Broccoli Curves & Broccoli Invariants and the relation to Welschinger invariants.

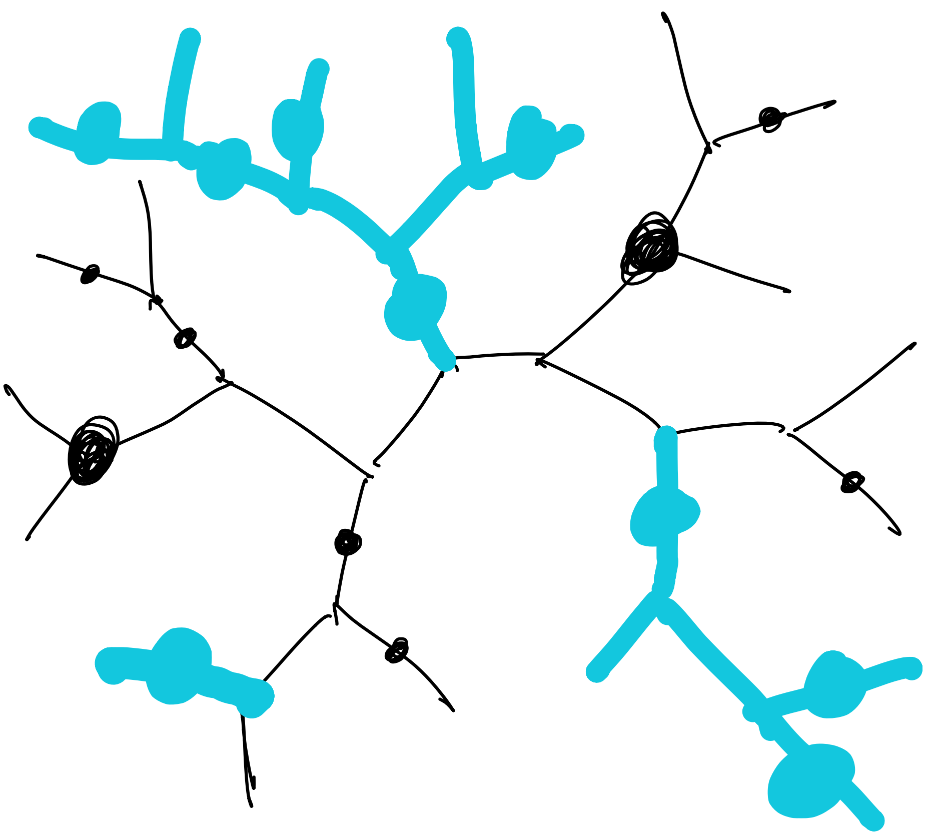
We've seen that one can compute Gromov-Witten invariants via tropical geometry.

complex curves passing through a given set of points

= # tropical curves passing through a set of points counted w/ multiplicity

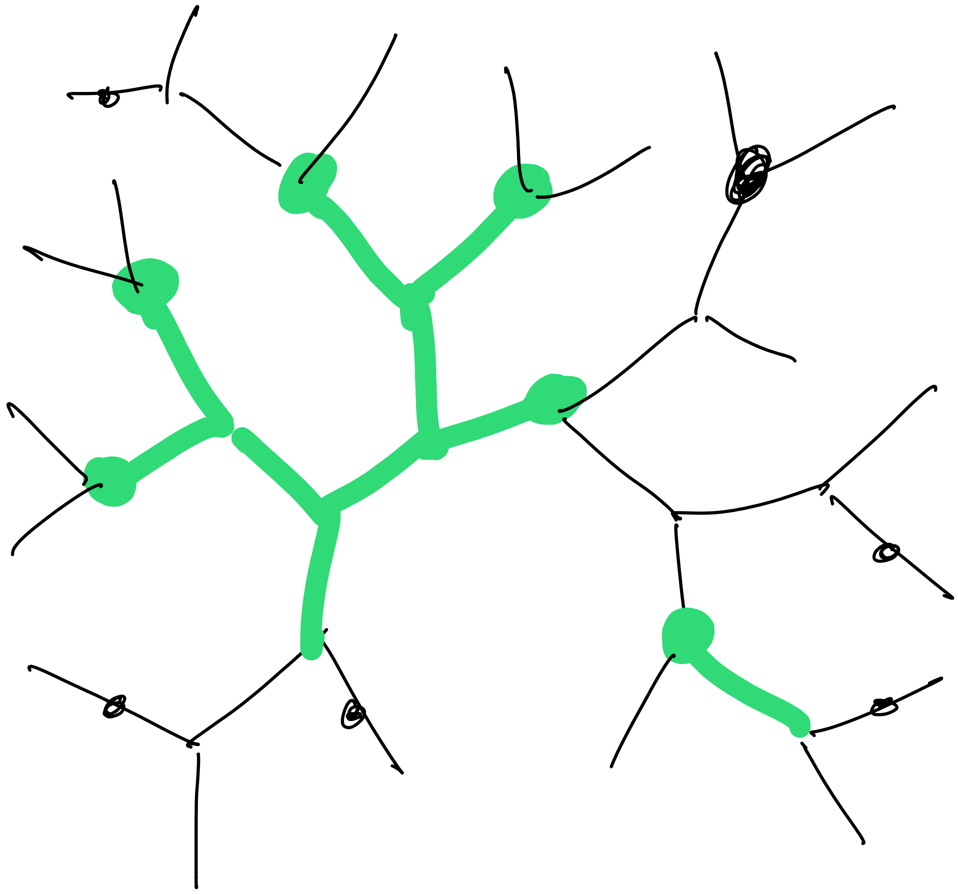
Today: Gathmann, Markwig, Schroeter
Broccoli Curves and tropical invariance of Welschinger Numbers

Example Welschinger Curve

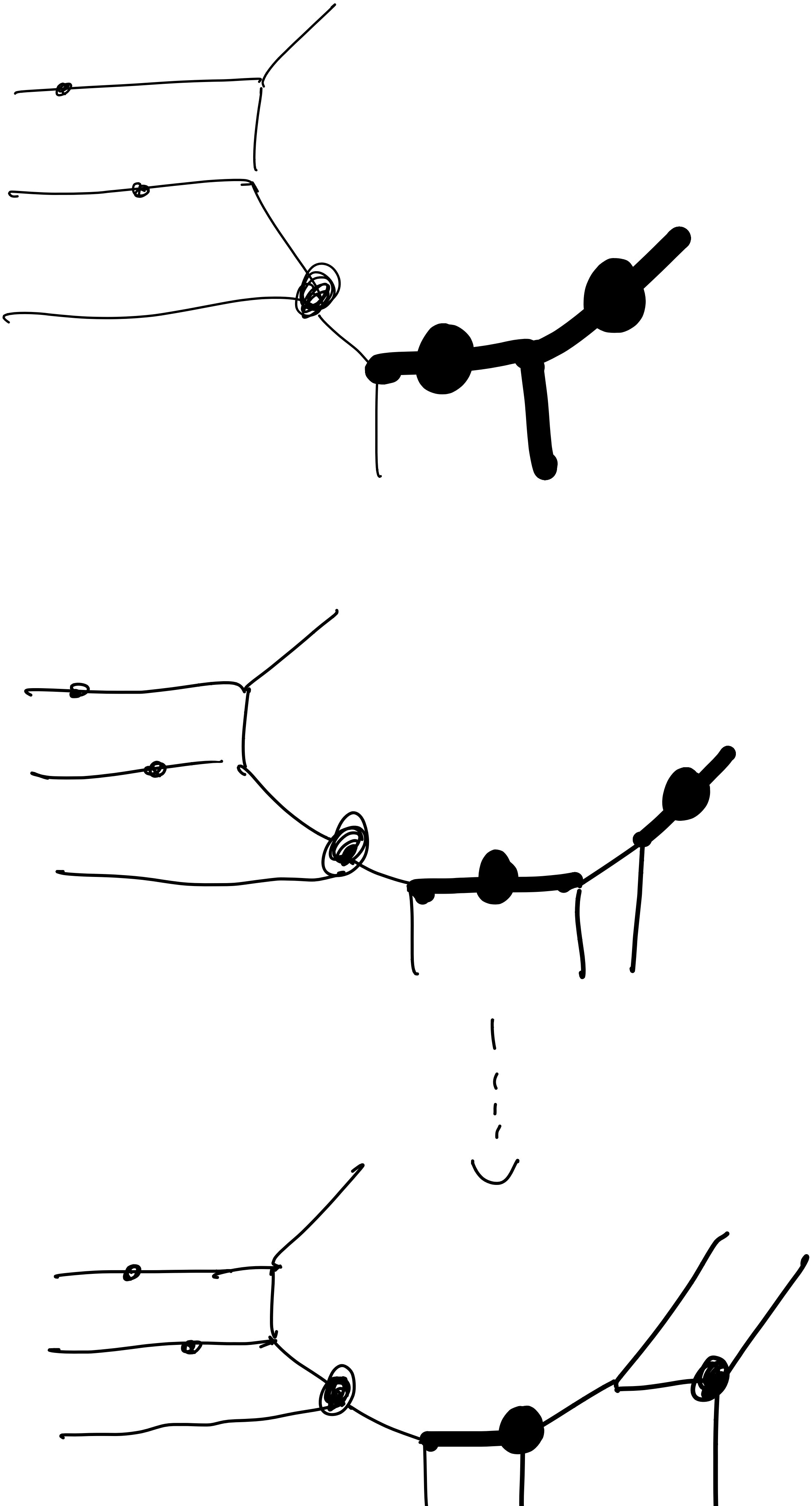


Welschinger invariant is not locally i.v.

Broccoli Curve



Broccoli invariant is locally invariant in the moduli space!



Definition (Marked curves)

An (r,s)-marked curve consists of

- a connected rational metric graph Γ
- labelings $x_1, \dots, x_{r+s}$ of edges Γ
- labelings $y_1 \to y_n$ of edges in Γ
- $h: \Gamma \longrightarrow \mathbb{R}^2$

s.t.
- every vertex in Γ has valence ≥ 3
- h is continuous and

  on each edge it is integer affine

  linear: $h(t) = \underset{\mathbb{Q}^2}{\underset{\cap}{a}} + t \cdot \underset{\mathbb{Z}^2}{\underset{\cap}{v}}$

$v(E,V)$ is the direction of E starting at V

at an end E you can just write $v(E)$

If $v(E)=0$, E is called "contracted"

- balancing condition at V

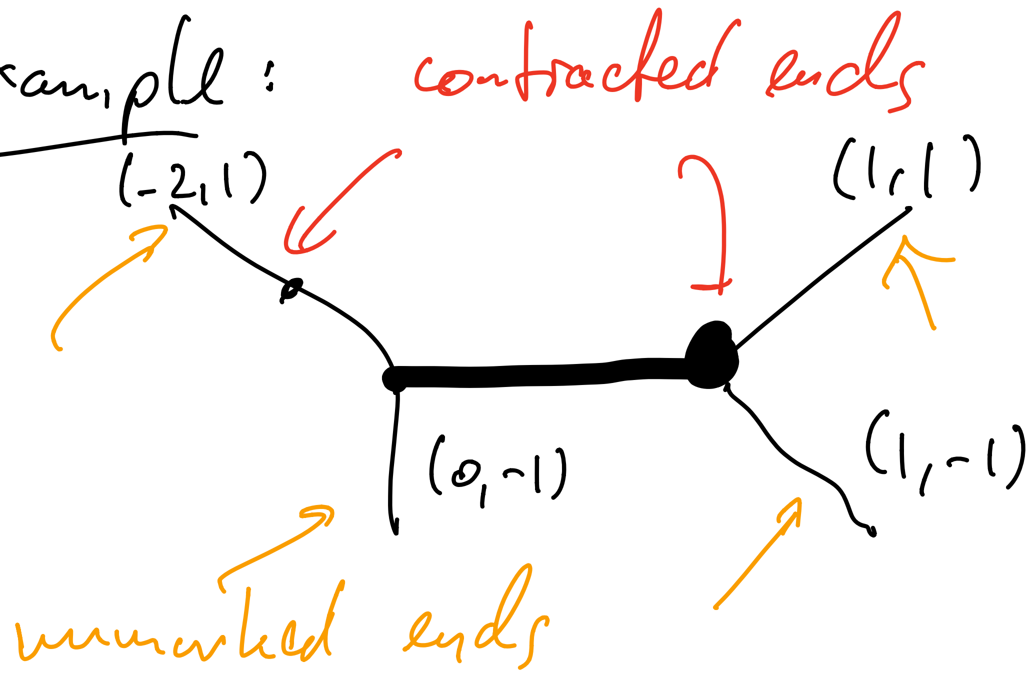
$$\sum_{V \in \partial E} v(E,V) = 0$$

- $x_1, \dots, x_{r+s}$ is a labeling of contracted ends

$y_1 \dots y_n$ is a labeling of the

unmarked ends

<u>Example:</u>



The degree of such a curve is the multiset of the directions of unmarked ends

Here: $\Delta = \{(-2,1), (0,-1), (1,-1), (1,1)\}$

The space of (r,s)-marked curves with degree Δ is $M_{r,s}(\Delta)$

<u>Def.:</u> The combinatorial type $\alpha(C)$ of

$$C = (\Gamma, x_1, \ldots, x_{r+s}, y_1, \ldots, y_n, h) \in \mathcal{M}_{0,s}(\Delta)$$

is the data

non-metric graph Γ

w/ labeling $x_1, \ldots, x_{r+s}, y_1, \ldots, y_n$

and directions of all edges

For a given combinatorial type α

you have $\mathcal{M}^{(\alpha)}_{r,s}(\Delta) \subset \mathcal{M}_{r,s}(\Delta)$

The dimension of a combinatorial type α is $\#(\text{bounded edges in } \Gamma) + 2$

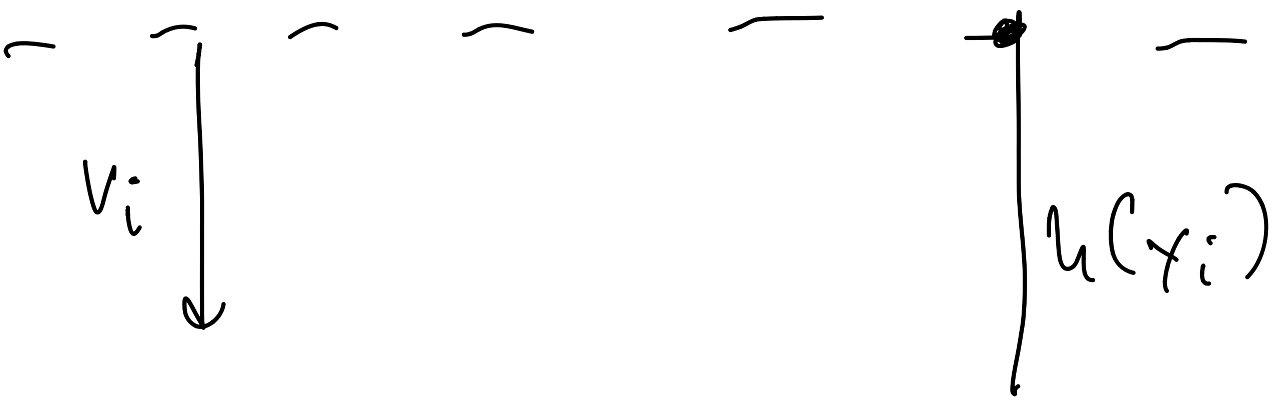
<u>Definition</u> $\Delta = (v_1, \ldots, v_n)$ a collection in $\mathbb{Z}^2 \setminus 0$ and $F \subset \{1, \ldots, n\}$

evaluation map ev_F:

$$\begin{array}{c} \mathbb{R}^{2(r+s)+|F|} \\ \cong \end{array}$$

$$M_{r,s}(\Delta) \longrightarrow (\mathbb{R}^2)^{r+s} \times \prod_{i \in F} \mathbb{R}^2 / \langle v_i \rangle$$

$$(\Gamma, x_1, \ldots, x_{r+s}, y_1, \ldots, y_n, h) \longmapsto (h(x_1), \ldots, h(x_{r+s}), (h(y_i) : i \in F))$$



- $G(\Delta, F) \subset S_n$ s.t. $\sigma(i) = i$ for $i \in F$,
  $v_{\sigma(i)} = v_i$ for all i

Configurations $P \in (\mathbb{R}^2)^{r+s} \times \prod_{i \in F} \mathbb{R}^2/\langle v_i \rangle$

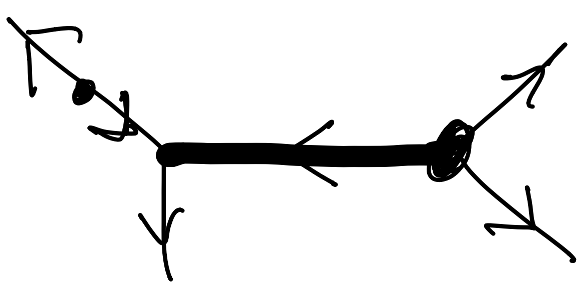
are called "collections of conditions".

Look at $ev_F(\mathcal{M}^{(\alpha)}_{r,s}(\Delta))$

$$\bigcup_{\alpha} ev_F(\mathcal{M}^{(\alpha)}_{r,s}(\Delta)) = \text{conditions in } \underline{\underline{\text{special position}}}$$

s.t. $\dim ev_F(\mathcal{M}^{(\alpha)}_{r,s}(\Delta)) < 2(r+s)+|F|$

Oriented marked curve:



Def: Consider an end fixed if it is oriented inward.

$$\begin{tikzcd} \mathcal{M}^{or}_{(r,s)}(\Delta, \mathcal{F}) \arrow[d, "fd"'] \\ \mathcal{M}_{(r,s)}(\Delta) \end{tikzcd}$$

$\rightsquigarrow$ $\mathcal{M}^{or}_{(r,s)}(\Delta, \mathcal{F})$ moduli space of oriented (r,s)-marked curves with fixed ends $\mathcal{F}$

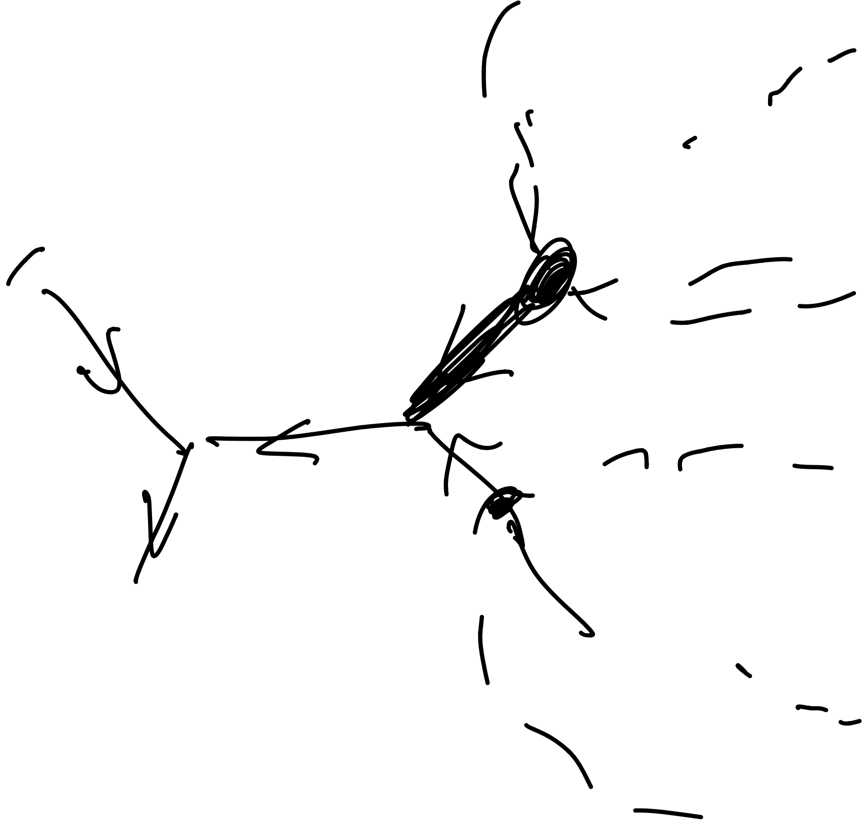
Lemma Let $M \subset \mathcal{M}_{(r,s)}(\Delta)$ a subcomplex and $\mathcal{P}$ a collection of conditions in general position (for M). Assume there is a curve $C \in ev_{\mathcal{F}}^{-1}(\mathcal{P})$

(a) Each conn. comp. of $\Gamma \setminus (x_1 \cup \dots \cup x_{r+s})$

has at least one unmarked end x_i
w/ $i \notin F$.

(b) if $\dim \alpha(C) = 2(r+s) + |F|$
and every vertex of C that is not adjacent to a marking is 3-valent,
then the unmarked ends in (a) are unique.

Lemma $\Rightarrow$ each curve satisfying these conditions carries an orientation:
every noncontracted edge points towards the unique unmarked non-fixed end.



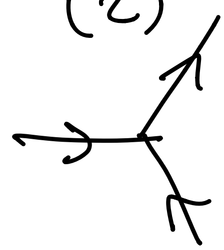
This justifies working w/ oriented curves instead of unoriented curves for counting purposes.

Def (Vertex types)

(1)

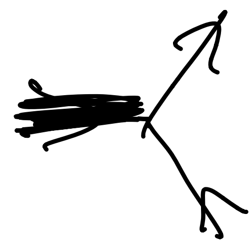
1

(2)



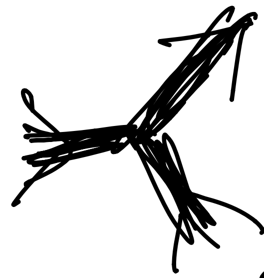
i^{a-1}

(3)

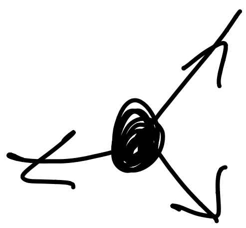


$a\, i^{a-1}$

(4)

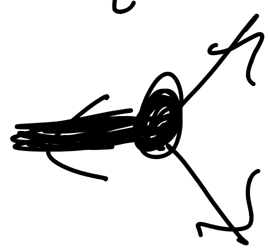


$a \cdot i^{a-1}$



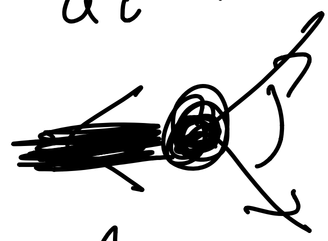
(5)

$a \cdot i^{a-1}$



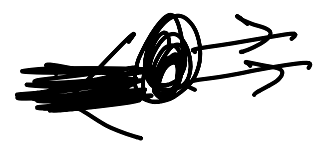
(6)

i^{a-1}



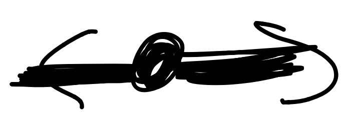
(6a)

i^{a-1}



(6b)

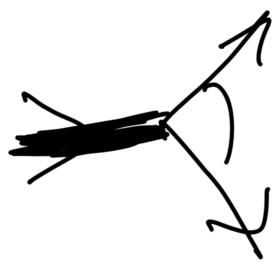
i^{-1}



(7)

1

(8) $-a$



(9) i^{a-1}

Each gets a multiplicity

where $a = \left|\det\begin{pmatrix}\text{two adj.}\\ \text{direction}\end{pmatrix}\right|$

Define a multiplicity for a curve C

containing only these types of vertices

$$m_C := \prod_{k=1}^{n} i^{w(y_k)-1} \prod_V m_V$$

weight of the unmarked end y_k

multiplicity of V as above

Def.: (Broccoli curve)

$\Delta = (v_1, \dots, v_n)$ a fixed degree, $F \subset \{1, \dots, n\}$

(a) an oriented curve $C \in M^{or}_{(r,s)}(\Delta, F)$ is a Broccoli curve if it only contains vertices of type (1) to (6).

(b) for $C = (\Gamma, x_1, \dots, x_{r+s}, y_1, \dots, y_n) \in M_{(r,s)}(\Delta)$ consider Γ_{even} = subgraph of all even edges

A stem in Γ_{even} is either a 1-valent vertex

or $x_i \subset \Gamma_{even}$ w/ $i \notin F$

C is a Broccoli curve if

(i) all complex markings are adjacent to 4-valent vertices

(ii) every connected component of Γ_{even} has exactly one stem.

<u>Proposition</u> if $r+2s+|F|=|\Delta|-1$ and $P \in \mathbb{R}^{2(r+s)+|F|}$ is in general position, then

$$ft: M^{or}_{(r,s)}(\Delta, F) \longrightarrow M_{(r,s)}(\Delta)$$

gives a bijection on Broccoli curves.

<u>Definition</u> (Broccoli invariants)

$\Delta = (v_1, \dots, v_n)$, $F \subset \{1, \dots, n\}$

$$r + 2s + |F| = |\Delta| - 1$$

$P \in \mathbb{R}^{2(r+s)+|F|}$ in general position

Then

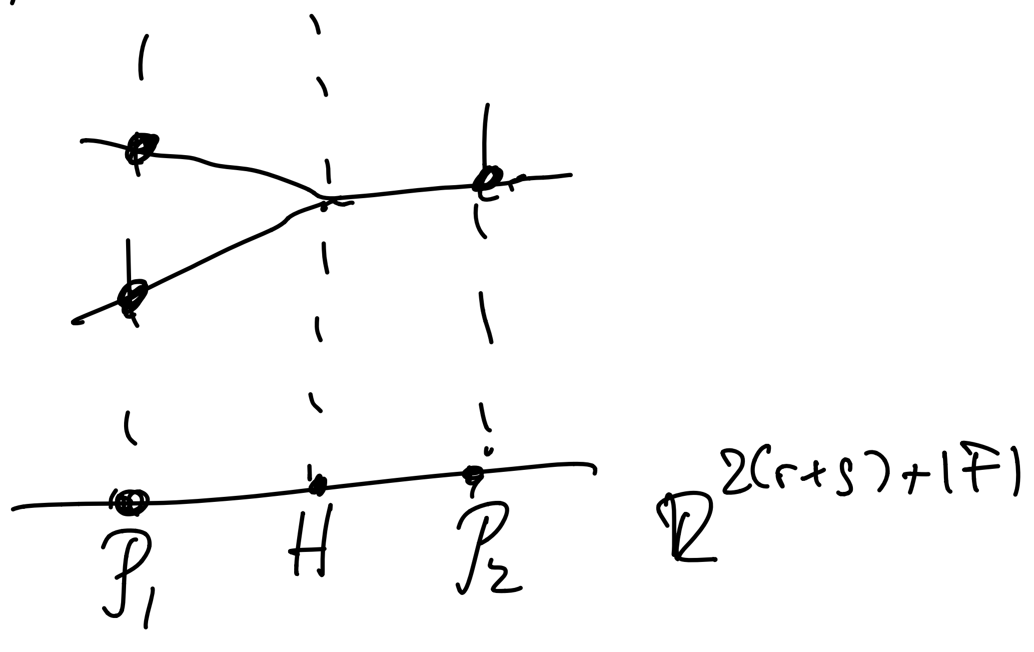
$$N^{B}_{(r,s)}(\Delta, F, P) := \frac{1}{|G(\Delta, F)|} \sum_{C \in ev_F^{-1}(P)} m_C$$

<u>Theorem</u> The Broccoli invariant is independent of the choice of P.

Proof Sketch:

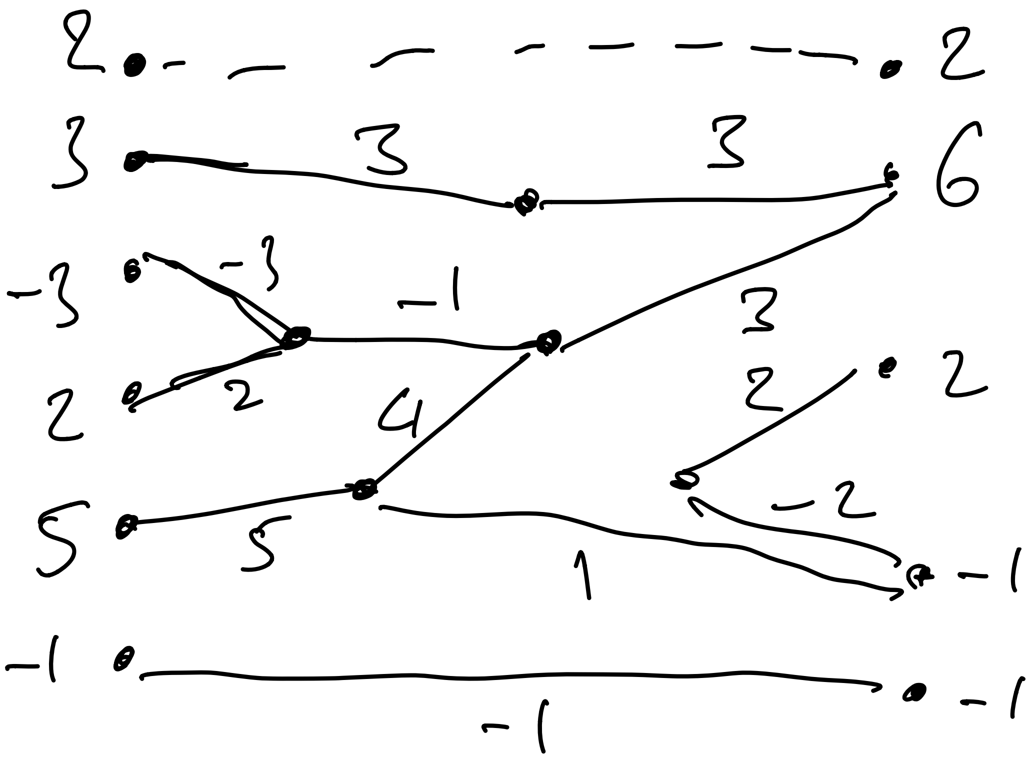
$N^{B}_{(r,s)}$ is locally constant on open cells

then you need to consider possible degenerations



To do this you list all possible degenerations of a single bdd edge for each allowed vertex type.

There's 18 cases.



Broccoli curves

Welschinger