Tropical methods in enumerative geometry

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Let d be a positive integer and let δ be an integer such that

$$0 \leq \delta \leq \frac{(d-1)(d-2)}{2}.$$

Put $n = \frac{d(d+3)}{2} - \delta$.

Let us define the function

$$\mathcal{N}_d^{\delta}\colon \left(\mathbb{CP}^2\right)^n \longrightarrow \mathbb{N} \cup \{\infty\},$$

where $\mathcal{N}_d^{\delta}(\mathcal{P})$ is the number of nodal curves $C \subset \mathbb{CP}^2$

- of degree d,
- with exactly δ non-degenerate double points,
- containing the points set $\mathcal{P} \subset C$

Example $d = 1$

For $d = 1$, we have that $\delta = 0$ and $n = 2$.
If $\mathcal{P} = (p_1, p_2) \in (\mathbb{CP}^2)^2$, then

$$\mathcal{N}_d^\delta(\mathcal{P}) = \begin{cases} 1 & p_1 \neq p_2 \\ \infty & p_1 = p_2 \end{cases}$$

Example $d = 2$

For $d = 2$, we have that $\delta = 0$ and $n = 5$.
If $\mathcal{P} \in (\mathbb{CP}^2)^5$, then

$$\mathcal{N}_d^\delta(\mathcal{P}) = \begin{cases} \infty & p_i = p_j \\ 0 & p_i, p_j, p_k \text{ are colinear} \\ 1 & \textit{otherwise} \end{cases}$$

Conic affine case

Given any 5 different (any two) points in the plane, there is one conic passing through them.

$$Ax_1^2 + Bx_1y_1 + Cy_1^2 + Dx_1 + Ey_1 + F = 0$$

$$\mathbf{A} = \begin{pmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{pmatrix}; \qquad \mathbf{x} = \begin{pmatrix} A \\ B \\ C \\ D \\ E \\ F \end{pmatrix}; \qquad \mathbf{Ax} = \mathbf{0}.$$

Given any $\dfrac{d(d+3)}{2}$ pairwise different points in the plane, there is one curve of degree d passing through them.

Moreover, if the point are in a **generic configuration** or in **general position**, there is *exactly* one curve of degree d passing through them.

- pairwise different points
- no three colinear points
- no six points belonging to a conic
- no ten points belonging to a cubic
- ...

Definition

$$N^\delta(d) := \mathcal{N}_d^\delta(\mathcal{P})$$

where $\mathcal{P}$ is a generic configuration of points.

Example $\delta = 0$

$$N^0(d) = 1, \forall d \geq 1.$$

$\delta = 1$

The space of degree d plane curves $C \subset \mathbb{CP}^2$ is the projective space $\mathcal{C}_d := \mathbb{CP}^{\frac{d(d+3)}{2}}$.
The discriminant of the space of degree d curves is the set

$$\mathcal{D}_d := \{C \in \mathcal{C}_d \mid C \text{ is singular}\}.$$

i.e., if $f \in \mathbb{C}[x,y,z]_d^{\text{homo}}$ such that $C = \{f = 0\}$, then there exist a point $p \in \mathbb{CP}^2$ such that

$$\begin{cases} \partial f/\partial x(p) = 0 \\ \partial f/\partial y(p) = 0 \\ \partial f/\partial z(p) = 0 \end{cases}$$

$\delta = 1$

- The discriminant $\mathcal{D}_d$ is an algebraic subvariety of $\mathcal{C}_d$ of codimension 1.
- $\mathcal{D}_d$ has a stratification given by "how singular" a curve can be, where the open strata correspond to curves having exactly one singular point of nodal type.

This means that a generic point of $\mathcal{D}_d$ represents a nodal curve of degree d with $\delta = 1$!.

Now, if $L \cong \mathbb{CP}^1$ is the projective line given by the restriction of the $\frac{d(d-3)}{2} - 1$ points in $\mathcal{P}$, then we have that

$$\begin{aligned} N^{\delta=1}(d) &= \text{number of curves in the intersection of } L \text{ and } \mathcal{D}_d. \\ &= L \cdot \mathcal{D}_d \\ &= deg(\mathcal{D}_d) \\ &= 3(d-1)^2 \end{aligned}$$

given that $d \geq 3$.

$N^{\delta}(d)$ is also know as a Severi degree.

Is there a polynomial $N_\delta \in \mathbb{Q}[d]$ such that

$$N^\delta(d) = N_\delta(d) \text{ for } d >> 0?$$

- $\delta = 1, 2, 3$ 19th-century ($\delta = 1$ J. Steiner 1848).
- $\delta = 4, 5, 6$ I. Vainsencher 1995.
- $\delta = 7, 8$ S. Kleiman and R. Piene in 2001.

S. Fomin and G. Mikhalkin showed that for every $\delta \geq 1$, there exists a node polynomial $N_\delta(d)$ of degree 2δ which satisfies $N^\delta(d) = N_\delta(d)$ for $d \geq 2\delta$.
Computations and more recent proofs show that this holds for $d \geq \delta + 2$.

Examples of node polynomials

$$
\begin{aligned}
N_0(d) &= 1,\\
N_1(d) &= 3(d-1)^2,\\
N_2(d) &= \frac{3}{2}(d-1)(d-2)(3d^2-3d-11),\\
N_3(d) &= \frac{9}{2}d^6 - 27d^5 + \frac{9}{2}d^4 + \frac{423}{2}d^3 - 229d^2 - \frac{829}{2}d + 525,\\
N_4(d) &= \frac{27}{8}d^8 - 27d^7 + \frac{1809}{4}d^5 - 642d^4 - 2529d^3 + \frac{37881}{8}d^2 + \frac{18057}{4}d - 8865,\\
N_5(d) &= \frac{81}{40}d^{10} - \frac{81}{4}d^9 - \frac{27}{8}d^8 + \frac{2349}{4}d^7 - 1044d^6 - \frac{127071}{20}d^5 + \frac{128859}{8}d^4 + \frac{59097}{2}d^3 - \frac{3528381}{40}d^2\\
&\quad - \frac{946929}{20}d + 153513
\end{aligned}
$$

Examples of node polynomials

$$
\begin{aligned}
N_{14}(d) = {} & \frac{19683}{358758400}d^{28} - \frac{19683}{12812800}d^{27} - \frac{6561}{2562560}d^{26} + \frac{1751787}{3942400}d^{25} - \frac{4529277}{1971200}d^{24} - \frac{562059}{9856}d^{23} \\
& + \frac{398785599}{788480}d^{22} + \frac{5214288411}{1254400}d^{21} - \frac{4860008991}{89600}d^{20} - \frac{63174295089}{358400}d^{19} + \frac{332872084467}{89600}d^{18} \\
& + \frac{3103879378581}{985600}d^{17} - \frac{4913807521304691}{27596800}d^{16} + \frac{899178800016807}{8968960}d^{15} + \frac{279086438050359453}{44844800}d^{14} \\
& - \frac{468967272863997483}{51251200}d^{13} - \frac{318443311640108577}{1971200}d^{12} + \frac{328351365725506869}{985600}d^{11} \\
& + \frac{1120586814080571923}{358400}d^{10} - \frac{9448861028448843949}{1254400}d^{9} - \frac{30880785216736406143}{689920}d^{8} \\
& + \frac{444525313669622586903}{3942400}d^{7} + \frac{11429038221675466251}{24640}d^{6} - \frac{269709254062572016617}{246400}d^{5} \\
& - \frac{74660630664748878665353}{22422400}d^{4} + \frac{140531359469510983018159}{22422400}d^{3} + \frac{16863931195154225977601}{1121120}d^{2} \\
& - \frac{64314454486825349085}{4004}d - 32644422296329680.
\end{aligned}
$$

Given X a projective complex surface

- can we define the invariant $N_X^\delta(d)$?

  Yes, but we need to redefine the degree.
- can we still define sequences $N_X(d)$?

  Why not, multiples of the new degree may work.
- are these sequences polynomials?

  In some cases, we can proof they are!
- Is there a One polynomial to rule them all, One polynomial to find them, One polynomial to bring them all and in the computations bind them?

  Wouldn't it be nice!

$$N_X^\delta(\mathcal{L}) = P_\delta(\mathcal{L}^2, \mathcal{L}\cdot K_X, K_X^2, c_2(X))$$

Gromov-Witten invariants

Nodal curves C lying in a smooth complex surface X satisfy

$$g + \delta = \frac{C^2 + C \cdot K_X + 2}{2}.$$

Definition

We call the Gromov-Witten invariant of a surface X the number

$$\mathrm{GW}_X(g, \delta, \bar{d}, \mathcal{P}) = \#\left(\left\{C \subset X \mid \mathrm{g}(C) = g, [C] = \bar{d}, \mathcal{P} \subset C\right\}\right)$$

where C is a nodal curve, $\bar{d} \in H_2(X, \mathbb{Z})$, $\mathcal{P}$ is a generic configuration of points in X.

Theorem

If X is a smooth projective **complex** surface, then the number

$$\mathrm{GW}_X(g,\delta,\bar{d},\mathcal{P})$$

is independent of $\mathcal{P}$ as long as $\mathcal{P}$ is a generic configuration of points.

$$\mathrm{GW}_X^g(\bar{d}) := \mathrm{GW}_X(g,\delta,\bar{d},\mathcal{P})$$

Gromov-Witten invariants examples

Examples:

- $GW^{0}_{\mathbb{CP}^2}(1) = 1$
- $GW^{0}_{\mathbb{CP}^2}(2) = 1$
- $GW^{1}_{\mathbb{CP}^2}(3) = 1$
- $GW^{\frac{(d-1)(d-2)}{2}}_{\mathbb{CP}^2}(d) = 1$
- $GW^{0}_{\mathbb{CP}^2}(3) = 12 = 3(3-2)^2$

Kontsevich's formula

Setting $N_d = \mathrm{GW}^0_{\mathbb{CP}^2}(d)$, we have that

$$N_d = \sum_{d_1+d_2=d} N_{d_1} N_{d_2} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right)$$

To get all the N_d, we only need to know N_1, the number of lines through two points.

- generalized to higher genus by Caporaso and Harris.
- generalized to *easy* toric surfaces using Tropical geometry.

Rational plane curves

$$
\begin{aligned}
N_1 &= 1\\
N_2 &= 1\\
N_3 &= 12\\
N_4 &= 620\\
N_5 &= 87304\\
N_6 &= 26312976\\
N_7 &= 14616808192\\
N_8 &= 13525751027392\\
N_9 &= 19385778269260800\\
N_{10} &= 40739017561997799680\\
N_{11} &= 120278021410937387514880\\
N_{12} &= 482113680618029292368686080
\end{aligned}
$$

Welschinger invariants

Over the reals it is a different story.

$$\mathrm{GW}_{\mathbb{RP}^2}(g = 0, \delta = 1, d = 3, \mathcal{P}) = 8, 10, \text{ or } 12$$

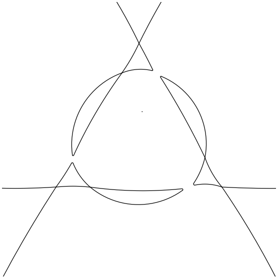
Theorem

$$W_{\mathbb{RP}^2}(d) := \sum_{C} (-1)^{h(C)}$$

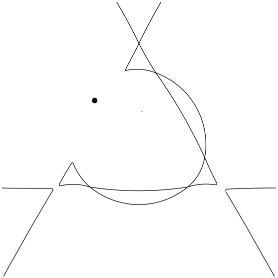
is invariant with respect to configurations of points $\mathcal{P} \subset \mathbb{RP}^2$ in general position.
The sum runs over all curves C of degree d and **genus** 0 containing $\mathcal{P}$. $h(C)$ is the number of hyperbolic nodes.

Hyperbolic nodes

Real genus 0 curves of degree 5.



$h(C) = 0$



$h(C') = 1$

Welschinger invariants for non-totally real configurations

Let $s, t \in \mathbb{N}$ such that $2s + 1 = 3d - 1$.

Let $\mathcal{P}$ a configuration of points in $\mathbb{CP}^2$ such that exactly t points belong to $\mathbb{RP}^2$ and s couples of points are complex conjugated.

Theorem

$$W_{\mathbb{RP}^2}(d; s) := \sum_C (-1)^{h(C)}$$

is invariant with respect to configurations of points $\mathcal{P} \subset \mathbb{RP}^2$ in general position.

The sum runs over all curves C of degree d and **genus** 0 containing $\mathcal{P}$. $h(C)$ is the number of hyperbolic nodes.

Welschinger invariants examples

- $W_{\mathbb{RP}^2}(1) = 1$
- $W_{\mathbb{RP}^2}(2) = 1$
- $W_{\mathbb{RP}^2}(d;0) = W_{\mathbb{RP}^2}(d)$

- $W_{\mathbb{RP}^2}(3;0) = 8$
- $W_{\mathbb{RP}^2}(3;1) = 6$
- $W_{\mathbb{RP}^2}(3;2) = 4$
- $W_{\mathbb{RP}^2}(3;3) = 2$
- $W_{\mathbb{RP}^2}(3;4) = 0$

- $W_{\mathbb{RP}^2}(4;0) = 240$
- $W_{\mathbb{RP}^2}(4;1) = 144$
- $W_{\mathbb{RP}^2}(4;2) = 80$
- $W_{\mathbb{RP}^2}(4;3) = 40$
- $W_{\mathbb{RP}^2}(4;4) = 16$
- $W_{\mathbb{RP}^2}(4;5) = 0$

Welschinger invariants examples

- $W_{\mathbb{RP}^2}(5;0) = 18264$
- $W_{\mathbb{RP}^2}(5;1) = 9096$
- $W_{\mathbb{RP}^2}(5;2) = 4272$
- $W_{\mathbb{RP}^2}(5;3) = 1872$
- $W_{\mathbb{RP}^2}(5;4) = 744$
- $W_{\mathbb{RP}^2}(5;5) = 248$
- $W_{\mathbb{RP}^2}(5;6) = 64$
- $W_{\mathbb{RP}^2}(5;7) = 64$

- $W_{\mathbb{RP}^2}(6;0) = 2845440$
- $W_{\mathbb{RP}^2}(6;1) = 1209600$
- $W_{\mathbb{RP}^2}(6;2) = 490368$
- $W_{\mathbb{RP}^2}(6;3) = 188544$
- $W_{\mathbb{RP}^2}(6;4) = 67968$
- $W_{\mathbb{RP}^2}(6;5) = 22400$
- $W_{\mathbb{RP}^2}(6;6) = 6400$
- $W_{\mathbb{RP}^2}(6;7) = 1536$
- $W_{\mathbb{RP}^2}(6;8) = 1024$

Welschinger invariants examples

- Are these invariants always positive?

  $W_{\mathbb{RP}^2}(7;10) = -14336$

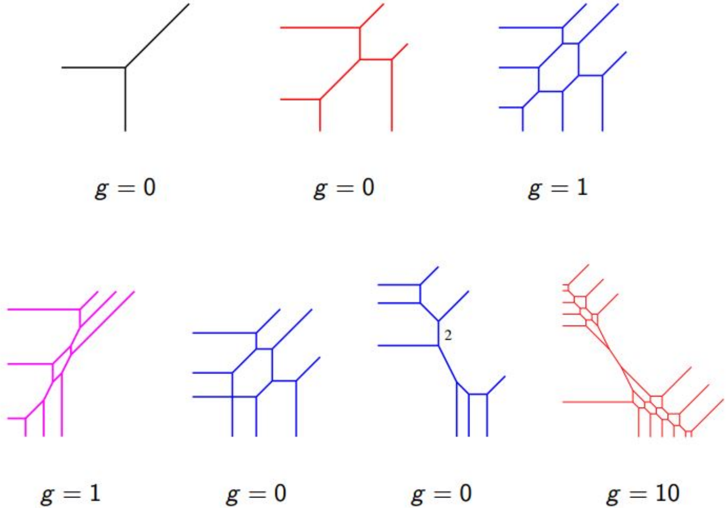
  $W_{\mathbb{RP}^2}(8;11) = -280576$

- Are these decreasing sequences?

  $W_{\mathbb{RP}^2}(9;12) = 3932160$

  $W_{\mathbb{RP}^2}(9;13) = 17326080$

Examples of tropical curves in $\mathbb{RP}^2$



Consider the function

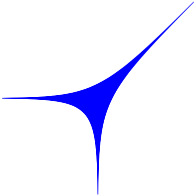
$$\mathrm{Log}\colon \begin{array}{ccc} (\mathbb{C}\setminus 0)^2 & \longrightarrow & \mathbb{R}^2 \\ (z_1, z_2) & \longmapsto & (\log|z_1|, \log|z_2|) \end{array}$$

If $C \subset \mathbb{CP}^2$ is a curve, we define its *amoeba* by

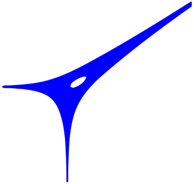
$$\begin{aligned} \mathcal{A}_C &= \mathrm{Log}(C) \\ &= \left\{\mathrm{Log}(z,w) \mid [z:w:1] \in C \text{ and } (z,w) \in (\mathbb{C}\setminus 0)^2\right\}. \end{aligned}$$

This amoeba contains a spine, that we would define as the tropicalization of the curve $\mathrm{Trop}(C)$.

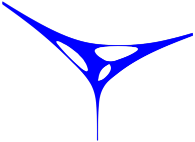
Examples of amoebas in $\mathbb{R}^2$



$w - 2z - 1$



$3z^2 + 5zw + w^3 + 1$



$1 + z + z^2 + z^3 + z^2w^3 + 10zw + 12z^2w + 10z^2w^2$

$50z^3 + 83z^2w + 24zw^2 + w^3 + 392z^2 + 414zw + 50w^2 - 28z + 59w - 100$

Tropical semifield

Let

$$\mathbb{T} = \mathbb{R} \cup \{-\infty\}$$

endowed with the operations

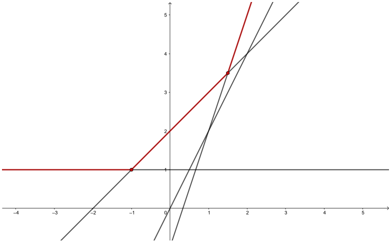
$$a \oplus b = \max\{a, b\}, \qquad a \otimes b = a + b.$$

Tropical polynomials are piecewise linear functions.

$$\bigoplus_{i=0}^{n} a_i \otimes x^{\otimes i} = \max_{i=0}^{n}\{a_i + i \cdot x\}$$

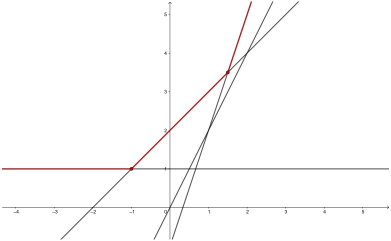
Examples of a tropical polynomial

$$
\begin{aligned}
f(x) &= 1 \oplus 2 \otimes x \oplus 0 \otimes x^2 \oplus (-1) \otimes x^3 \\
&= \max\{1, 2 + x, 2x, -1 + 3x\}
\end{aligned}
$$

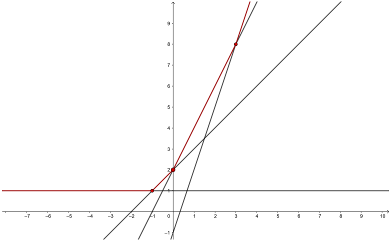


The roots of the polynomials are the points were the graph is not linear, i.e., the values x where the maximum $f(x)$ is achieved at least twice.
The multiplicity of the root is the change of slope of the adjacent linear pieces.

Examples of a tropical roots



$1 \oplus 2 \otimes x \oplus 0 \otimes x^2 \oplus (-1) \otimes x^3$



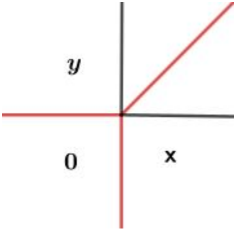
$1 \oplus 2 \otimes x \oplus 0 \otimes 2 \otimes x^2 \oplus (-1) \otimes x^3$

Tropical variety of a polynomial

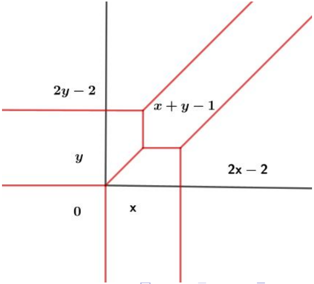
Let $f \in \mathbb{R}[x, y]$ be a polynomial. We define

$$V_{Trop}(f) = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) \text{ is achieved at least twice}\}$$

$0 \oplus x \oplus y$



$0 \oplus x \oplus y \oplus (-1)xy \oplus -2x^2 \oplus -2y^2$



Connection to classical algebraic geometry

Let K be algebraically closed field with a non-trivial valuation $\text{val} : K \longrightarrow \mathbb{R} \cup \{\infty\}$. Let $f = \sum_{(i,j)\in I} a_{i,j} x^i y^j \in K[x,y]$ be a polynomial. We define

$$\text{Trop}(f) = \bigoplus_{(i,j)\in I} \text{val}(a_{i,j}) \otimes x^{\otimes i} \otimes y^{\otimes j}$$

Kapranov's theorem

$$\text{Trop}\left(V(f)\right) = V_{Trop}\left(\text{Trop}(f)\right)$$

Enumeration of tropical curves I

$GW^{\mathbb{T}}_{\mathbb{C}P^2}(d,g) =$ number of tropical curves of degree d and genus g passing through $3d+g-1$ given points in $\mathbb{R}^2$.

Example

$d=1$. *How many lines through 2 points?*

Enumeration of tropical curves I

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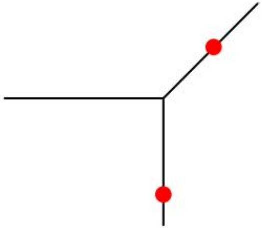


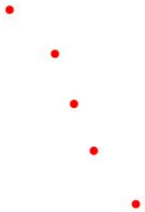
$$GW^{\mathbb{T}}_{\mathbb{C}P^2}(1,0) = 1$$

Enumeration of tropical curves II

$GW^{\mathbb{T}}_{\mathbb{C}P^2}(d,g) =$ number of tropical curves of degree d and genus g passing through $3d+g-1$ given points in $\mathbb{R}^2$.

Example

$d = 2$. *How many conics through 2 points?*



Enumeration of tropical curves II

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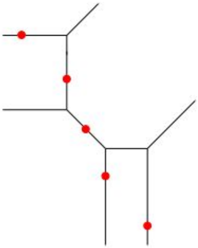
Example

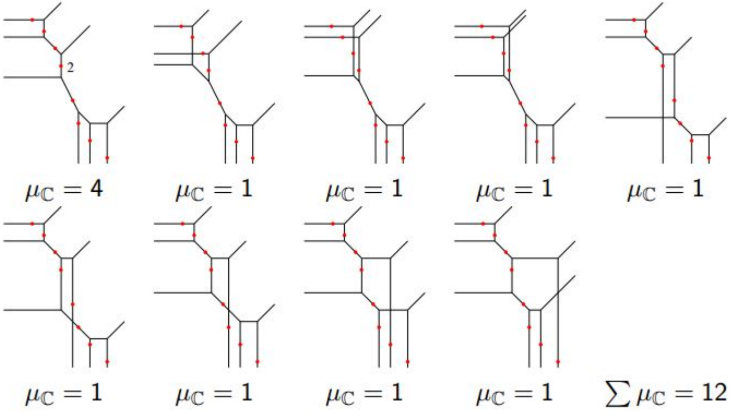
$d = 2$. *How many conics through 2 points?*



$$GW^{\mathbb{T}}_{\mathbb{C}P^2}(2,0) = 1$$

Enumeration of tropical curves III

$GW^{\mathbb{T}}_{\mathbb{C}P^2}(3,0) = ??$ $GW_{\mathbb{C}P^2}(3,0) = 12$



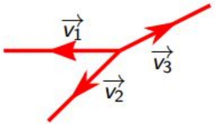
Tropical invariants

Theorem (Mikhalkin)

$GW_{\mathbb{C}P^2}(d,g) =$ *number of tropical curves counted with multiplicity*

$$\mu(C) = \prod_{v \text{ vertex}} \mu(v)$$

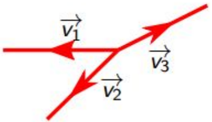
$$\mu_{\mathbb{C}}(v) = |w_1 w_2 \det(\vec{v_1}, \vec{v_2})|$$



Refined invariants

$$\mu_{\mathbb{C}}(v) = |w_1 w_2 \det(\vec{v_1}, \vec{v_2})|$$

$$\mu_G(v) = \frac{q^{\frac{\mu_{\mathbb{C}}(v)}{2}} - q^{-\frac{\mu_{\mathbb{C}}(v)}{2}}}{q^{\frac{1}{2}} - q^{\frac{-1}{2}}}$$



Theorem (Block-Göttsche, Itenberg-Mikhalkin)

Counting tropical curves with multiplicity $\mu_G(C)$ yields an invariant $G_{\mathbb{C}P^2}(d,g) \in \mathbb{Q}[q^{\pm 1}]$

Remark

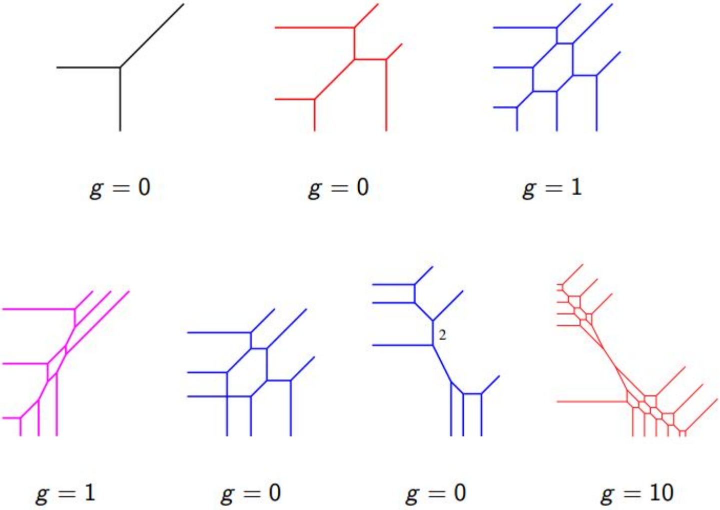
$G_{\mathbb{C}P^2}(d,g)(1) = GW_{\mathbb{C}P^2}(d,g)$

Example

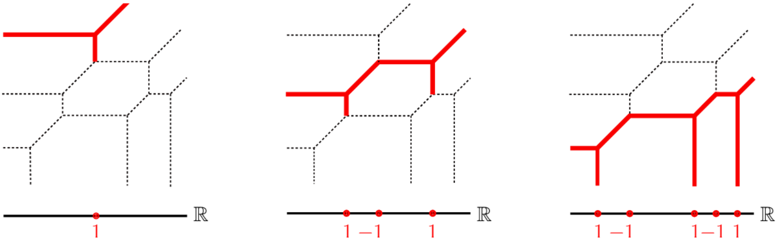
$G_{\mathbb{C}P^2}(1,0) = 1 \qquad G_{\mathbb{C}P^2}(2,0) = 1 \qquad G_{\mathbb{C}P^2}(3,0) = q^{-1} + 10 + q$

$$G_{\mathbb{C}P^2}(4,0) = q^{-3} + 13q^{-2} + 94q^{-1} + 404 + 94q + 13q^2 + q^3$$

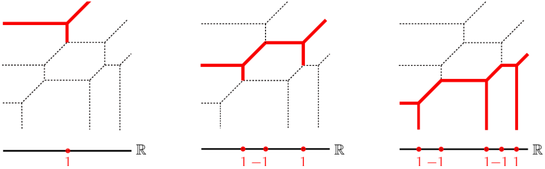
Examples of tropical curves in $\mathbb{RP}^2$

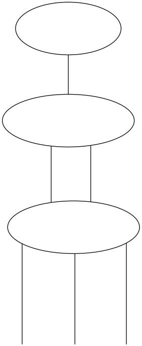


Systematic study of graphs: floor diagrams

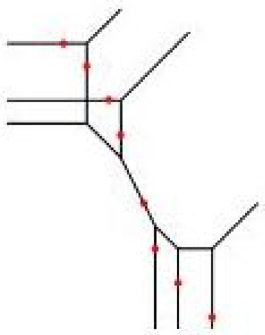


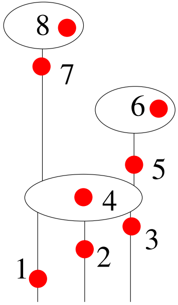
Floor diagram of a cubic curve



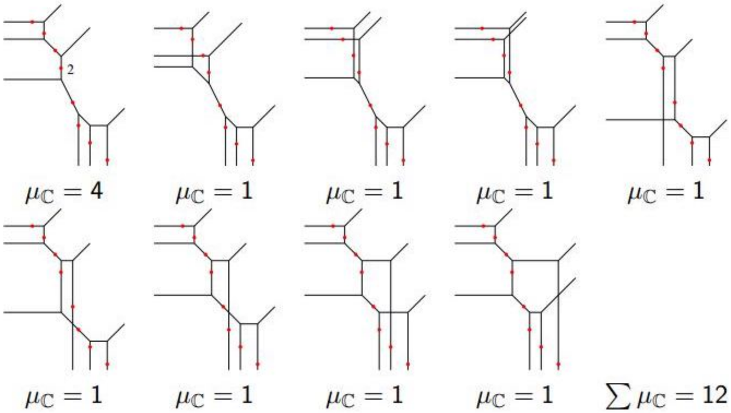


Marked floor diagram of a rational cubic curve

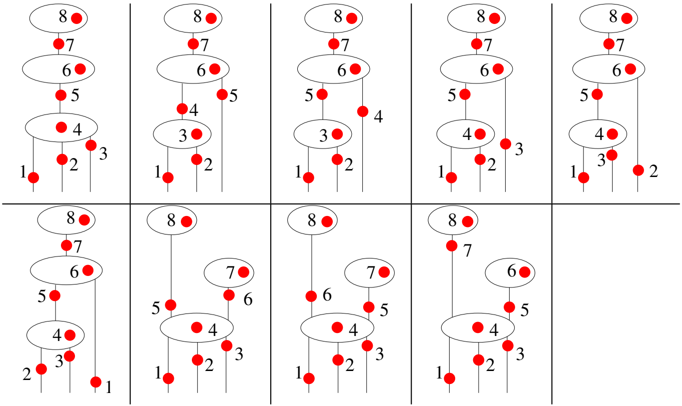




Rational cubics



Floor diagrams rational cubics



$$GW^{\mathbb{T}}(d,g) = \sum_{(\mathcal{D},m)} \mu_{\mathbb{C}}(\mathcal{D})$$

where the sum runs over all marked floor diagrams of degree d and genus g, and

$$\mu_{\mathbb{C}} = \prod_{e \in \mathrm{Edge}(\mathcal{D})} (\omega(e))^2 .$$

In the same fashion,

$$G(d,g)(q) = \sum_{(\mathcal{D},m)} \prod_{e \in \mathrm{Edge}(\mathcal{D})} \left(\frac{q^{\frac{\omega(e)}{2}} - q^{-\frac{\omega(e)}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \right)^2 .$$

Refined invariants for rational curves

For genus 0 curves we have

$$G_{\mathbb{CP}^2}(d,g=0)(-1)=W_{\mathbb{CP}^2}(d).$$

For a partition $2s+t=3d-1$ we can define purely *combinatorially*

$$G_{\mathbb{CP}^2}(d,g=0;s)(q)$$

as refine invariant satisfying

$$G_{\mathbb{CP}^2}(d,g=0;s)(-1)=W_{\mathbb{RP}^2}(d;s).$$

Pairs of complex conjugated points are represented by a set S of couples of consecutive marked points $\{i, i+1\}$ that can be infinitely close without changing the genus of the diagram.

The refined S-multiplicity of a marked floor diagram $(\mathcal{D}, m)$ is defined by

$$\mu_S(\mathcal{D}, m)(q) = \prod_{e \in E_0} [\omega(e)]^2(q) \prod_{e \in E_1} [\omega(e)](q^2)$$

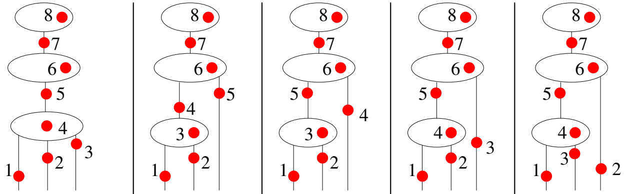
$$\cdot \prod_{\{e,e'\} \in E_2} \frac{[\omega(e)] \times [\omega(e')] \times [\omega(e) + \omega(e')]}{[2]}(q)$$

if $(\mathcal{D}, m)$ is compatible with S, and by

$$\mu_S(\mathcal{D}, m)(q) = 0$$

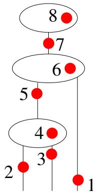
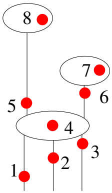
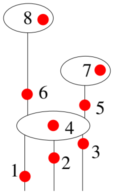
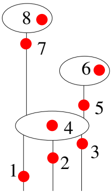
otherwise.

S-multiplicities of floor diagrams rational cubics



μ	$q+2+q^{-1}$	1	1	1	1
μ_{S_1}	$q+2+q^{-1}$	1	1	1	1
μ_{S_2}	$q+q^{-1}$	1	1	1	1
μ_{S_3}	$q+q^{-1}$	1	0	0	1
μ_{S_4}	$q+q^{-1}$	1	0	0	0

S-multiplicities of floor diagrams rational cubics II

				
μ	1	1	1	1
μ_{S_1}	1	0	0	1
μ_{S_2}	1	0	0	1
μ_{S_3}	1	0	0	1
μ_{S_4}	0	0	0	1

Refined node polynomiality

If we fix δ the number of nodal points of the curves, we can calcute for every degree d:

- $N_{\delta=0}(d) = 1$
- $N_{\delta=1} = 3(d-1)^2$
- ...

The polynomiality property holds for the refined version

- $GW_{\mathbb{CP}^2}(\delta = 0)(d) = 1$
- $GW_{\mathbb{CP}^2}(\delta = 1)(q)(d) =$

$$\frac{(d-1)(d-2)}{2}q + (d-1)(2d-1) + \frac{(d-1)(d-2)}{2}q^{-1}$$

Polynomiality with respect to the degree

- $G^{g=0}_{\mathbb{CP}^2}(d=1)(q) = 1$
- $G^{g=0}_{\mathbb{CP}^2}(d=2)(q) = 1$
- $G^{g=0}_{\mathbb{CP}^2}(d=3)(q) = q + 10 + q^{-1}$
- $G^{g=0}_{\mathbb{CP}^2}(d=4)(q) = q^{-3} + 13q^{-2} + 94q^{-1} + 404 + 94q + 13q^2 + q^3$

For a fixed genus, the coefficient of fixed codegree are polynomial with respect to the degree.

Polynomiality with respect to s

$$
\begin{aligned}
G_{\Delta_4}(0;0) &= q^{-3} + 13q^{-2} + 94q^{-1} + 404 + 94q + 13q^2 + q^3\\
G_{\Delta_4}(0;1) &= q^{-3} + 11q^{-2} + 70q^{-1} + 264 + 70q + 11q^2 + q^3\\
G_{\Delta_4}(0;2) &= q^{-3} + 9q^{-2} + 50q^{-1} + 164 + 50q + 9q^2 + q^3\\
G_{\Delta_4}(0;3) &= q^{-3} + 7q^{-2} + 34q^{-1} + 96 + 34q + 7q^2 + q^3\\
G_{\Delta_4}(0;4) &= q^{-3} + 5q^{-2} + 22q^{-1} + 52 + 22q + 5q^2 + q^3\\
G_{\Delta_4}(0;5) &= q^{-3} + 3q^{-2} + 14q^{-1} + 24 + 14q + 3q^2 + q^3
\end{aligned}
$$

For a genus 0 and degree d, the coefficient of fixed codegree are polynomial with respect to the number of complex conjugated pairs s.

Examples of polynomials for $\mathbb{CP}^2$

$$\forall d \geq 3,\ \langle G_{\Delta_d}(0;s)\rangle_1 = t+2$$

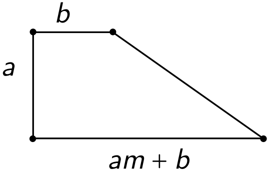
$$\forall d \geq 4,\ \langle G_{\Delta_d}(0;s)\rangle_2 = \frac{t^2+4t+y+11}{2}$$

$$\forall d \geq 5,\ \langle G_{\Delta_d}(0;s)\rangle_3 = \frac{t^3+6t^2+(3y+35)t+6y+72}{3!}$$

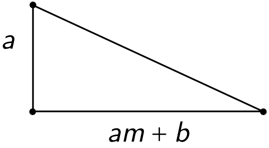
$$\forall d \geq 6,\ \langle G_{\Delta_d}(0;s)\rangle_4 = \frac{t^4+8t^3+(6y+74)t^2+(24y+304)t+3y^2+72y+621}{4!}$$

$$\forall d \geq 7,\ \langle G_{\Delta_d}(0;s)\rangle_5 = \frac{1}{5!}\times\big(t^5+10t^4+(10y+130)t^3+(60y+800)t^2+(15y^2+380y+3349)t$$
$$+30y^2+780y+6030\big)$$

Universal polynomials



a) $\Delta_{a,b,n}$



b) $\Delta_{a,0,n}$

Examples of polynomials for $\mathbb{CP}^2$

$$
\begin{aligned}
p_0(a,b,m) &= 1,\\
p_1(a,b,m) &= 3b^2m + 6ab - 2bm - 4a - 4b + 4,\\
p_2(a,b,m) &= \frac{9}{2}b^4m^2 + 18ab^3m - 6b^3m^2 + 18a^2b^2 - 24ab^2m - 12b^3m + 2b^2m^2 - 24a^2b - 24ab^2 + 8abm - b^2m + 8a^2\\
&\quad - 2ab + 8b^2 + \frac{23}{2}bm + 23a + 23b - 30,\\
p_3(a,b,m) &= \frac{9}{2}b^6m^3 + 27ab^5m^2 - 9b^5m^3 + 54a^2b^4m - 54ab^4m^2 - 18b^5m^2 + 6b^4m^3 + 36a^3b^3 - 108a^2b^3m - 72ab^4m\\
&\quad + 36ab^3m^2 - 21b^4m^2 - \frac{4}{3}b^3m^3 - 72a^3b^2 - 72a^2b^3 + 72a^2b^2m - 84ab^3m - 8ab^2m^2 + 24b^4m + \frac{137}{2}b^3m^2\\
&\quad + 48a^3b - 84a^2b^2 - 16a^2bm + 48ab^3 + 274ab^2m + 137b^3m - 31b^2m^2 - \frac{32}{3}a^3 + 274a^2b + 274ab^2 - 124abm\\
&\quad - \frac{32}{3}b^3 - 68b^2m - 124a^2 - 136ab - 124b^2 - \frac{374}{3}bm - \frac{748}{3}a - \frac{748}{3}b + 452.
\end{aligned}
$$