Triple product p-adic L-functions

A generalization and some applications

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Broad picture (after a talk by Andreatta)

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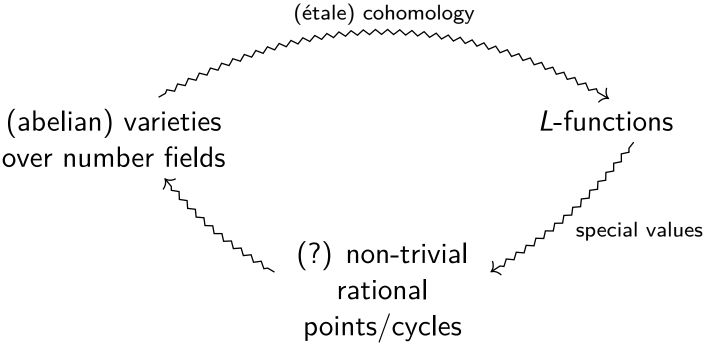

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- $E/\mathbb{Q}$ an elliptic curve;
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Conjecture (Galois equivariant BSD conjecture)

The function $L(E,\rho,s)$ admits analytic continuation and satisfies a functional equation $s \leftrightarrow 2-s$. Moreover:

$$\mathrm{ord}_{s=1} L(E,\rho,s) = \dim_L \left(\mathrm{Hom}_{L[G_{\mathbb{Q}}]}(V_\rho, E(H)\otimes L) \right).$$

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When the global picture is poorly understood, one can try to move to the local setting and to implement p-adic methods.

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STEP 1: construct a p-adic L-function via p-adic interpolation of (the algebraic part of) special values of classical L-functions.

Key words: congruences, p-adic measures, interpolation range/region.

STEP 2: approach arithmetically meaningful p-adic L-values via p-adic limit formulas and relate them to (local/hopefully global) points/cycles.

Key words: explicit reciprocity law, p-adic derivatives

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- We let $K/\mathbb{Q}$ be a quadratic imaginary field where p is inert and we consider two Galois characters η_1, η_2 of K of conductor $cp^r\mathcal{O}_K$ with $c \in \mathbb{Z}$, $(c, p) = 1$ and $r \geq 1$.

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- Minor technical assumptions.

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(ii) $\rho_1 = \rho_g$, $\rho_2 = \rho_h$, where g (resp. h) is the theta series attached to η_1 (resp. η_2). The newforms g and h have weight 1, level divisible by p^{2r} and infinite p-slope (i.e. $a_p(g) = 0 = a_p(h)$).

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(iii) We can identify

$$L(E, \rho, s) = L(f_E \times g \times h, s)$$

- $f_E \in S_2(\Gamma_0(N_E))$ newform attached to E via modularity.
- $L(f_E \times g \times h, s)$ Garrett-Rankin triple product L-function (for which analytic continuation and functional equation are known!).

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(iv) The decomposition in (i) yields a factorization

$$L(f_E \times g \times h, s) = L(f_E/K, \varphi, s) \cdot L(f_E/K, \psi, s) \qquad \varphi := \eta_1\eta_2,\ \psi := \eta_1\eta_2^{\sigma}.$$

Families of modular forms I

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We associate to f_E the unique *Hida family* $\boldsymbol{f}$ passing through f_E, i.e.

$$\boldsymbol{f} = \sum_{n\geq 1} a_n(\boldsymbol{k})q^n, \quad a_n(\boldsymbol{k}) \in \Lambda_{\boldsymbol{f}}$$

where $\Lambda_{\boldsymbol{f}}$ is a suitable Iwasawa algebra (in this case $\Lambda_{\boldsymbol{f}} \cong \mathbb{Z}_p[[T]]$) and one thinks about the coefficients $a_n(\boldsymbol{k})$ as p-adic analytic functions of the weight variable $\boldsymbol{k}$.

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The formal q-expansion $\boldsymbol{f}$ satisfies the following interpolation property:

(i) for all $k \geq 2$,

$$\boldsymbol{f}(k) := \sum_{n\geq 1} a_n(\boldsymbol{k})|_{\boldsymbol{k}=k}\, q^n$$

is the q-expansion at the cusp ∞ of a p-ordinary modular form of weight k and level N_E;

(ii) $\boldsymbol{f}(2) = f_E$.

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One can similarly associate to g (resp. h) a p-adic family of modular forms $\boldsymbol{g}$ (resp. $\boldsymbol{h}$) passing through g (resp. h). The families $\boldsymbol{g}$ and $\boldsymbol{h}$ essentially come from a p-adic deformation of the characters η_1 and η_2.

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Remark

(i) There is no good general theory for families of ∞ p-slope.

(ii) The corresponding Iwasawa algebras $\Lambda_{\boldsymbol{g}}$ and $\Lambda_{\boldsymbol{h}}$ are *bigger* than $\Lambda_{\boldsymbol{f}}$. More precisely, they are abstractly isomorphic to a ring of the form $\mathcal{O}_F[[X, Y]]$, with $F/\mathbb{Q}_p$ a large enough finite extension. The two variables morally come from the fact that the units $\mathcal{O}_{K,p}^{\times}$ of the p-adic completion of $\mathcal{O}_K$ are a rank two $\mathbb{Z}_p$-module (up to torsion), since p is inert in K.

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Our aim is to interpolate p-adically (square roots of) the special values

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in the so-called $\boldsymbol{f}$*-unbalanced region*, i.e. for $k \geq l + m$ and $l, m \in \mathbb{Z}_{\geq 1}$.

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Theorem (M., in progress)

There exists an element $\mathscr{L}_p^{\boldsymbol{f}}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \in \Lambda_{\boldsymbol{f}} \hat{\otimes}_{\mathbb{Z}_p} \Lambda_{\boldsymbol{g}} \hat{\otimes}_{\mathbb{Z}_p} \Lambda_{\boldsymbol{h}}$ such that, for all $\boldsymbol{f}$-unbalanced triples (k, l, m), it holds

$$\left(\mathscr{L}_p^{\boldsymbol{f}}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})(k, l, m)\right)^2 = \mathscr{E}_p(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})(k, l, m) \cdot L^{\mathrm{alg}}(\boldsymbol{f}(k) \times \boldsymbol{g}(l) \times \boldsymbol{h}(m), c_{k,l,m}),$$

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The main idea is to adapt the constructions of Darmon-Rotger and Hsieh for the case in which also $\boldsymbol{g}$ and $\boldsymbol{h}$ are Hida families, relying on previous works of Hida and on Ichino's formula.

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The factorization

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suggests a factorization of the form

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- $(\diamond)$ denotes an explicit factor never vanishing for $k = 2$.
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Theorem (M., in progress)

The above factorization holds (in a precise sense).

The idea of the proof is to compare the interpolation formulas for both sides.

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Corollary (factorization + previous works of Bertolini-Darmon)

If, moreover, $p\mathcal{O}_K$ divides the conductor of ψ and (as one expects in most cases) $L(f_E/K, \psi, 1) \neq 0$, then one can characterise the fact that P_φ is of infinite order in terms the non-vanishing of certain p-adic partial derivatives of $\mathscr{L}_p^{\mathbf{f}}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ at $(2, 1, 1)$.

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Why do we need to pass to derivatives?

(i) With the above hypothesis, the Euler factor $\mathscr{E}_p(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ vanishes at $(2,1,1)$.

(ii) In our setting $L(f_E/K,\varphi,s)$ has sign -1 (due to the *Heegner hypothesis*), hence $L(f_E/K,\varphi,1) = 0$.

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Following works of Darmon-Rotger and Bertolini-Seveso-Venerucci, one expects a geometric interpretation/construction of $\mathscr{L}_p^{\boldsymbol{f}}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ in terms of diagonal cycles/classes on a product of three modular curves, in the so-called *geometric balanced region*, i.e. for $k, l, m \in \mathbb{Z}_{\geq 2}$ such that they can be the sizes of the edges of a triangle.

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The nice p-adic variation of such classes should allow to obtain a class $\kappa_{2,1,1}$ as a limit of geometric classes (note that $(2, 1, 1)$ is NOT in the balanced region) and one expects to relate such a class to the behaviour of $\mathscr{L}_p^{\boldsymbol{f}}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ at $(2, 1, 1)$.

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Main difficulty: one has to work with modular curves whose reduction modulo p is not smooth, so that the cohomological machinery becomes more complicated.