# Recap: Caporaso-Harris recursion formula

finite sequences of non-negative integers

$\alpha = (\alpha_1, \alpha_2, \dots)$ assigned pts

$\beta = (\beta_1, \beta_2, \dots)$ unassigned pts

$e_k = (0, \dots, 0, \underset{\uparrow}{1})$

kt position

$|\alpha| := \sum_i \alpha_i$ without multiplicity

$I\alpha := \sum_i i \cdot \alpha_i$ with multiplicity

$\alpha' \leq \alpha :\Leftrightarrow \alpha_i' \leq \alpha_i \quad \forall i$

$$\binom{\alpha}{\alpha'} := \prod_i \binom{\alpha_i}{\alpha_i'}$$

$$I^\alpha := \prod_i i^{\alpha_i}$$ product of multiplicities

$D \subset \mathbb{P}^2$ a fixed line and $I\alpha + I\beta = d$

$N^{d,\delta}(\alpha, \beta) :=$ # complex reduced plane curves of degree $d$ and genus $\binom{d-1}{2} - \delta$ that

$\shortparallel$
relative Severi degree

- intersect $D$ in $\alpha_i$ <u>fixed</u> pts with multiplicity $i$ $\forall i \geq 1$
- intersect $D$ in $\beta_i$ <u>arbitrary</u> pts with mult $i$ $\forall i \geq 1$
- pass through $\binom{d+1}{2} - \delta + |\beta|$

## <u>Caporaso - Harris's recursion formula</u>

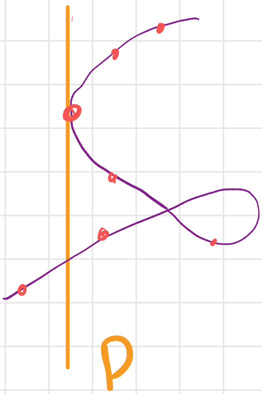
assign one more pt

$$N^{d,\delta}(\alpha,\beta) = \sum_{k : \beta_k > 0} k \cdot N^{d,\delta}(\alpha + e_k, \beta - e_k)$$

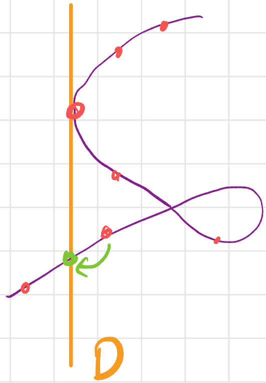
$$+ \sum_{\substack{\alpha' \leq \alpha \\ \beta' \geq \beta \\ \delta - \delta' + |\beta' - \beta| = d-1 \\ \delta' \leq \delta}} I^{\beta' - \beta} \binom{\alpha}{\alpha'} \binom{\beta'}{\beta} N^{d-1,\delta'}(\alpha', \beta')$$

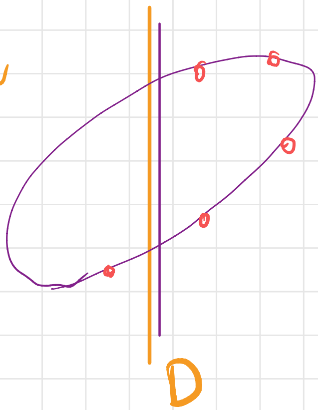
curve degenerates into $C'$ of degree $d-1$ and $D$

Ex: $d=3,\ \delta=1,\ \alpha=(0,1),\ \beta=(1)$




move one point to D:




$$N^{3,1}((0,1),(1))$$
$$= N^{3,1}((1,1),0)$$
$$+ \binom{2}{1} N^{2,0}((0),(2))$$

Today we will see the Caporaso-Harris recursion formula for

- $S = \mathbb{P}^2$
- $S = \Sigma_m$ Hirzebruch surface
- $S = \mathbb{P}(1,1,m)$ weighted proj space

and prove it for tropical curves with quantum multiplicities [Gathmann-Markwig, Bloch-Göttsche]

and we will do some computations with floor diagrams.

$\rightsquigarrow$ node polynomials

Will replace • $S = \mathbb{P}^2$ by $S = \Sigma_m$

or $S = \mathbb{P}(1,1,m)$

• and $d$ $(= \mathcal{O}_{\mathbb{P}^2}(d)$
$= d \cdot H$ $)$

class of a line in $\mathbb{P}^2$

by a line bundle on $S$ / class in $H^2(S, \mathbb{Z})$

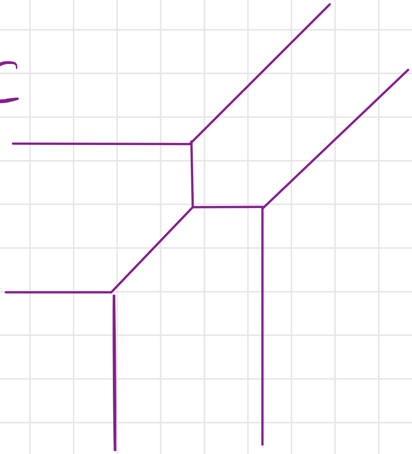
and count curves $\in |L| = \mathbb{P}\, H^0(S, L)$

$1^{st}$ case $S = \mathbb{P}^2$, $L = d \cdot H$

$H$ = class of a line in $\mathbb{P}^2$

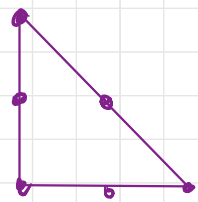
$\Delta = \Delta_d = \mathrm{conv}\,((0,0), (0,d), (d,0))$

Ex: $C$



degree $(C)$

$=$



$= \Delta_2$

2nd case: $S = \Sigma_m = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-m))$

$H$ = class of a section with $H^2 = m$

$F$ = class of a fiber

$L = d \cdot H + cF$

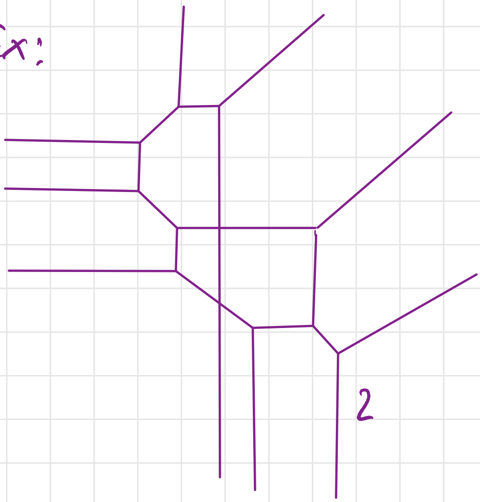
$E^2 = -m$

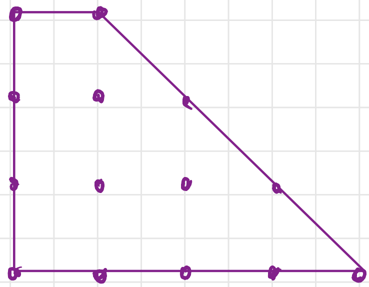
$\downarrow$

$E, F$

$H = E + mF$

Ex:



$d = 3, \quad c = 1, \quad m = 1$

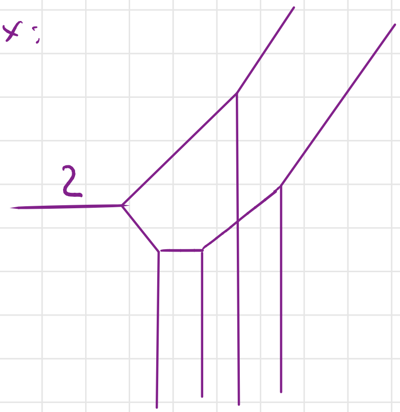
$$\Delta = \mathrm{conv}((0,0), (0,d), (c,d), (c+md, 0))$$

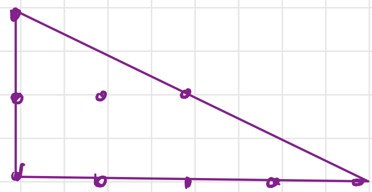
3rd case: $S = \mathbb{P}(1,1,m)$

$H$ = class of a line $H^2 = m$

$L = d \cdot H$

Ex:



$d = m = 2$

$\Delta = \mathrm{conv}\{(0,0), (0,d), (dm, 0)\}$

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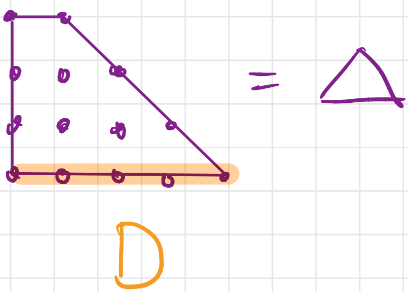
$\Delta$ = lattice polygon

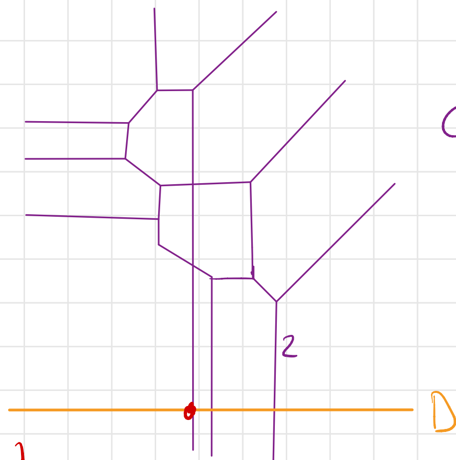
$h : C \to \mathbb{R}^2$ parametrized tropical curve with degree $\Delta$

$D$ = an edge of $\Delta$ (D replaces the fixed line in the classical set up)

Def: • tropical boundary divisor of $D$ $(=: \mathcal{D})$ is a classical line in $\mathbb{R}^2$ parallel to $D$ and sufficiently far away dual to $D$ (all intersections with $C$ are orthogonal)

- $C$ is tangent to $D$ of order $\beta$ if $\beta$ "counts" the unbounded edges orthogonal to $D$




$\alpha = (1)$
$\beta = (1,1)$ Here $\beta = (2,1)$

Let $\alpha, \beta$ st $I\alpha + I\beta = \text{length}(D)$

$\Pi :=$ $n = \#(\Delta \cap \mathbb{Z}^2) - 1 - \delta$
$\quad \| \qquad\qquad - I\alpha - I\beta + |\alpha| + |\beta|$
$\{p_1, \ldots, p_n\}$

tropically generic pts with precisely $|\alpha|$ pts on $D$

**Def:** $C$ is $(\alpha, \beta)$-tangent to $D$ if $\alpha_i + \beta_i$ unbounded edges of $C$ are orthogonal to $D$ and have mult $i$ and $\alpha_i$ of these pass through $D \cap \Pi$

Recall: mult $C$ = $\prod_{\substack{3\text{-valent} \\ \text{vertices}}} 2\,\mathrm{Area}(\Delta_v)$

triangle dual to $v$ in dual subdivision

refined multiplicity

$$\mathrm{mult}(C, y) = \prod_{\substack{3\text{-valent} \\ \text{vertices}}} [2\,\mathrm{Area}(\Delta_v)]_y$$

where $[i]_y = \dfrac{y^{i/2} - y^{-i/2}}{y^{1/2} - y^{-1/2}}$

refined relative multiplicity

$$\mathrm{mult}_{\alpha,\beta}(C,y) = \frac{1}{\prod_{i\geq 1} ([i]_y)^{\alpha_i}} \cdot \mathrm{mult}(C,y)$$

refined relative Severi degree $N^{\Delta,\delta}(\alpha,\beta)(y)$

:= # $\delta$-nodal tropical curves of degree $\Delta$ passing through $\Pi$ that are $(\alpha,\beta)$-tangent to $D$ counted with $\mathrm{mult}_{\alpha,\beta}(C,y)$

Rmk: This depends on the choice of $D$.

<u>Thm 7.3 in Bloch-Göttsche:</u>

$N^{\Delta,\delta}(\alpha,\beta)(y)$ is independent of the choice of $\Pi$ (as long as generic.)

<u>Thm 7.5 Bloch-Göttsche</u> /

<u>Thm 4.3 Gathmann-Markwig</u>

Let $\Delta$ be as in one of the three cases above

For $S = X(\Delta) \leftarrow \mathbb{P}^2, \Sigma_m, \mathbb{P}(1,1,m)$

$L = L(\Delta)$

$$N^{(S,L),\delta}(\alpha,\beta)(y) = \sum_{k:\,\beta_k>0} [k]_y \cdot N^{(S,L),\delta}(\alpha+e_k,\beta-e_k)(y)$$

$$+ \sum_{\substack{\beta'\geq\beta \\ \alpha'\leq\alpha}} \prod_i [i]_y^{\beta_i'-\beta_i} \binom{\alpha}{\alpha'}\binom{\beta'}{\beta} N^{(S,L-H),\delta'}(\alpha',\beta')(y)$$

$\delta' = \delta - H(L-H) + |\beta'-\beta|$

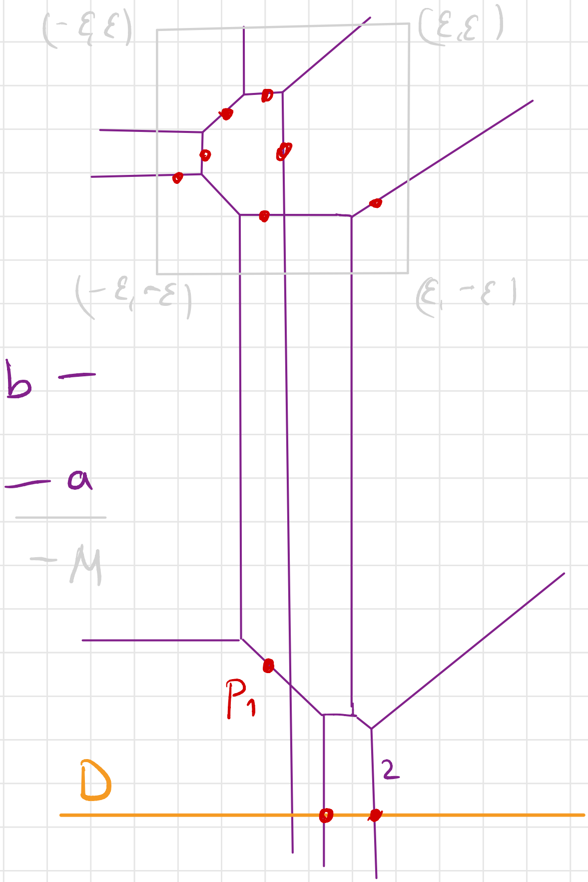
$I\alpha' + I\beta' = H(L-H)$

$\varepsilon > 0$, $M >> 0$, $\Pi = \{p_1, \dots, p_n\}$ st

1) $x$-coord of $p_i \in (-\varepsilon, \varepsilon)$

2) $p_1$ is not on $D$ and its $y$-coord is $< -M$

3) all $p_i \neq p_1$ not lying on $D$ have $y$-coord in $(-\varepsilon, \varepsilon)$




Lemma (Gathmann-Markwig)

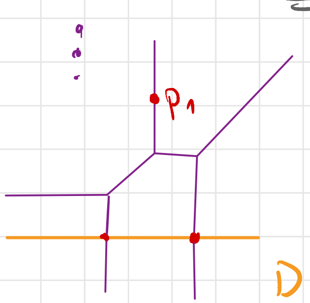
1) all vertices have x-coord $\in (-\varepsilon, \varepsilon)$

2) $\exists\ -M < a < b < -\varepsilon$ st in $\mathbb{R} \times [a,b]$ all edges of C are vertical

Pf of Recursion formula:

Case 1: $p_1$ lies on a vertical edge of weigth k

$\Rightarrow$ all edges with y-coord $\leq -\varepsilon$ are vertical:

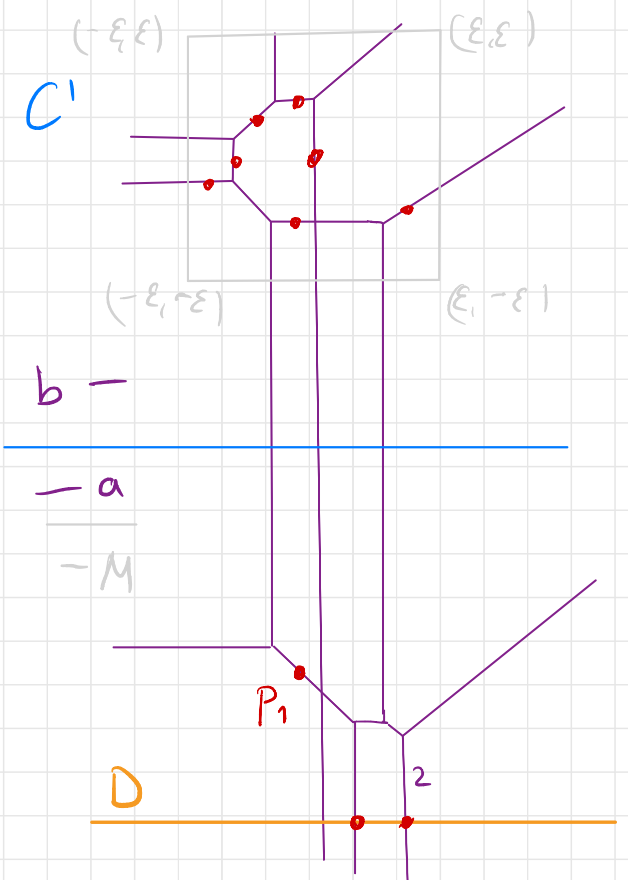



By Mikhalkin Lemma 4.20 one cannot go from one unbounded edge to another without passing a marking

$\Rightarrow$ can move $p_1$ to $D$

$$\mathrm{mult}_{\alpha,\beta}(C,y) = [k]_y \, \mathrm{mult}_{\alpha+e_k,\beta-e_k}(C,y)$$

$$\rightsquigarrow \sum_{k:\beta_k>0} [k]_y \, N^{(S,L),s}(\alpha+e_k,\beta-e_k)(y)$$




Case 2: $p_1$ does not lie on a vertical unbounded edge:

Can divide $C$ into upper $C'$ and lower part

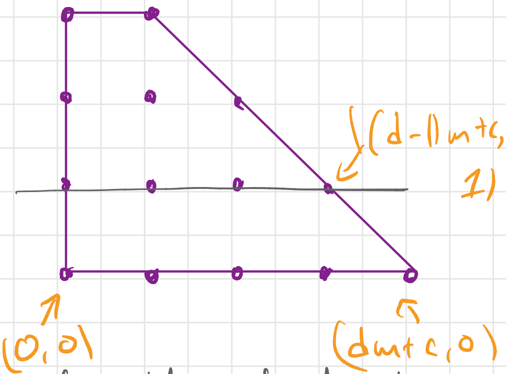
$C'$ is a tropical curve that is

$\binom{\alpha}{\alpha'} \leftarrow \alpha' \leq \alpha$ $\qquad$ $(\alpha', \beta')$ - tangent to $D$.

$\binom{\beta'}{\beta} \leftarrow \beta' \geq \beta$

$C'$ has degree $\Delta'$

obtained from removing the bottom strip of $\Delta$




To show $I\alpha' + I\beta' =$ length of bottom edge of $\Delta'$

$$\overset{?}{=} H(L-H)$$

and $\Delta' \leftrightarrow (S, L-H)$

$S = \mathbb{P}^2$: $\underset{\substack{\text{class}\\\text{of a}\\\text{line}}}{H}(\underset{d \cdot H}{L}-H) = d-1$

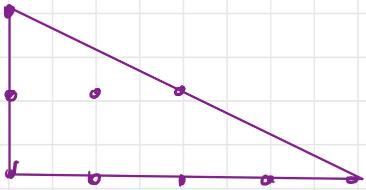
$$\Delta = \Delta_d$$

$$\Delta' = \Delta_{d-1}$$

$S = \Sigma_m$: $\underset{H^2=m}{H}(\underset{dH+cF}{L}-H) = md+c-m$

$$= (d-1)m+c$$

$S = \mathbb{P}(1,1,m)$: $\underset{H^2=m}{H}(\underset{d \cdot H}{L}-H) = dm-m = (d-1)m$



$d = m = 2$

$\delta - \delta'$ = # parallelograms in bottom strip of $\Delta$ in dual subdivision $\Delta_C$

= # unbounded edges of $C'$ that intersect $D$ and are also unbounded in $C$

= # unbounded edges of $C'$ that intersect $D$

length of lower edge of $\Delta'$

− # edges in $C'$ that become bounded in $C$

$$= H(L-H) - |\beta' - \beta|$$

Multiplicities:

$$\mathrm{mult}_{\alpha,\beta}(C,y) = \frac{1}{\prod_i ([i]_y)^{\alpha_i}}\, \mathrm{mult}(C,y)$$

$$= \frac{\prod_i ([i]_y)^{\cancel{\alpha_i} - \cancel{\alpha_i'} + \beta_i' - \beta_i}}{\prod_i ([i]_y)^{\cancel{\alpha_i}\ \alpha_i'}} \operatorname{mult}(C', y)$$

$$= \prod_i ([i]_y)^{\beta_i' - \beta_i} \operatorname{mult}_{\alpha', \beta'}(C, y)$$

$\square$

Rmk: Gathmann–Markwig give another proof ($S = \mathbb{P}^2$, $S = \mathbb{P}^1 \times \mathbb{P}^1$) using a relative version of $\lambda$-increasing paths.

__Floor diagrams__ (Block – relative node polynomials for plane curves)

Now $S = \mathbb{P}^2$

__Prop 7.7__ (Block–Göttsche)

$$N^{d,\delta}(\alpha, \beta)(y) = \sum_{D \in FD(d,\delta)} \operatorname{mult}_\beta(D, y)\, \nu_{\alpha,\beta}(D)$$

Need to define

- $FD(d,\delta)$
- $\mathrm{mult}_\beta(D,y)$
- $\nu_{d,\beta}(D)$

**Def** floor diagrams = directed graphs on $\{1,\dots,d\}$ and weights $w(e) \in \mathbb{Z}_{>0}$

st 1) $i \to j \Rightarrow i < j$

2) $\mathrm{div}(j)$

$$:= \sum_{j \xrightarrow{e} k} w(e) - \sum_{i \xrightarrow{e} j} w(e) \leq 1$$

degree $d(D) := d$

genus $g(D) :=$ genus of the graph

cogenus $\delta(D) := \binom{d-1}{2} - g$ if $D$ is connected

Otherwise

$$\delta(D) = \sum \delta_j + \sum_{j<j'} d_j d_{j'}$$

$FD(d,\delta) = \{$ floor diagrams of degree $d$ and cogenus $\delta\}$

Ex:

$$\begin{tikzcd} \circ \arrow[r] & \circ \arrow[r, "2"] & \circ \arrow[r, bend left] \arrow[r, bend right] & \circ \end{tikzcd}$$

$d = 4$

$g = 1$

$\delta = 2$

| div | 1 | 1 | 0 | -2 |
|---|---|---|---|---|

$$\mathrm{mult}(D, y) := \prod_{\text{edges}} \left([w(e)]_y\right)^2$$

$$\mathrm{mult}_\beta(D, y) := \prod_{i \geq 1} \left([i]_y\right)^{\beta_i} \cdot \mathrm{mult}(D, y)$$

Def: $(\alpha, \beta)$-marking of a floor diagram $D$, $\quad I\alpha + I\beta = d$

Step 1: Fix $\{\alpha^i\}, \{\beta^i\}$ $\quad i = 1, \dots, d$

st $\quad$ 1) $\sum \alpha^i = \alpha \qquad \sum \beta^i = \beta$

$\qquad$ 2) $I\alpha^i + I\beta^i = 1 - \mathrm{div}(i) \quad \forall i$

$$\begin{tikzcd} \circ \arrow[r] & \circ \arrow[r, "2"] & \circ \arrow[r, bend left] \arrow[r, bend right] & \circ \end{tikzcd}$$

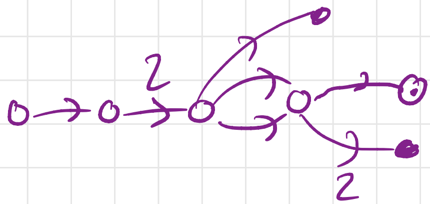
$\alpha = (1)$

$\beta = (1, 1)$

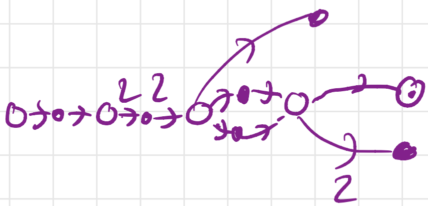
| | | | | |
|---|---|---|---|---|
| div | 1 | 1 | 0 | -2 |
| 1-div | 0 | 0 | 1 | 3 |
| $\alpha^i$ | 0 | 0 | 0 | (1) |
| $\beta^i$ | 0 | 0 | (1) | (0,1) |

Step 2: $i$ = vertex

$\forall j$ create $\beta_j^i$ and $\alpha_j^i$ new vertices and connect them to $i$ with an edge of weight $j$ directed away from $i$



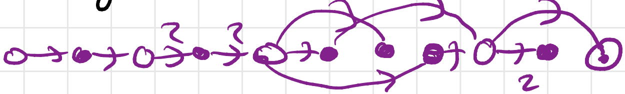

Step 3: Subdivide each edge in original graph




$=: \tilde{D}$

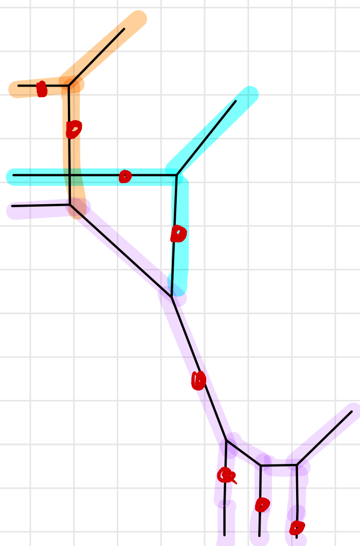
Step 4: Linearly order vertices of $\tilde{D}$ st $i \to j \Rightarrow i < j$ and st $\alpha$-vertices are largest

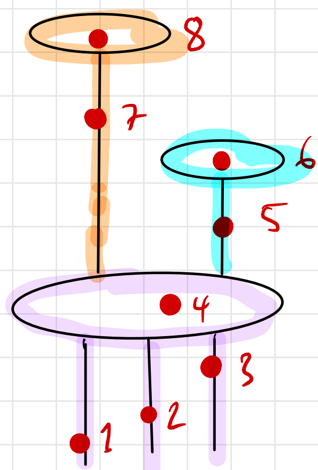
$v_{\alpha,\beta}(D) = 5$



$\nu_{(\alpha,\beta)}(D) := \#\{(\alpha,\beta)\text{-marking}\} \Big/ \left(\begin{array}{l}\text{weight preserving}\\ \text{auto that fixes } D\end{array}\right)$

Rmk: This looks different from the floor diagrams we have seen before.






New floor diagram

## Thm 7.8 (Block–Göttsche)

For $\delta \geq 1$ there is a polynomial $N_\delta(\alpha,\beta,y) \in \mathbb{Q}[y^{\pm 1}][\alpha_1,\dots,\alpha_\delta,\beta_1,\dots,\beta_\delta]$ st $\forall \alpha,\beta$ with $|\beta| > \delta$ we have

$$N^{d,\delta}(\alpha,\beta)(y) = \prod_i \left([i]_y\right)^{\beta_i} \frac{(|\beta|-\delta)!}{\beta_1!\,\beta_2!\cdots} N_\delta(\alpha,\beta,y)$$

Ex (non-refined):

- $N_0(\alpha,\beta)(y) = 1$
- $N_1(\alpha,\beta) = 3d^2|\beta| - 8d|\beta| + d\beta_1 + |\beta|\alpha_1 + |\beta|\beta_1 + 4|\beta| - \beta_1$