, \Lambda_2 \in \mathcal{C}(\mathfrak{A}, \psi_\alpha)$ intertwine in G if an only if Λ_1 and Λ_2 are equivalent. Again, this follows from a previous result that we have omitted and lies behind the constructions of the previous section. The proof for n odd is rather straightforward, but the one for n even would take us too far afield.
Finally, if n is odd, the construction of Λ_i in the previous section implies that $\chi_i = \Lambda_i|_{E^\times}$. If n is even, we can recover χ_i from Λ by using point (2) and (3) of Corollary 8.
This concludes the proof of injectivity of the map $\mathbb{P}_2(F) \to \mathcal{A}_2^\circ(F)$.

<u>Step 3</u>: Surjectivity of the map $\mathbb{P}_2(F) \to \mathcal{A}_2^\circ(F)$, $(E/F, \chi) \mapsto \varpi_\chi$.
The key result is the following:
<u>Proposition 14</u>: Let $\varpi \in \mathcal{A}_2^\circ(F)$ satisfy $0 < \ell(\varpi) \leq \ell(\chi\varpi)$ for all characters χ of $F^\times$. Then there exists a minimal pair $(E/F, \chi) \in \mathbb{P}_2(F)$ such that $\varpi \cong \varpi_\chi$.
<u>Proof</u>: ϖ is associated with a cuspidal type $(\mathfrak{A}, J, \Lambda)$ attached to a simple stratum $(\mathfrak{A}, n, \alpha)$. Thus, $J = J_\alpha$ and $\Lambda|_{U_{\mathfrak{A}}^{\lfloor \frac{n+1}{2} \rfloor}}$ is a multiple of ψ_α. Since α is minimal, $E = F[\alpha]$ is a field and we have $J = E^\times U_{\mathfrak{A}}^{\lfloor \frac{n+1}{2} \rfloor}$.
If n is odd, then we can set $\chi = \Lambda|_{E^\times}$, and it follows immediately that the pair $(E/F, \chi)$ is mapped to ϖ via $\mathbb{P}_2(F) \to \mathcal{A}_2^\circ(F)$.
If n is even, we have to work harder. Let ϑ be the unique character of H_α^1 occurring in $\Lambda|_{H_\alpha^1}$. Let now χ' be any character of $E^\times$ agreeing with ϑ on U_E^1 and with ω_ϖ on $F^\times$. It is easily checked that the pair $(E/F, \chi')$ is minimal and admissible. By Corollary 8, there is a unique representation $\Lambda' \in \mathcal{C}(\psi_\alpha, \mathfrak{A})$ such that $\operatorname{Tr}\Lambda'(\zeta) = -\chi'(\zeta)$ for all $\zeta \in \mu_E \setminus \mu_F$.
Then, there is a unique character ϕ of $J_\alpha / F^\times J_\alpha^1 \cong \mu_E/\mu_F$ such that $\Lambda \cong \phi \otimes \Lambda'$. Via the isomorphism $\mu_E/\mu_F \cong E^\times / F^\times U_E^1$, we can regard ϕ as a character of $E^\times$ by inflation. Then the pair $(E/F, \phi\chi')$ is the one we are looking for. $\square$

Now it is just a matter to extend this result to any $\varpi \in \mathcal{A}_2^\circ(F)$.
We can take a character ϕ of $F^\times$ such that $\rho = \phi^{-1}\varpi$ has minimal level among its twists, i.e. $\ell(\rho) \le \ell(\chi\rho)$ for any $\phi \in \widehat{F^\times}$. If we end up with $\ell(\rho) = 0$, then we already have a bijection $\mathbb{P}_2(F)_0 \xrightarrow{\sim} \mathcal{A}_2^\circ(F)_0$ by Proposition 5, so we can find a minimal admissible pair $(E/F, \xi)$ such that $\rho \cong \varpi_\xi$. Otherwise, if $\ell(\rho) > 0$ we can apply Proposition 14 to end up with $(E/F, \xi)$ such that $\rho \cong \varpi_\xi$ all the same. If we set $\chi = \phi_E \cdot \xi$, we obtain $\varpi \cong \varpi_\chi$, which proves surjectivity.

<u>Step 4:</u> Proof of Lemma 13.
This is the only missing ingredient in the proof of the Parametrization Theorem. Recall that we had started with two cuspidal types $(\mathfrak{A}, J, \Lambda_1)$ and $(\mathfrak{A}, J, \Lambda_2)$ associated with simple fundamental strata $(\mathfrak{A}, n, \alpha_1)$ and $(\mathfrak{A}, n, \alpha_2)$. We were under the hypothesis that the field $E = F[\alpha_1]$ and $F[\alpha_2]$ coincide, and that the characters ψ_{α_1} and ψ_{α_2} of $U_{\mathfrak{A}}^{\lfloor \frac{n}{2} \rfloor + 1}$ are conjugate in $U_{\mathfrak{A}}$, i.e., the cosets $\alpha_1 U_{\mathfrak{A}}^{\lfloor \frac{n}{2} \rfloor + 1}$ and $\alpha_2 U_{\mathfrak{A}}^{\lfloor \frac{n}{2} \rfloor + 1}$ are $U_{\mathfrak{A}}$-conjugate. In particular, the cosets $\alpha_1 U_{\mathfrak{A}}^1$ and $\alpha_2 U_{\mathfrak{A}}^1$ are $U_{\mathfrak{A}}$-conjugate.
<u>Lemma 15:</u> There exists $\sigma \in Gal(E/F)$ such that $\alpha_2 \equiv \alpha_1^\sigma \pmod{U_E^1}$
<u>Proof:</u> Suppose E/F unramified. Then $\alpha_i \equiv \pi^{-n}\zeta_i \pmod{U_E^1}$ for some root of unity $\zeta_i \in \mu_E \setminus \mu_F$. Since $\pi \in Z(G)$, it follows that $\zeta_1 U_{\mathfrak{A}}^1$ and $\zeta_2 U_{\mathfrak{A}}^1$ are also $U_{\mathfrak{A}}$-conjugate. Reducing modulo $\mathfrak{P}$, the images $\tilde{\zeta}_1$ and $\tilde{\zeta}_2$ of ζ_1 and ζ_2 in $\mathfrak{A}/\mathfrak{p} \cong M_2(k)$ are conjugate under $GL_2(k)$. Moreover, $\tilde{\zeta}_1$ and $\tilde{\zeta}_2$ must generate the residue extension of E/F (because $(\mathfrak{A}, n, \alpha_i)$ is simple, so α_i is minimal), hence $\tilde{\zeta}_1$ and $\tilde{\zeta}_2$ are Galois conjugate in $k[\tilde{\zeta}_1] = k[\tilde{\zeta}_2] = k_E$ over k. It follows that ζ_1 and ζ_2 are Galois conjugate in E/F. Thus, we may find $\sigma \in Gal(E/F)$ such that $\zeta_2 = \zeta_1^\sigma$. Since $\pi \in F$, we have:
$$\alpha_2 \equiv \pi^{-n}\zeta_2 \equiv \pi^{-n}\zeta_1^\sigma \equiv (\pi^{-n}\zeta_1)^\sigma = \alpha_1^\sigma \pmod{U_E^1}.$$
Suppose instead that E/F is totally ramified, so $U_E = U_F U_E^1$. We may therefore find $y \in U_F$ such that $\alpha_2 \equiv y\alpha_1 \pmod{U_E^1}$. By using the fact that α_2 and α_1 are $U_{\mathfrak{A}}$-conjugate and taking norms, we must have $y \equiv \pm 1 \pmod{U_E^1}$. If σ is the non-trivial Galois automorphism of E/F, it follows that $\alpha_1^\sigma \equiv -\alpha_1 \pmod{U_E^1}$ $\square$

Lemma 16: There exists $u \in \mathcal{U}_{\mathfrak{A}}$ such that $uEu^{-1} = E$ and $u\alpha_2 u^{-1} \equiv \alpha_1 \pmod{U_E^1}$.

Proof: We know that there is $\sigma \in \mathrm{Gal}(E/F)$ such that $\alpha_2 \equiv \alpha_1^\sigma \pmod{U_E^1}$ by the previous lemma. Denote by $\jmath$ the embedding $E \hookrightarrow A$. The conjugate embedding $\jmath^\sigma : E \hookrightarrow A$, $x \mapsto \jmath(x^\sigma)$ will map α_1 to $\alpha_2 \cdot x$ for some $x \in U_E^1$. Since $E = F[\alpha_1] = F[\alpha_2]$, it follows that $\jmath^\sigma(E^\times) \subseteq \mathcal{K}_{\mathfrak{A}}$. It follows that $\jmath$ and $\jmath^\sigma$ are $\mathcal{U}_{\mathfrak{A}}$-conjugate, which implies the statement. $\square$

In order to prove Lemma 13, up to replacing α_2 by $u\alpha_2 u^{-1}$ we may assume that $\alpha_2 \equiv \alpha_1 \pmod{U_E^1}$. The final step is to show:

Lemma 17: Suppose that $\alpha_1 \equiv \alpha_2 \pmod{U_E^1}$ and that $\alpha_1 \mathcal{U}_{\mathfrak{A}}^{\lfloor \frac{n+1}{2} \rfloor}$ and $\alpha_2 \mathcal{U}_{\mathfrak{A}}^{\lfloor \frac{n+1}{2} \rfloor}$ are $\mathcal{U}_{\mathfrak{A}}$-conjugate. Then $\alpha_1 \equiv \alpha_2 \pmod{U_E^{\lfloor \frac{n+1}{2} \rfloor}}$

Proof: If $\mathfrak{P} = \mathrm{rad}\,\mathfrak{A}$, there is $u \in \mathcal{U}_{\mathfrak{A}}$ such that $u\alpha_2 u^{-1} \equiv \alpha_1 \pmod{\mathfrak{P}^{1-n}}$. It is not too difficult but rather technical to see that under these assumptions $u \in U_{\mathfrak{A}}^1 U_E$. We may actually assume that $u \in U_{\mathfrak{A}}^1$, since if $u = ab$ with $a \in U_{\mathfrak{A}}^1$ and $b \in U_E$, then $u\alpha_2 u^{-1} = ab\alpha_2 b^{-1} a^{-1} = a\alpha_2 a^{-1}$ because $\alpha_2, b \in E$, so they commute with each other. Let us then write $u = 1 + x$ for $x \in \mathfrak{P}$.

We now argue by induction. Let $m \geq 1$; suppose that $\alpha_1 \equiv \alpha_2 \pmod{\mathfrak{p}_E^{m-n}}$ and that there exists $x \in \mathfrak{P}^m$ such that $(1+x)\alpha_2(1+x)^{-1} \equiv \alpha_1 \pmod{\mathfrak{P}^{m+1-n}}$. Then we claim $x \in \mathfrak{p}_E^m + \mathfrak{P}^{m+1}$ and $\alpha_1 \equiv \alpha_2 \pmod{\mathfrak{p}_E^{m+1-n}}$.

Write $\alpha_2 = \alpha_1 + c$ with $c \in \mathfrak{p}_E^{m-n}$. Since $U_{\mathfrak{A}}^m / U_{\mathfrak{A}}^{m+1} \cong \mathfrak{P}^m/\mathfrak{P}^{m+1}$, we have $(1+x)^{-1} = 1 - x + q$ for some $q \in \mathfrak{P}^{m+1}$. Thus:

$$\alpha_1 \equiv (1+x)\alpha_2(1+x)^{-1} \equiv (1+x)(\alpha_1 + c)(1 - x + q) \equiv \alpha_1 + c - \alpha_1 x + x\alpha_1 \pmod{\mathfrak{P}^{m+1-n}},$$

where the last congruence follows by expanding the product and noticing that many of the terms which appear lie in $\mathfrak{P}^{m+1-n}$. Hence $\alpha_1 x - x\alpha_1 \equiv c \pmod{\mathfrak{P}^{m+1-n}}$

Now consider the character of $\mathfrak{p}_E^{m-n}$ given by $y \mapsto \psi_E(cy)$ for $y \in \mathfrak{p}_E^{m-n}$. Using the fact that $\psi_A = \psi \circ \mathrm{tr}_A$ and that the trace is a central function (i.e., it is constant on conjugacy classes of A), we get

$$\begin{aligned}\psi_E(cy) &= \psi_A(cy) = \psi_A((\alpha_1 x - x\alpha_1)y) = \psi_A(\alpha_1 x y)\,\psi_A(x\alpha_1 y)^{-1} = \\ &= \psi_A(y \cdot (\alpha_1 x y) \cdot y^{-1}) \cdot \psi_A(\alpha_1 y \cdot (x\alpha_1 y)(\alpha_1 y)^{-1})^{-1} = \psi_A(y\alpha_1 x)\,\psi_A(\alpha_1 y x)^{-1} = \\ &= \psi_A((y\alpha_1 - \alpha_1 y)x).\end{aligned}$$

But α_1 and y both lie in E, so they commute with one another. As a result $\psi_E(cy)=1$ for all $y\in \mathfrak{p}_E^{m-n}$. The fact that ψ_E has level one, we infer that $c\in \mathfrak{p}_E^{m+1-n}$, which means $\alpha_2\equiv\alpha_1 \pmod{\mathfrak{p}_E^{m+1-n}}$. We also obtain $\alpha_1 x - x\alpha_1 \in \mathfrak{P}^{m+1-n}$. This finally implies that $x\in \mathfrak{p}_E^{m}+\mathfrak{P}^{m+1}$.

From this induction process Lemma 17 follows, and Lemma 13 together with it $\square$

§5. Some Fourier analysis.

We need to recall some elementary results about Fourier transforms which will be used in future talks.

Fix a character ψ of F, $\psi\neq 1$, and a Haar measure μ for F. We recall that all characters of F are of the form $a\psi$ for $a\in F$, where $(a\psi)(x)=\psi(ax)$ for all $x\in F$. Recall that $C_c^\infty(F)$ denotes the set of compactly supported smooth functions over F, i.e. functions $F\to\mathbb{C}$ which are locally constant and are zero on the complement of a compact subset of F.

Def: The "Fourier transform" of a function $\Phi\in C_c^\infty(F)$ is the function $\hat{\Phi}:F\to\mathbb{C}$ defined by $\hat{\Phi}(x)=\int_F \Phi(y)\psi(xy)\,d\mu(y)$.

Remark: This definition mimics the one in classical analysis. If $f\in L^1(\mathbb{R})$ (with respect to the Lebesgue measure over $\mathbb{R}$), its Fourier transform is $\hat{f}(x)=\int_{\mathbb{R}} f(y)e^{-2\pi i xy}\,dy$, and notice that the map $\mathbb{R}\to\mathbb{C}^\times$, $y\mapsto e^{-2\pi i y}$ is a character of $\mathbb{R}$ as an additive group.

Let us recall some properties of the classical Fourier transform.

If $f\in L^1(\mathbb{R})$, then $\|\hat{f}\|_{L^\infty}\leq\|f\|_{L^1}$, so $\hat{f}$ is bounded; however, $\hat{f}$ need not lie in $L^1(\mathbb{R})$ in general. In case it happens that $\hat{f}\in L^1(\mathbb{R})$, then Fourier inversion formula holds: $f(x)=\int_{\mathbb{R}}\hat{f}(y)e^{2\pi i xy}\,dy$, which means $\hat{\hat{f}}(x)=f(-x)$

If $\Phi\in C_c^\infty(F)$, the function $y\mapsto \Phi(y)\psi(xy)$ is locally constant, so $\hat{\Phi}$ consists of a finite sum.

<u>Proposition 18</u>: Let $\Phi \in C_c^\infty(F)$

1) $\hat{\Phi} \in C_c^\infty(F)$;
2) There exists some positive real number $c = c(\psi, \mu_\psi)$ such that $\hat{\hat{\Phi}}(x) = c\,\Phi(-x)$ for all $x \in F$.
3) For a given ψ, there exists a unique Haar measure μ_ψ such that $c(\psi, \mu_\psi) = 1$. Moreover, if ψ has level ℓ, $\mu_\psi(\mathcal{O}_F) = q^{\ell/2}$.
4) For all $a \in F^\times$, $\mu_{a\psi} = \|a\|^{1/2} \mu_\psi$.

<u>Proof</u>: Let ℓ be the level of ψ and denote by Φ_J the characteristic function of $\mathfrak{p}^J$. For $a \in F$, the character $a\psi|_{\mathfrak{p}^J}$ is trivial if and only if $a \in \mathfrak{p}^{\ell - J}$.
Notice that

$$\hat{\Phi}_J(x) = \int_F \Phi_J(y)\psi(xy)\,d\mu(y) = \int_{\mathfrak{p}^J} \psi(xy)\,d\mu(y) =$$

$$= \begin{cases} \mu(\mathfrak{p}^J) = \mu(\mathcal{O}_F)\cdot q^{-J} & \text{if } x \in \mathfrak{p}^{\ell-J} \\ \displaystyle\sum_{\substack{y \in \mathfrak{p}^J/\mathfrak{p}^{J+1} \\ \text{(coset repres. of } \mathfrak{p}^J/\mathfrak{p}^{J+1})}} (x\psi)(y) \int_{y+\mathfrak{p}^{J+1}} d\mu = \int_{\mathfrak{p}^{J+1}} d\mu \cdot \sum_{y \in \mathfrak{p}^J/\mathfrak{p}^{J+1}} (x\psi)(y) = 0 & \text{if } x \notin \mathfrak{p}^{\ell - J} \end{cases}$$

In the last equality, we are using the fact that if A is a finite group and χ a character of A, then $\sum_{a \in A} \chi(a) = |A|$ if χ is trivial, while $\sum_{a \in A} \chi(a) = 0$ if χ is non-trivial.
This computation easily leads to (1) and (2) in the case when $\Phi = \Phi_J$.
Now, for $\Phi \in C_c^\infty(F)$ and $a \in F$, consider the function $\Psi \in C_c^\infty(F)$ given by $\Psi(x) = \Phi(x - a)$ for all $x \in F$. We have

$$\hat{\Psi}(x) = \int_F \Phi(y-a)\psi(xy)\,d\mu(y) = (a\psi)(x) \int_F \Phi(y-a)\psi(x(y-a))\,d\mu(y) = (a\psi)(x)\,\hat{\Phi}(x).$$

Since $a\psi$ is locally constant, $\hat{\Psi} \in C_c^\infty(F)$ if and only if $\hat{\Phi} \in C_c^\infty(F)$.
Moreover, if $\Phi \in C_c^\infty(F)$, then

$$\hat{\hat{\Psi}}(x) = \int_F \hat{\Phi}(y)\psi(ay)\psi(xy)\,d\mu(y) = \int_F \hat{\Phi}(y)\,\psi(y(x+a))\,d\mu(y) = \hat{\hat{\Phi}}(x+a).$$

This shows that (1) and (2) hold for Φ if and only if they hold for Ψ (with the same value of $c = c(\psi, \mu)$ for (2)).

But (1) and (2) hold for the functions Φ_j for all $j \in \mathbb{Z}$. Thus, they hold also for the functions $x \mapsto \Phi_j(x-a)$ for all $a \in F, j \in \mathbb{Z}$. These functions generate $C_c^\infty(F)$ as a $\mathbb{C}$-vector space, so (1) and (2) hold for every $\Phi \in C_c^\infty(F)$, using the fact that the Fourier transform is $\mathbb{C}$-linear.
We may also take $c = c(\psi, \mu) = \mu(\mathcal{O}_F)^2 q^{-\ell}$.
For any $b > 0$ we have $c(\psi, b\mu) = b^2 \mu(\mathcal{O}_F)^2 q^{-\ell} = b^2 c(\psi, \mu)$. Thus, if we fix ψ and we start with any Haar measure μ of F, the measure μ_ψ in (3) is given by $\mu_\psi = \mu(\mathcal{O}_F) q^{-\ell/2} \cdot \mu$. Since all Haar measures of F are multiples of μ, uniqueness in (3) follows. Now (4) comes from a straightforward computation. ◻

Def: The measure μ_ψ as in Proposition 18 is called the "self dual Haar measure on F, relative to ψ".
Using μ_ψ for computing Fourier transforms gives the Fourier inversion formula: $\hat{\hat{\Phi}}(x) = \Phi(-x)$ for all $\Phi \in C_c^\infty(F)$ and $x \in F$.

We now pass to the case of $A = M_2(F)$.
Fix a character ψ of F, $\psi \neq 1$, and set $\psi_A = \psi \circ \mathrm{Tr}_A$. Fix a Haar measure μ of A.
Def: The Fourier transfor of $\Phi \in C_c^\infty(A)$ is the function $\hat{\Phi} : A \to \mathbb{C}$ defined by
$$\hat{\Phi}(x) = \int_A \Phi(y)\, \psi_A(xy)\, d\mu(y) \quad \text{for all } x \in A.$$
Exactly as in Proposition 18, one can check that $\hat{\Phi} \in C_c^\infty(A)$ and there is a unique Haar measure μ_ψ^A for which the inversion formula holds, that is $\hat{\hat{\Phi}}(x) = \Phi(-x)$ for all $x \in A$.
Such μ_ψ^A is called the "self dual Haar measure on A, relative to ψ".
For this measure, we have $\mu_\psi^A(\mathfrak{M}) = q^{2\ell}$ and $\mu_{a\psi}^A = \|a\|^2 \mu_\psi^A$, where $a \in F^\times$ and ℓ is the level of ψ.