Babyseminar WS 2022-23 - Talk 11

Reductive groups I - (Split) reductive groups of semisimple rank 1

Ref: [M] J.S. Milne, 'Algebraic Groups', chapters 13 and 20

§1. Introduction

$k = \bar{k}$ alg. closed field, G/k affine algebraic group

Def: (a) G is called reductive if G is smooth and connected and $R_u(G) = e$ ($R_u(G)$ is the unipotent radical of G, i.e. the maximal smooth normal connected unipotent subgroup of G)

(b) If G is a reductive group, its semisimple rank is defined as the dimension of a maximal torus in $G/R(G)$ (recall that $R(G)$ is the radical of G, i.e. the maximal smooth connected normal solvable subgroup of G)

Ex: $G = GL_n$ $\quad R(G) = Z(G) = \mathbb{G}_m \quad G/R(G) \cong PGL_n$

and maximal tori in PGL_n have dim. $n-1$

$\Rightarrow GL_n$ has semisimple rank $n-1$

Rmk: G reductive of semisimple rank 0 $\iff$ G reductive and solvable

$\iff$ G is a torus

Niklas' Talk

("$\Leftarrow$" obvious

"$\Rightarrow$" $G/R(G)$ unipotent (it contains no nontrivial torus) and semisimple, hence trivial)

AIM OF TODAY:

Thm A ([M], prop. 20.32) Let G be a (split) reductive group of semisimple rank 1 with central kernel. Then there exists a homomorphism $v: (SL_2, T_2) \to (G, T)$ and any two. Every such v is a central isogeny from SL_2 onto $\mathcal{D}G = [G, G]$ differ by the inner automorphism defined by an element of $(N/\mu_2)(k)$. Here $T_2 \leq SL_2$ maximal diagonal torus, $N = N_{SL_2}(T)$, $\mu_2 = Z(SL_2)$.

Actually one can then deduce that there are not many possibilities

Thm B ([M], thm 20.33) Every (split) reductive group over k of semisimple rank 1 is isomorphic to exactly one of the following groups:

$\mathbb{G}_m^r \times SL_2$, $\quad \mathbb{G}_m^r \times PGL_2$, $\quad \mathbb{G}_m^r \times GL_2$ $\quad$ some $r \in \mathbb{Z}_{\geq 0}$

Rmk: The groups appearing above are pairwise non-isomorphic, since:

(i) $\mathbb{G}_m^r \cong \mathbb{G}_m^s \iff r=s$

(ii) $\mathcal{D}(SL_2) = \mathcal{D}(GL_2) = SL_2$ $\quad$ $\mathcal{D}(PGL_2) = PGL_2$ $\quad$ (one can check this on k-points)

(iii) $Z(SL_2) = \mu_2$, $Z(GL_2) = \mathbb{G}_m$, $Z(PGL_2) = e$

§2. Cocharacters and limits in algebraic groups

Def Let X be a separated alg. k-scheme and assume that we have an action $\mathbb{G}_m \times X \xrightarrow{\mu} X$

Given $x \in X(R)$ we have an orbit map $\mu_x : \mathbb{G}_{m,R} \to X_R$, $t \mapsto t\cdot x$

If μ_x extends (nec. uniquely) to a map $\mathbb{A}^1_R \xrightarrow{\tilde{\mu}_x} X_R$, we say that $\lim_{t\to 0} t\cdot x \in X(R)$ exists and it is given by $\tilde{\mu}_x(0)$.

Now let G be an alg. group/k (always affine), $\lambda : \mathbb{G}_m \to G$ a cocharacter; then λ defines an action $\mathbb{G}_m \times G \to G$ via inner automorphisms, i.e. $t\cdot g = \lambda(t)\, g\, \lambda(t)^{-1}$.

Def: In the above setting we let (i) $Z(\lambda) := C_G(\mathrm{Im}(\lambda)) = G^{\mathbb{G}_m}$

(ii) $P(\lambda)(R) := \{ g \in G(R) \mid \lim_{t\to 0} t\cdot g \text{ exists} \}$

(iii) $U(\lambda)(R) := \{ g \in G(R) \mid \lim_{t\to 0} t\cdot g \text{ exists and equals } 1_R \in G(R) \}$

Prop 1: In the above setting we have:

(a) $Z(\lambda)$, $P(\lambda)$, $U(\lambda)$ define algebraic subgroups of G and $U(\lambda)$ is a normal unipotent subgroup of $P(\lambda)$.

Assume that G is smooth, then:

(b) $Z(\lambda)$, $P(\lambda)$, $U(\lambda)$ are smooth; $P(\lambda)$ (resp. $U(\lambda)$) is the unique smooth algebraic subgroup of G st $P(\lambda)(k) = \{ g \in G(k) \mid \lim_{t\to 0} t\cdot g \text{ exists} \}$ (resp. $U(\lambda)(k) = \{ g\in G(k) \mid \lim_{t\to 0} t\cdot g = 1_k \}$)

(c) The multiplication map $U(\lambda) \rtimes Z(\lambda) \to P(\lambda)$ is an iso of alg. groups

(d) " " $U(-\lambda) \times P(\lambda) \to G$ is an open immersion of algebraic varieties

(e) G connected $\Rightarrow$ $Z(\lambda)$, $P(\lambda)$, $U(\lambda)$ are connected and $U(\lambda)$, $Z(\lambda)$, $U(-\lambda)$ generate G

Pf: Omitted, cf. [M], section 13.d.

§3. Reductive groups of semisimple rank 1

Lemma 2: Let G be a smooth connected semisimple nonsolvable algebraic group of rank 1.
Fix an isomorphism $\lambda: \mathbb{G}_m \xrightarrow{\sim} T$ $\quad T \subseteq G$ a maximal torus. Then:

(a) $\exists B^+$ Borel subgroup of G s.t. $T \subseteq B^+$ and $U(\lambda) \subseteq B^+$
$\quad \exists B^-$ ——————————— $T \subseteq B^-$ and $U(-\lambda) \subseteq B^-$

(b) It cannot happen $U(\lambda) \subseteq B^-$ or $U(-\lambda) \subseteq B^+$ (with the notation of (a))

(c) There exists an element of $N_G(T)(k)$ which acts on T as $t \mapsto t^{-1}$

Pf: (a) $Z(\lambda) = \underbrace{C_G(T) = T}_{\text{Marc's Talk}}$, $\quad P(\lambda) \cong U(\lambda) \rtimes Z(\lambda)$ (prop. 1 (c))

$\implies P(\lambda)$ solvable and connected $\implies \exists B^+ \leq G$ Borel subgroup st $P(\lambda) \subseteq B^+$.
Similarly $\exists B^-$ st $P(-\lambda) \subseteq B^-$.

(b) Since G is not solvable and since $U(\lambda), Z(\lambda), U(-\lambda)$ generate G, we deduce that $U(-\lambda) \nsubseteq B^+$ and $U(\lambda) \nsubseteq B^-$.

(c) Recall that (Lukas' talk) $N_G(T)(k)$ acts transitively on the set $\mathcal{B}^T(k)$ of Borel subgroups containing T. Pick $x \in N_G(T)(k)$ sending B^+ to B^-; then the iso $B^+ \xrightarrow{\sim} B^-$ given by conjugation by x induces an automorphism $T \xrightarrow{\sim} T$ which must be nontrivial, since $W(G,T) = \dfrac{N_G(T)(k)}{C_G(T)(k)}$ acts simply transitively on $\mathcal{B}^T(k)$. Hence the automorphism $T \xrightarrow{\sim} T$ must be given by $t \mapsto t^{-1}$ (unique non trivial automorphism of $\mathbb{G}_m \cong T$) ∎

Lemma 3: Let C be a smooth proj. alg. curve over k; if C admits a nontrivial action by a smooth and connected alg. group $G \implies C \cong \mathbb{P}^1$

Pf: Assume first that G is solvable $\iff$ split solvable; hence C admits a nontrivial action by $\mathbb{G}_a$ (or $\mathbb{G}_m$); then for some $x \in C(k)$ the orbit map
$\mu_x: \mathbb{G}_a \to C$ (or $\mu_x: \mathbb{G}_m \to C$) is non constant, hence dominant

$\implies \underset{k}{\overset{\cup\!\!\!\times}{R(C)}} \hookrightarrow R(T) \implies R(C) \cong k(\mathbb{P}^1) \implies C \cong \mathbb{P}^1$

Now for general G recall that $G(k) = \bigcup_{B \text{ Borel}} B(k)$ and since G is smooth we see that G acts nontrivially on $C \iff \underset{\text{Borel}}{B}$ acts nontrivially on C for some C

So we are reduced to the case G solvable ∎

Thm 4: G reductive nonsolvable group, $T \leq G$ maximal torus; TFAE:

(a) G has semisimple rank 1

(b) T lies exactly in two Borel subgroups

(c) $\dim(\mathcal{B}) = 1$ (recall $\mathcal{B} \cong G/B$, WLOG B Borel subgr., $B \supseteq T$)

(d) there is an isogeny $G/R(G) \to PGL_2$

Pf: (a) $\Rightarrow$ (b) we can replace G by $G/R(G)$ and assume that G is semisimple. Since in this case $T \cong \mathbb{G}_m$ and $\mathrm{Aut}(\mathbb{G}_m) = \{\pm 1\}$ and $W(G,T)$ acts simply transitively on the set of Borel subgroups containing T $\Rightarrow$ at most 2 of those.

Lemma 2 (b) $\Rightarrow$ at least two of those, so (b) follows.

(b) $\Rightarrow$ (c) one can show that there must be at least $\dim(\mathcal{B})+1$ fixed points for the action of T on $\mathcal{B}$, and we know that there are exactly two of them

$\Rightarrow \dim(\mathcal{B}) = 1$.

(c) $\Rightarrow$ (d) $\mathcal{B}$ smooth proj curve with a nontrivial action of G $\Rightarrow$ (lemma 3) $\mathcal{B} \cong \mathbb{P}^1$ $\Rightarrow$ we get a homomorphism $G \xrightarrow{\varphi} \underline{\mathrm{Aut}(\mathbb{P}^1)} \cong PGL_2$ with

$$R(G) = \Big(\bigcap_{\substack{B \leq G \\ \text{Borel}}} B\Big)^0_{\text{red}} \subseteq \bigcap_{\substack{B \subseteq G \\ \text{Borel}}} B \underset{\text{Lukas' talk}}{=} \bigcap_{\substack{B \subseteq G \\ \text{Borel}}} N_G(B) = \ker\varphi \qquad (*)$$

Note that φ is surjective, since every proper alg. subgroup of PGL_2 is solvable (having $\dim \leq 2$) and G is nonsolvable by assumption.

Finally $\bar{\varphi}: G/R(G) \twoheadrightarrow PGL_2$ is an isogeny because by $(*)$ we see that $R(G) \trianglelefteq \ker\varphi$ has finite index.

(d) $\Rightarrow$ (a) $PGL_2 = \dfrac{GL_2}{Z(GL_2)} = \dfrac{GL_2}{R(GL_2)}$ is semisimple of rank 1.

Since we have an isogeny $G/R(G) \to PGL_2$, the same holds for $G/R(G)$.

Rmk: The surjective hom. $\varphi: G \twoheadrightarrow PGL_2$ in the above proof actually satisfies $\ker\varphi = Z(G)$. Indeed since φ is surjective we see that $\varphi(Z(G)) \leq Z(PGL_2) = e$ $\Rightarrow Z(G) \leq \ker\varphi$. On the other hand $\ker\varphi = \bigcap_{\substack{B \leq G \\ \text{Borel}}} B \underset{\text{Marc's Talk}}{=} \Big(\bigcap_{\substack{T \leq G \\ \text{max torus}}} T\Big) R(G)$ is a normal diagonalizable subgroup of G.

By rigidity (cf. Giulia's talk) we deduce that $\ker\varphi \leq Z(G)$.

Prop 5: SL_2 is simply connected (i.e. every isogeny $G \to SL_2$ with G smooth and conn. and with diagonalizable kernel is an isomorphism) and the projection map $SL_2 \twoheadrightarrow PGL_2$ is the universal covering of PGL_2 (in particular, since SL_2 and PGL_2 are also perfect, it holds that, for every $\varphi: G \to PGL_2$ isogeny of connected group varieties with diagonalizable kernel, there exists a unique $\alpha: SL_2 \to G$ st

$$\begin{tikzcd} & SL_2 \arrow[d, two heads] \\ G \arrow[ur, dashed, "\exists!\alpha"] \arrow[r, two heads, "\varphi"'] & PGL_2 \end{tikzcd}$$

).

Pf: Assume $G \overset{\varphi}{\twoheadrightarrow} PGL_2$ is an isogeny with G sm. and conn. and $\ker\varphi$ diagonalizable. Then G is reductive ($R_u(G)$ maps isomorphically onto its image in PGL_2, which is trivial). Pick $T \subseteq G$ max. torus. As in the previous remark, by rigidity $\ker\varphi \subseteq Z(G)$ $\Rightarrow$ $\ker\varphi \subseteq T$ and $T/\ker\varphi$ maps isom. onto a max torus in PGL_2, so $T/\ker\varphi \cong \mathbb{G}_m$

$\Rightarrow T \cong \mathbb{G}_m$ and $\ker\varphi \cong \mu_n$ for some n. We claim that $n \leq 2$.

The element $\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] \in PGL_2(k)$ normalizes the diagonal torus in PGL_2 and acts on it as $t \mapsto t^{-1}$. Hence a lift to $G(k)$ of a suitable conjugate on $\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right]$ normalizes T and acts on it as $t \mapsto t^{-1}$. Since $\ker\varphi$ is central, the action is trivial on $\ker\varphi$

$\Longrightarrow n \leq 2$.

Now if $G \to SL_2$ is an isogeny with G smooth and conn. and with diag. kernel, then $G \to SL_2 \twoheadrightarrow PGL_2$ has degree at most 2, so $G \to SL_2$ must be an isomorphism ($SL_2 \twoheadrightarrow PGL_2$ has already deg. 2) ∎

Proof of thm A

We have produced an exact sequence: $e \to Z(G) \to G \overset{q}{\twoheadrightarrow} PGL_2 \to e$ where we can assume that q maps the maximal torus T onto the diagonal torus in PGL_2. $T' := (T \cap \mathcal{D}G)_t$ is a maximal torus of $\mathcal{D}G$ (easy to see). Note that $\mathcal{D}G$ is not solvable (otherwise G would be solvable), hence $q|_{\mathcal{D}G}$ must be surjective (every proper alg subgr. of PGL_2 is solvable). $q|_{\mathcal{D}G}: \mathcal{D}G \longrightarrow PGL_2$.

Moreover $\ker(q|_{\mathcal{D}G}) = Z(G) \cap \mathcal{D}G$ is finite. Indeed choose a faithful rep. $G \overset{\rho}{\hookrightarrow} GL_V$, recall that $R(G) = Z(G)_t$ (max torus in $Z(G)$). Since $R(G)$ torus, we can diagonalize its action on V, hence $V = V_{\chi_1} \oplus \cdots \oplus V_{\chi_n}$ $\chi_i \in X^*(R(G))$

with $\chi_i \neq \chi_j$ if $i \neq j$. Choosing a suitable basis of V we see that the images of elements $t \in R(G)(k)$ are of the form $\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_r \end{pmatrix}$ $A_i = \begin{pmatrix} \chi_i(t) & & \\ & \ddots & \\ & & \chi_i(t) \end{pmatrix}$

Images of elements of $\mathcal{D}G(k)$ lie in $\begin{pmatrix} SL_{V_{\chi_1}}(k) & & 0 \\ & \ddots & \\ 0 & & SL_{V_{\chi_r}}(k) \end{pmatrix}$ (here we use that, since $R(G) \subseteq Z(G)$, the decomp. $V = \bigoplus V_{\chi_i}$ is preserved by G)

but $SL_{V_{\chi_i}}(k)$ contains only finitely many scalar matrices for all $i = 1 \dots r$

$\Rightarrow R(G) \cap \mathcal{D}G$ finite $\Rightarrow Z(G) \cap \mathcal{D}G$ finite.

We conclude that $q|_{\mathcal{D}G} : \mathcal{D}G \twoheadrightarrow PGL_2$ is an isogeny with diagonalizable kernel and such that $q(T')$ is the maximal diagonal torus in PGL_2.

Hence by prop. 5 we deduce that $\exists!\, v' : (SL_2, T_2) \to (\mathcal{D}G, T')$ isogeny with diag. kernel s.t.

$$\begin{tikzcd} & (SL_2, T_2) \arrow[dl, dashed, "\exists!\, v'"'] \arrow[d] \\ (\mathcal{D}G, T') \arrow[r, two heads, "q|_{\mathcal{D}G}"'] & (PGL_2, \overline{T}_2) \end{tikzcd}$$

We let $v : (SL_2, T_2) \xrightarrow{v'} (\mathcal{D}G, T') \subseteq (G, T)$ to be this composition.

Clearly $\ker v = \begin{cases} e & \text{if } \mathcal{D}G \text{ simply conn.} \\ \mu_2 & \text{else} \end{cases}$

and any two v's differ by an automorphism of (SL_2, T_2) and one can check that such automorphisms are the inner autom. defined by an element of $\frac{N}{\mu_2}(k)$ $\quad N = N_{SL_2}(T_2)$. $\blacksquare$

Appendix: what happens if k is NOT separably closed?

Now assume k is any field and that G/k is a reductive group of semisimple rank 1 (i.e. $G_{\bar{k}}/R(G_{\bar{k}})$ is semisimple of rank 1).

If $T \subseteq G$ is a maximal torus, then T splits over k^{sep}

$\Rightarrow$ G is a k^{sep}/k form of one of the groups appearing in thm B.

if $\Gamma = Gal(k^{sep}/k)$, one can show that

$$\{k^{sep}/k \text{ forms of an alg group } \hat{G}/k^{sep}\}/_{\cong} \xleftrightarrow{1:1} H^1(\Gamma, Aut_{k^{sep}}(\hat{G}))$$

$\Gamma \curvearrowright Aut_{k^{sep}}(\hat{G})$ naturally as $\sigma \cdot \alpha = \sigma \circ \alpha \circ \sigma^{-1}$ $\sigma \in \Gamma$ $\alpha \in Aut_{k^{sep}}(\hat{G})$

If G/k is a k^{sep}/k form of $\hat{G}$ then there is an isomorphism

$$f: \hat{G} \xrightarrow{\cong} G_{k^{sep}} \rightsquigarrow \begin{bmatrix} \Gamma \longrightarrow Aut_{k^{sep}}(\hat{G}) \\ \sigma \longmapsto a_\sigma = f^{-1} \circ \sigma f \end{bmatrix} \in H^1(\Gamma, Aut_{k^{sep}}(\hat{G}))$$

If $\hat{G} = GL_{2/k^{sep}}$, one can show that:

- $Aut_{k^{sep}}(\hat{G}) \cong PGL_2(k^{sep}) \Rightarrow \{k^{sep}/k \text{ forms of } GL_2\}/_{\cong} \xleftrightarrow{1:1} H^1(\Gamma, PGL_2(k^{sep}))$
- $H^1(\Gamma, PGL_2(k^{sep})) \xleftrightarrow{1:1} \{\text{iso classes of quaternion algebras over } k\}$

$$[\tau \mapsto c_\tau = a^{-1} \circ \tau a] \longleftarrow [A]$$

$$a: M_2(k^{sep}) \xrightarrow{\cong} A \otimes_k k^{sep}$$

Hence $\{k^{sep}/k \text{ forms of } GL_2\}/_{\cong} \xleftrightarrow{1:1} \{\text{iso classes of quaternion alg.}/k\}$

$$[R \longmapsto G^A(R) := (A \otimes_k R)^\times] \longleftarrow [A]$$

Given a quaternion algebra A over k, one can define also:

$S^A: R \longmapsto (A \otimes R)^{Norm = 1}$

k^{sep}/k form of SL_2/k^{sep}

$P^A = G^A/Z(G^A)$

k^{sep}/k form of PGL_2/k^{sep}

Not surprisingly the classification then is as follows:

Thm B+ : Let T be a torus and A be a quaternion algebra $/k$. Then $T \times S^A$, $T \times P^A$, $T \times G^A$ are reductive groups of semisimple rank 1 over k. Every reductive group of semisimple rank 1 over k is isomorphic to one of these groups.