Kontsevich's formula for enumeration of rational curves

1) Introduction to the moduli space of genus 0 stable curves

2) Introduction to the moduli space of stable maps

3) Kontsevich's formula

§ Moduli space of stable curves

<u>Def.</u> · A n-pointed smooth rational curve $(C; p_1, \dots, p_n)$ is a proj. smooth rational curve C equipped with a choice of n distinct marked points $p_1, \dots, p_n \in C$

· An isom. between two n-pointed rational curves $\varphi: (C; p_1, \dots, p_n) \to (C', p_1', \dots, p_n')$ is an isom. $\varphi: C \xrightarrow{\sim} C'$ such that $\varphi(p_i) = p_i'$.

· $n \geq 3$, $M_{0,n} :=$ moduli space of n-pointed smooth rational curves up to isom.

<u>Ex.</u> · $n=3$. Given (C, p_1, p_2, p_3) there exists unique an isom. from (C, p_1, p_2, p_3) to $(\mathbb{P}^1, 0, 1, \infty)$

$\rightsquigarrow M_{0,3} = pt$ (representing $(\mathbb{P}^1, 0, 1, \infty)$)

- $n=4$. Given $(C, p_1, \dots, p_4)$, there exists an unique isom. to $(\mathbb{P}^1, 0, 1, \infty, q)$ with $q \in \mathbb{P}^1 \setminus \{0,1,\infty\}$

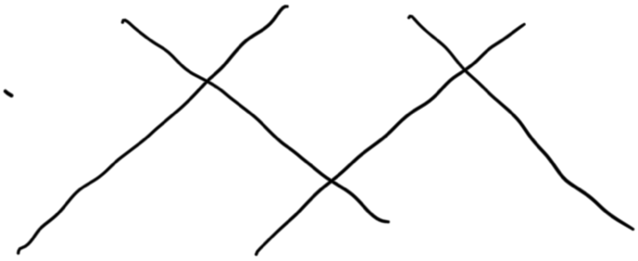
$\rightsquigarrow M_{0,4} \simeq \mathbb{P}^1 \setminus \{0,1,\infty\}$, the point $q \in \mathbb{P}^1 \setminus \{0,1,\infty\}$ represents the curve $(\mathbb{P}^1, 0, 1, \infty, q)$.

- $n \geq 4$, $M_{0,n} \simeq (\mathbb{P}^1 \setminus \{0,1,\infty\})^{n-3} \setminus$ diagonals.

Def. A genus 0 nodal curve is a (possibly reductible) curve such that:

1) Each irred. component is isom. to $\mathbb{P}^1$
2) The points of intersection of the components are ordinary double pts (i.e. nodes, locally isomorphic to $\{xy=0\} \subseteq \mathbb{C}^2$)
3) There are no closed circuits: if we remove a node, the curve becomes disconnected.

Picture.

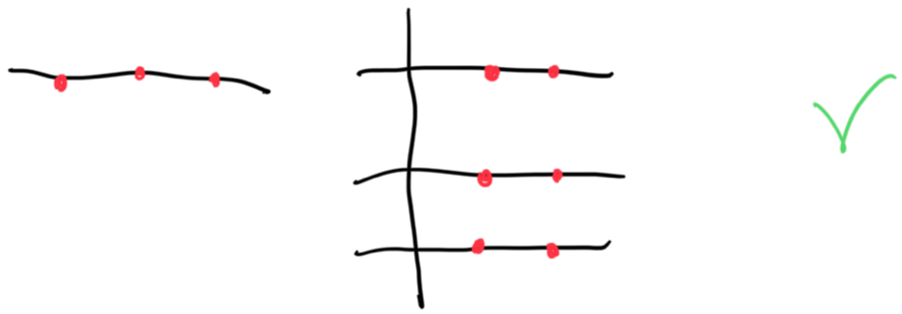


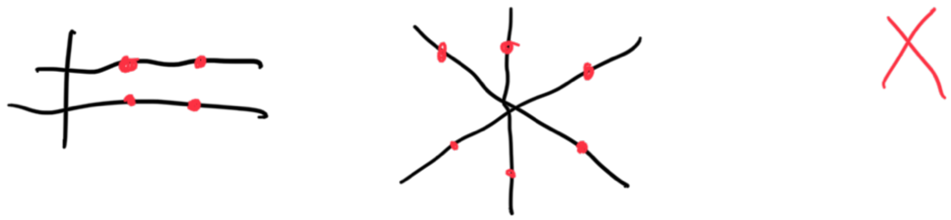
(geometric picture)

(topological picture)

Def. Let $n \geq 3$. A stable n-pointed genus 0 curve is a genus nodal curve, with n distinct marked pts that are smooth points of C, such that every irred. component contains at least 3 special points (either a node or marked pt)

Ex.





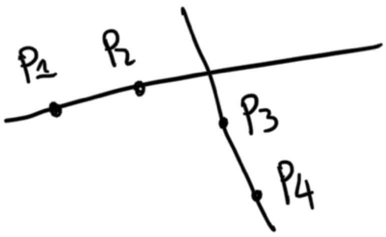
Def.
- An isom. of two n-pointed curves $(C, p_1, \ldots, p_n)$ and $(C', p_1', \ldots, p_n')$ is an isom. $\varphi : C \to C'$ sending $\varphi(p_i) = p_i'$.
- We say that a curve is automorphism-free if the only autom. is the identity.

Prop. A n-pointed genus 0 curve is stable $\Longleftrightarrow$ it is autom. free

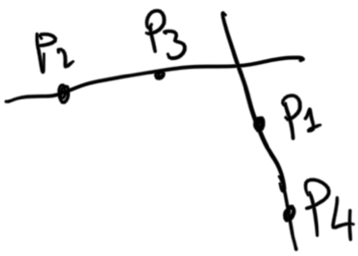
Thm. (Knudsen) For $n \geq 3$, there is a smooth projective variety $\overline{M}_{0,n}$ parametrizing genus 0 n-ptd stable curves. It contains $M_{0,n}$ a dense open subset.

Ex. · $M_{0,3} = \overline{M}_{0,3}$

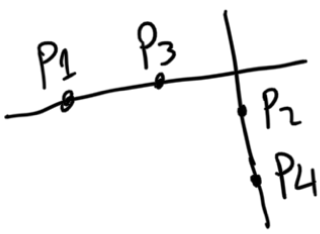
· $\overline{M}_{0,4} \simeq \mathbb{P}^1$. We have added 3 pts, which correspond to



$\leftarrow$ over ∞



$\leftarrow$ over 0



$\leftarrow$ over 1

§ Combinatorial structure of $\overline{M}_{0,n}$

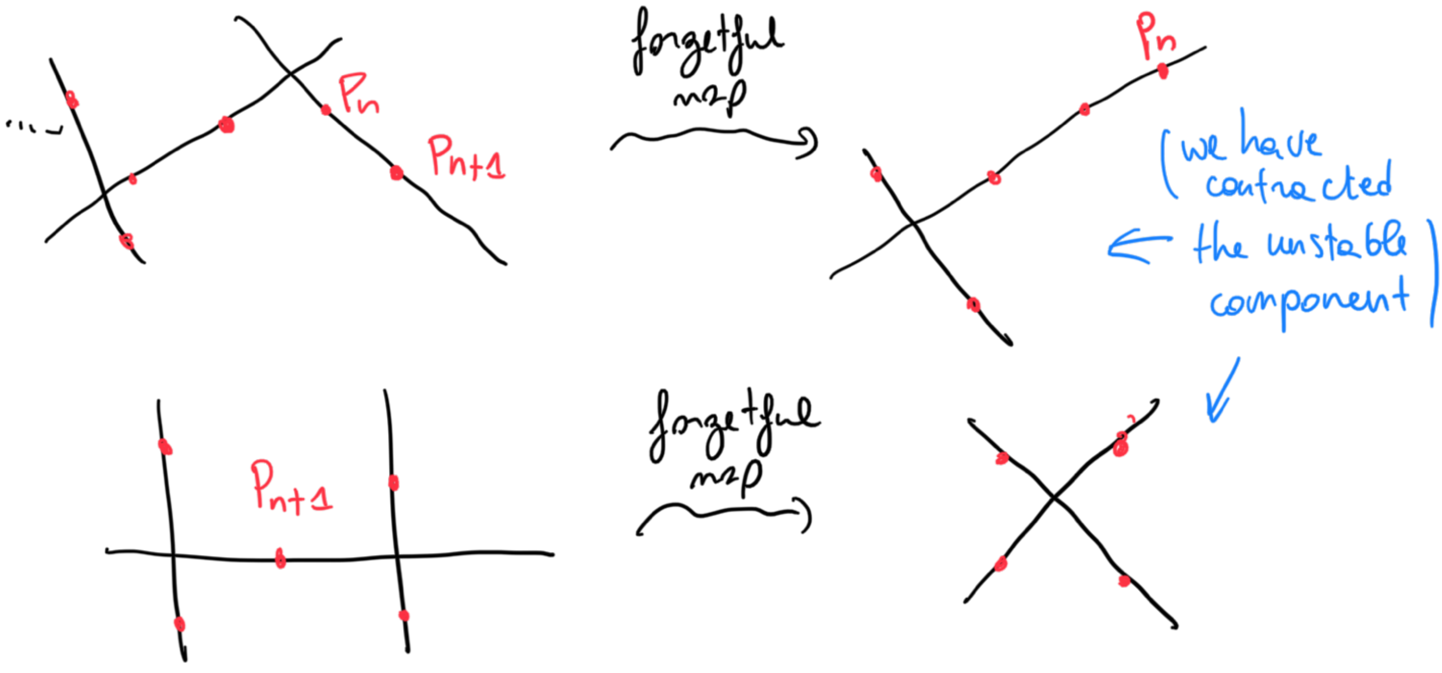
We are going to define some natural maps between $\overline{M}_{0,n}$ for different values of n.

Forgetful maps

$$\varepsilon : \overline{M}_{0,n+1} \longrightarrow \overline{M}_{0,n}$$

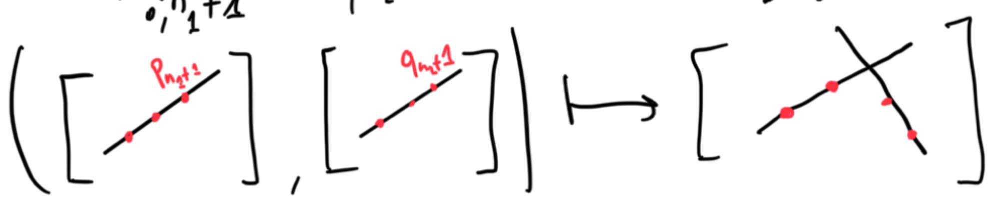
$$(C, p_1, \dots, p_{n+1}) \longmapsto \mathrm{Stab}\,(C, p_1, \dots, p_n)$$

⚠ It may happen that after we forget p_{n+1} some component is not stable anymore.
We have to consider the stabilization.



Gluing maps.

$$\overline{M}_{0,n_1+1} \times \overline{M}_{0,n_2+1} \longrightarrow \overline{M}_{n_1+n_2}$$



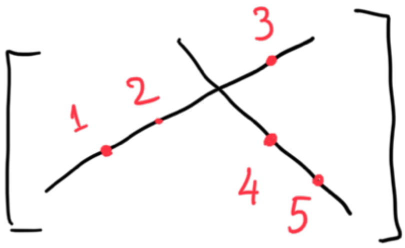
The image of gluing maps is included in the boundary of the moduli space.

Boundary: $\partial \overline{M}_{0,n} := \overline{M}_{0,n} \setminus M_{0,n}$ corresponds to reducible curves.

There is a stratification of $\overline{M}_{0,n}$ by the number of nodes.

Case of one node: Consider a partion $A \sqcup B = \{1, \dots, n\}$, with $|A| \geq 2$, $|B| \geq 2$, we have a divisor $D(A|B)$ whose general point represents a curve with two irreducible components such that the labelling A lies on one component and the labelling of B on the other one.

Example $n = 5$, $\{1,2,3\} \sqcup \{4,5\}$

$D(123|45) \ni$ 

Prop. • The union of all the divisors $D(A|B)$ forms the boundary $\partial \overline{M}_{0,n}$

• Each $D(A|B)$ is a smooth divisor

- The boundary $\partial \overline{M}_{0,n}$ is a normal crossings divisor.

Prop. $$D(A|B) \underset{\text{gluing map}}{\simeq} \overline{M}_{0,A\cup\{x\}} \times \overline{M}_{0,B\cup\{x\}}$$

Pull-back of boundary divisors under forgetful maps

Let $\varepsilon : \overline{M}_{0,n+1} \to \overline{M}_{0,n}$ be the forgetful map, forgetting the mark q.

Then $$\varepsilon^* D(A|B) = D(A\cup\{q\}|B) + D(A|B\cup\{q\})$$

We can compose two or more forgetful maps and consider the map

$$\varepsilon : \overline{M}_{0,n} \longrightarrow \overline{M}_{0,4}$$

The boundary in $\overline{M}_{0,4}$ is composed by three points

$$D(ij|kh) \quad D(ik|jh) \quad D(ih|jk)$$

These three divisors are linearly equivalent!

$$\Rightarrow \quad \varepsilon^* D(ij|kh) = \varepsilon^* D(ik|jh) = \varepsilon^* D(ih|jk)$$

$$\Longrightarrow \quad \varepsilon\, D(ij|kh) - \varepsilon\, D(ik|jh) = \varepsilon \cdot D(ih|jk)$$

$$\underset{\substack{A \sqcup B = \{1,\dots,n\} \\ i,j \in A,\ k,h \in B}}{\sum} D(A|B) \qquad \qquad \underset{\substack{A \sqcup B = \{1,\dots,n\} \\ i,k \in A \\ j,h \in B}}{\sum}$$

§ Moduli space of stable maps

Def. The <u>degree</u> of a map $\mu : \mathbb{P}^1 \to \mathbb{P}^2$ is the homology class of $\mu_*[\mathbb{P}^1] \in H^2(\mathbb{P}^2, \mathbb{Z}) \simeq \mathbb{Z}$.
In particular a constant map has degree 0.

Rmk. To give a map $\mu : \mathbb{P}^1 \to \mathbb{P}^2$ of degree d is to specify (up to a constant factor) 3 degree d homog. forms in 2 variables which are not allowed to vanish simultaneously at any point.

We then have a Zariski open subset

$$W(2,d) \subseteq \mathbb{P}\left(\bigoplus_{i=1}^{3} H^0(\mathbb{P}^1, \mathcal{O}(d)) \right)$$

The dimension of $W(2,d)$ is $3d+2$.

Def. An <u>isom.</u> between maps $\mu : C \to \mathbb{P}^2$, $\mu' : C' \to \mathbb{P}^2$ is an isom. $\varphi : C \xrightarrow{\sim} C'$ such that $\mu' \circ \varphi = \mu$

Lemma. $\mu : \mathbb{P}^1 \to \mathbb{P}^2$ non constant map. Then there is only a finite number of autom. $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ such that $\mu = \mu \circ \varphi$.

If μ is birational onto its image, then $\mathrm{Aut}(\mu)$ is trivial.

Def. We define the moduli space $M_{0,0}(\mathbb{P}^2, d)$ to be the quotient $W(2,d) / \mathrm{Aut}(\mathbb{P}^1)$

Thm.
- There is an open subset $M^*_{0,0}(\mathbb{P}^2, d)$ which parametrizes autom-free degree d maps.
  $\dim M^*_{0,0}(\mathbb{P}^2, d) = \dim M_{0,0}(\mathbb{P}^2, d) = 3d-1$.
- $\mathrm{codim}\left(M_{0,0}(\mathbb{P}^2, d) \setminus M^*_{0,0}(\mathbb{P}^2, d)\right) \geq d-1$

Def.
- A n-pointed map is a morphism $\mu : C \to \mathbb{P}^2$ where C is a n-ptd genus nodal curve
- An isom. of n-ptd maps $\mu : C \to \mathbb{P}^2$ and $\mu' : C' \to \mathbb{P}^2$ is an isom. $\varphi : C \to C'$ such that $\varphi(p_i) = p_i'$
- A map $\mu : C \to \mathbb{P}^2$ is stable iff $\mathrm{Aut}(\mu) < +\infty$.

Equiv, if an irreducible component of C is mapped to a point (it is contracted), then there must be at least three special pts on it.

Rmk. $\mu: C \to \mathbb{P}^2$ is stable $\not\Rightarrow$ C is stable. Indeed any non-constant map $\mu: \mathbb{P}^1 \to \mathbb{P}^2$ is stable.

Thm (Kontsevich). There exists a projective variety $\overline{M}_{0,n}(\mathbb{P}^2, d)$ parametrizing n-pointed stable maps.

- It contains $\overline{M}^*_{0,n}(\mathbb{P}^2, d)$ which is an open dense smooth subvariety parametrizing maps without automorphism.
- If $n \neq 0$ or $d \geq 2$, then $\operatorname{codim}\left(\overline{M}_{0,n}(\mathbb{P}^2, d) \setminus \overline{M}^*_{0,n}(\mathbb{P}^2, d)\right) \geq 2$

§ Combinatorial structure of $\overline{M}_{0,n}(\mathbb{P}^2, d)$

- Evaluation maps For each mark p_i there is a natural map
$$ev_i : \overline{M}_{0,n}(\mathbb{P}^2, d) \to \mathbb{P}^2$$
$$(C; p_1, \dots, p_n, \mu) \mapsto \mu(p_i)$$

- Forgetful maps $\varepsilon: \overline{M}_{0,n+1}(\mathbb{P}^2, d) \longrightarrow \overline{M}_{0,n}(\mathbb{P}^2, d)$

(irreducible components that become unstable must be contracted)

- Forgetting the map to $\mathbb{P}^2$ For $n \geq 3$ there is a forgetful map $\overline{M}_{0,n}(\mathbb{P}^2, d) \longrightarrow \overline{M}_{0,n}$

§ Boundary of $\overline{M}_{0,n}(\mathbb{P}^2, d)$

Def. A d-weighted partion of $\{1,\dots,n\}$ constinst of a partition $A \sqcup B = \{1,\dots,n\}$ with a partion $d_A + d_B = d$.

For each d-weighted partition there is an irreducible divisor $D(A,B;d_A,d_B)$ whose general point represent a curve with two irred components C_A and C_B and such that $\deg(\mu|_{C_A}) = d_A$ and $\deg(\mu|_{C_B}) = d_B$.

Prop. $\overline{M}_{0,A\cup\{x\}}(\mathbb{P}^2, d_A) \times_{\mathbb{P}^2} \overline{M}_{0,B\cup\{x\}}(\mathbb{P}^2, d_B) \xrightarrow{\cong} D(A,B;d_A,d_B)$

where the fiber product is taken via the evaluation maps

$$\overline{M}_{0,A\cup\{x\}}(\mathbb{P}^2, d_A) \to \mathbb{P}^2, \quad \mu \mapsto \mu(x)$$

$$\overline{M}_{0,B\cup\{x\}}(\mathbb{P}^2, d_B) \to \mathbb{P}^2, \quad \mu \mapsto \mu(x)$$

Special boundary divisors. For $n \geq 4$, let

$$\overline{M}_{0,n}(\mathbb{P}^2, d) \to \overline{M}_{0,n} \to \overline{M}_{0,4}.$$ (composition of forgetful maps)

let $D(ij|kh)$ be a point divisor in $M_{0,4}$ representing the curve 

Then $$\varepsilon^* D(ij|kh) = \sum_{\substack{A \sqcup B = \{1,\dots,n\} \\ i,j \in A,\ k,h \in B \\ d_A + d_B = d}} D(A,B;d_A,d_B)$$

Recalling that $\bar{M}_{0,4} \simeq \mathbb{P}^1$ and the three boundary divisors in $\bar{M}_{0,4}$ are equivalent we obtain

$$\sum_{\substack{A \sqcup B = \{1,\dots,n\} \\ i,j \in A,\ k,h \in B \\ d_A + d_B = d}} D(A,B;d_A,d_B) \equiv \sum_{\substack{A \sqcup B = \{1,\dots,n\} \\ i,k \in A\ \ j,h \in B \\ d_A + d_B = d}} D(A,B;d_A,d_B)$$

§ Kontsevich formula.

let $N_d :=$ # degree d rational curves passing through $3d-1$ general pts in $\mathbb{P}^2$.

Thm (Kontsevich 1994)

$$N_d + \sum_{\substack{d_A + d_B = d \\ d_A \geq 1\ d_B \geq 1}} \binom{3d-4}{3d_A-1} d_A^2 N_{d_A} N_{d_B} d_A d_B = \sum_{\substack{d_A + d_B = d \\ d_A \geq 1\ d_B \geq 1}} \binom{3d-4}{3d_A-2} d_A^2 d_B^2 N_{d_A} N_{d_B}$$

<u>proof</u>. let ℓ_1, ℓ_2 generic lines in $\mathbb{P}^2$ and
$q_3, \dots, q_{3d}$ generic points in $\mathbb{P}^2$.

Let $Y := ev_1^{-1}(\ell_1) \cap ev_2^{-1}(\ell_2) \cap ev_3^{-1}(q_3) \cap \dots \cap ev_{3d}^{-1}(q_{3d})$
$\subseteq \overline{M}_{0,3d}(\mathbb{P}^2, d)$

is a smooth curve intersecting transversally the boundary $\partial \overline{M}_{0,3d}(\mathbb{P}^2, d)$ and such that $Y \subset \overline{M}^*_{0,3d}(\mathbb{P}^2, d)$.

Recall the forgetful map $\varepsilon : \overline{M}_{0,3d}(\mathbb{P}^2, d) \to \overline{M}_{0,4}$

$$\# Y \cap \varepsilon^* D(12|34) = \# Y \cap \varepsilon^* D(13|24) \quad (*)$$

<u>Claim</u>. the formula will follow from $(*)$

Let's compute the left-hand side:

$$Y \cap \varepsilon^* D(12|34) = Y \cap \sum_{\substack{1,2 \in A \\ 3,4 \in B \\ d_A + d_B = d}} D(A, B; d_A, d_B)$$

• $d_A = 0$, • the component C_A is contracted to $\ell_1 \cap \ell_2$,

• $A = \{1,2\}$, because C_A is conctracted to $\ell_1 \cap \ell_2$ and the other marked points must be send to $q_3, \dots, q_{3d}$ that are disjoint from ℓ_1 and ℓ_2

• $p_3, \dots, p_{3d}$ lies on C_B and $\mu(p_i) = q_i$.
We also know that $C_A \cap C_B$ must be mapped to $\ell_1 \cap \ell_2$.
We have $3d-1$ special pts on C_B that must be send to $3d-1$ generic pts on $\mathbb{P}^2$

$\Longrightarrow$ There are N_d such maps!

• $d_B = 0$. We know that $\{3,4\} \subseteq B$ and we know that $\mu(p_3) = q_3$ $\mu(p_4) = q_4$.
$q_3 \neq q_4$ so there are no such maps!

• $d_A \geq 1$ $d_B \geq 1$. By a dimension count, we can see that the only contributions are given by $|A| = 3d_A + 1$, $|B| = 3d_B - 1$.

• We must choose $3d_A - 1$ marks for A, there are exactly $\binom{3d-4}{3d_A-1}$ possible choices.

• Then for each choice there are exactly

- Then, for each choice, there are exactly N_{d_A} maps sending these $3d_A-1$ marks on C_A to the corresponding $3d_A-1$ general point on $\mathbb{P}^2$
- For each such curve, we have d_A intersection points with ℓ_1 and d_A intersection points with ℓ_2.

  So we have d_A^2 possible choices for the marks 1 and 2

- For the curve C_B, there are N_{d_B} curves passing through the points corresponding to the marks of B.

- We have to choose how to glue C_A and C_B together. There are $d_A \cdot d_B$ possible choices for this,

$$\Longrightarrow \text{L.H.S. of } (*) \quad N_d + \sum_{\substack{d_A+d_B=d \\ d_A>0 \\ d_B>0}} \binom{3d-4}{3d_A-1} N_{d_A} N_{d_B} d_A^2 d_A d_B$$

let us look at the R.H.S. of $(*)$

$$Y_n \sum_{\substack{A \cup B = \{1,\dots,3d\} \\ 1,3 \in A \quad 2,4 \in B}} D(A,B;d_A;d_B)$$

$d_A + d_B = d$

- $Y \cap \varepsilon^* D(13|24)$ is non-empty only when $|A| = 3d_A$, $|B| = 3d_B$
- There are no curves with d_A or $d_B = 0$.
- Let $d_A \geq 1$. We have to choose $3d_A - 2$ among $3d-4$.
  We have $\binom{3d-4}{3d_A-2}$ possibilities of choosing such pts.
- There are N_{d_A} curves curves passing through q_3 and the other $3d_A - 2$ chosen pts.

  For each such curve, there are d_A intersection with the line ℓ_1. There are d_A choices where to put the mark p_1.

  For C_A, we have $N_{d_A} \cdot d_A$ degree d_A curves.
- By symmetry, we also have $N_{d_B} \cdot d_B$ degree d_B maps for the component C_B.
- We have to glue together C_A and C_B. There are $d_A \cdot d_B$ points for the intersection $\mu(C_A) \cap \mu(C_B)$.
  $\Rightarrow$ there are $d_A \cdot d_B$ choices where to put the node.

We finally obtain R.H.S. of (*)

$$\sum_{\substack{d_A+d_B=d \\ d_A>0\ d_B>0}} \binom{3d-4}{3d_A-2} d_A^2 d_B^2 N_{d_A} N_{d_B}.$$

$\square$