Caporasso - Harris recursion formula for plane curves of any genus

$/\mathbb{C}$

Goal: Count plane curves of degree d and genus g through $3d+g-1$ general points

Tool: "relative Gromov-Witten invariants"
= count of curves with specified local contact orders to a fixed line $L \subset \mathbb{P}^2$

Notation:

assigned pts

$\alpha = (\alpha_1, \alpha_2, \dots)$
$\beta = (\beta_1, \beta_2, \dots)$

unassigned pts

sequences of non-negative integers st only finitely many are $\neq 0$

$e_k = (0, \dots, 0, 1, 0, \dots)$

kth position

$|\alpha| = \alpha_1 + \alpha_2 + \dots$ without multiplicity

$I\alpha = \alpha_1 + 2\alpha_2 + 3\alpha_3$ with multiplicity

$I^\alpha = 1^{\alpha_1} \cdot 2^{\alpha_2} \cdot 3^{\alpha_3} \dots$ product of multiplicities

$\alpha \geq \alpha'$ if $\alpha_k \geq \alpha_k' \; \forall k$

$$\binom{\alpha}{\alpha'} := \binom{\alpha_1}{\alpha_1'} \cdot \binom{\alpha_2}{\alpha_2'} \cdot \dots$$

$d, \delta \in \mathbb{Z}_{\geq 0}$, α, β st $I\alpha + I\beta = d$

degree #nodes

$\mathbb{P}^N :=$ degree d plane curves

$$N = \binom{d+2}{2} - 1$$

$L \subset \mathbb{P}^2$ line , $\Omega = \{p_{i,j}\}_{1 \leq j \leq \alpha_i} \subseteq L$

Def: generalized Severi variety $V^{d,\delta}(\alpha, \beta)$
:= closure of locus of reduced plane curves of genus $g = \binom{d-1}{2} - \delta$ not containing L and st

$$\nu : X^\nu \to X$$

normalization of such a curve

$\exists\ |\alpha|$ pts $q_{i,j} \in X^\nu$ st $\nu(q_{i,j}) = p_{i,j}$
and $\exists\ |\beta|$ pts $r_{i,j} \in X^\nu$
st

$$\nu^* L = \sum i \cdot q_{i,j} + \sum i \cdot r_{i,j}$$

Rmk: Classically the Severi variety $V_{d,\delta}$ consists of irreducible curves

Ex: Fix d, δ $\qquad V^{d,\delta}(\alpha, \beta)$

- $\alpha = 0$, $\beta = (d, 0, \dots)$

  $\rightsquigarrow$ plane curves of degree d with δ nodes

- $\alpha = (1, 0, \dots)$, $\beta = (d-1, 0, \dots)$

  $\rightsquigarrow$ curves through 1 fixed pt on L

- $\alpha = 0$, $\beta = (d-2, 1, 0, \dots)$

  $\rightsquigarrow$ curves tangent to L

Dimension

Prop 2.1: $\dim V^{d,\delta}(\alpha, \beta) = \binom{d+1}{2} - \delta + |\beta|$

$$= 2d + g - 1 + |\beta|$$

Pf: "$\geq$"

$$\dim V^{d,\delta}(\alpha, \beta) \geq \underbrace{\binom{d+2}{2} - 1}_{N} - \delta - \underbrace{I\alpha - (I\beta - |\beta|)}_{d}$$

$$= \binom{d+1}{2} - \delta + |\beta|$$

"$\leq$" much harder

Degree

$N^{d,\delta}(\alpha,\beta) := \text{degree } V^{d,\delta}(\alpha,\beta)$

Caporasso-Harris recursion formula

$$N^{d,\delta}(\alpha,\beta) = \sum_{k:\beta_k>0} k\cdot N^{d,\delta}(\alpha+e_k, \beta-e_k)$$

← assign one more pt

$$+ \sum_{\substack{\alpha'\leq\alpha \\ \beta'\geq\beta}} I^{\beta'-\beta}\binom{\beta'}{\beta}\binom{\alpha}{\alpha'} N^{d-1,\delta'}(\alpha',\beta')$$

$X_0 = L \cup X$

$I\alpha' + I\beta' = d-1$

$\delta - \delta' + |\beta' - \beta| = d-1$

$0 \leq \delta' \leq \delta$

Proof idea: Let $H_p \subseteq \mathbb{P}^N$ be the hyperplane of curves containing $p \in L$

C-H study $V^{d,\delta}(\alpha,\beta) \cap H_p$

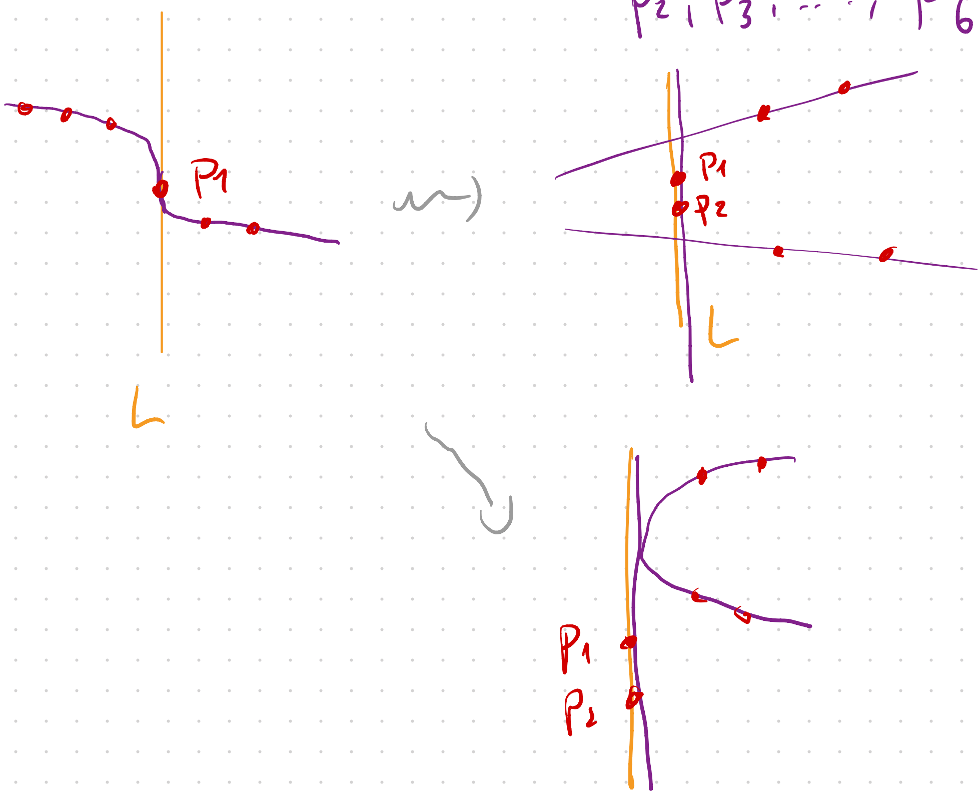
Example (Gathmann - Markwig)

$d = 3, \quad \delta = 1$

$\alpha = (0, 0, 1, 0, \dots)$

$\beta = 0$

$N^{d,\delta}(\alpha, \beta) = \#$ plane degree 3 with 1 node curves of contact order 3 to L at a fixed pt $p_1 \in L$ and passing through 5 additional general pts $p_2, p_3, \dots, p_6 \in \mathbb{P}^2$



Goal: Find components of
$V^{d,\delta}(\alpha,\beta) \cap H_p$ and intersection multiplicities

$V' :=$ irred comp of $V^{d,\delta}(\alpha,\beta) \cap H_p$

$[X_0] \in V'$ general pt

$\updownarrow$

$X_0 \subset \mathbb{P}^2$

1^{st} case: $L \not\subset X_0$

$[X_0] \in V^{d,\delta}(\alpha+e_k, \beta-e_k) \supseteq V'$

Prop 4.5: $V^{d,\delta}(\alpha,\beta)$ contains $[X_0]$ and is smooth there and has intersection multiplicity k with H_p along V'

Pf idea:

prove the analogous statement for the linear series of divisors on L

$$\pi : |\mathcal{O}_{\mathbb{P}^2}(d)| = \mathbb{P}^N \longrightarrow |\mathcal{O}_L(d)| = \mathbb{P}^d$$

$H_p \leftarrow$ $H := \{ D \in |O_L(d)| : D - p \geq 0 \}$

$$\Phi := \{ D \in |O_L(d)| : D = \sum i \cdot p_{i,j} + \sum i \cdot p'_{i,j} \text{ for some } p'_{i,j} \}$$

$\Phi \leftrightarrow V^{dis}(\alpha, \beta)$

$$\Psi := \{ D \in |O_L(d)| : D = \sum i \cdot p_{i,j} + k \cdot p + \sum i \cdot p'_{i,j} \text{ for some } p'_{i,j} \}$$

$\Psi \leftrightarrow V^{dis}(\alpha + e_k, \beta - e_k)$

$$D_0 = X_0 \cdot L \in |O_L(d)|$$

parametrize nbhd of D_0 in Φ

$x =$ coord on L

$p = \{x = 0\}$

coord for $p_{i,j}$

$$(\varepsilon, \varepsilon_{i,j}) \mapsto [f(x)] = [(x-\varepsilon)^k \cdot \prod (x - \lambda_{i,j})^i \prod (x - \mu_{i,j} - \varepsilon_{i,j})^i]$$

coord of $p'_{i,j}$ in D_0

$$f(x) = x^d + b_{d-1} x^{d-1} + \dots + b_0$$

defining equation for H is $b_0 = 0$

which pulls back to

$\varepsilon^{k} \cdot$ (poly in $\varepsilon_{i,j}$ $\neq 0$ at origin)

2nd case : $V' \subset V^{d,\delta}(\alpha, \beta) \cap H_p$ irred comp

st $[X_0] \in V'$ general pt

$$X_0 = X \cup L$$

$$X \in V^{d-1,\delta'}(\alpha', \beta')$$

Q: What is δ', α', β' ?

$\Gamma :=$ $\{[X_\gamma]\} \subset V^{d,\delta}(\alpha, \beta)$ curve through $[X_0]$

$\rightsquigarrow$ family of curves

$$\mathcal{X} \to \Gamma$$

In a nbhd of $[X_0]$ C-H define a family of curves $\mathcal{Y} \to B$

$$\xrightarrow{\pi}$$

nice section $Q_{i,j}, R_{i,j}$ $\subset$

$$
\begin{tikzcd}
\mathcal{Y} \arrow[r] \arrow[d, "f"'] & X^\nu \arrow[r] \arrow[d] & X \subset \Gamma \times \mathbb{P}^2 \arrow[r] \arrow[d] & \mathbb{P}^2 \\
B \arrow[r] & \Gamma^\nu \arrow[r] & \Gamma &
\end{tikzcd}
$$

$b_0 \in B$, $b_0 = b_0 \in \Gamma^\nu$, $b_0 \longmapsto [X_0] \in \Gamma$

$Q_{i,j}, R_{i,j}$ are closures of sections

$Q^*_{i,j}, R^*_{i,j} \subset$ section $\mathcal{Y}^* = f^{-1}(B \setminus b_0)$

s.t. $\pi(Q_{i,j}) = P_{i,j}$

and

$$\pi^* L \cap \mathcal{Y}^* = \sum i \cdot Q^*_{i,j} + \sum i \cdot R^*_{i,j}$$

$$Y_0 = f^{-1}(b_0) = \tilde{L} \cup Y \cup Z$$

π non-constant

$\leftarrow \pi$ constant

π maps $\tilde{L}$ to L

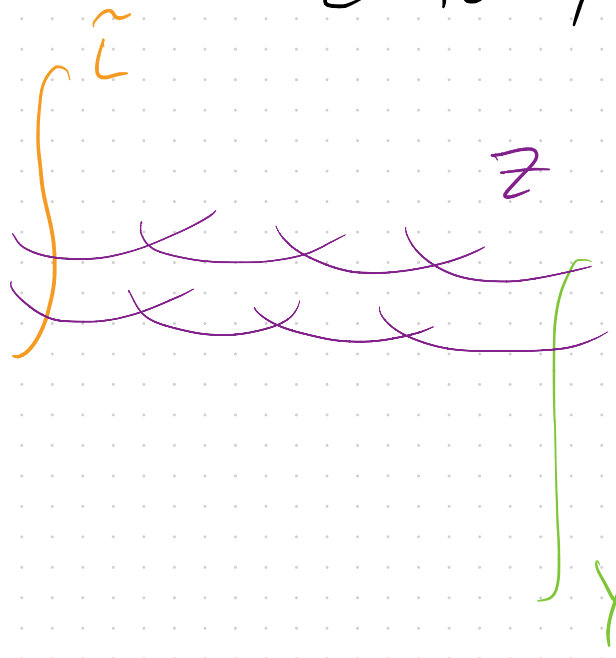
π maps those to X

$X_0 = X \cup L$

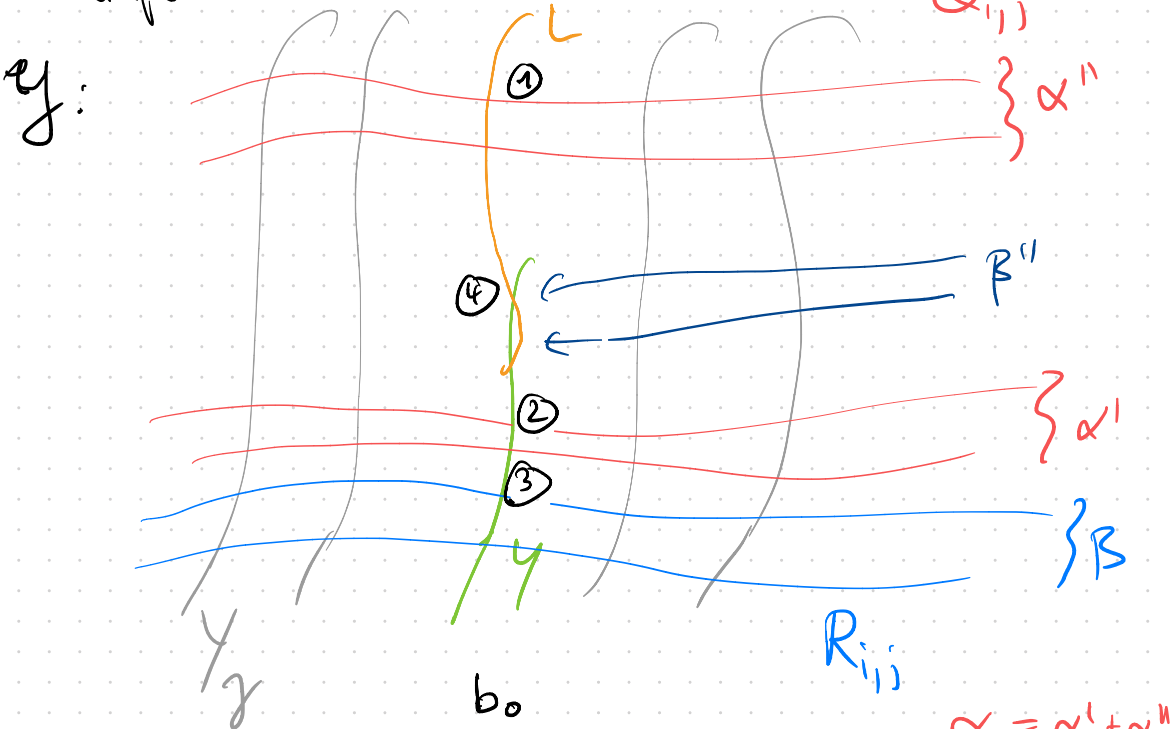
Assumption 1: $\pi|_{\tilde{L}}$ is an iso

Assumption 2: $Q_{i,j}$, $R_{i,j}$ do not meet Z

Assumption 3: Z consists of a disjoint union of chains of rational curves joining $\tilde{L}$ to Y

$Y_0 =$ 

Collapse Z:

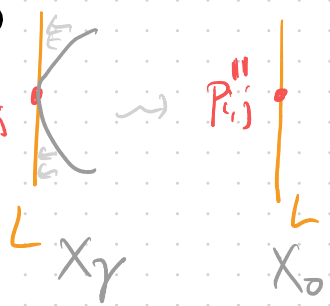


$\alpha = \alpha' + \alpha''$

$X_0 = X \cup L$

Fact: all $R_{i,j}$ must go through y

1



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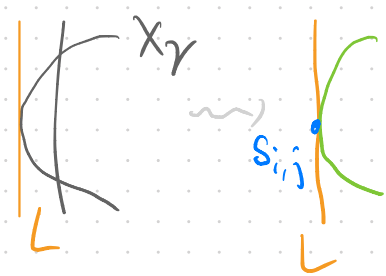

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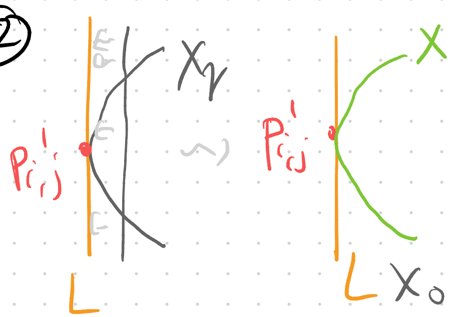
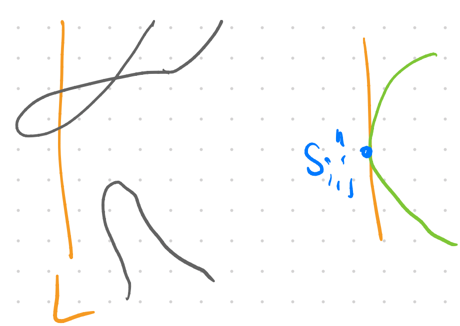


4



$\alpha' \leq \alpha$, $\beta' = \beta + \beta'' \geq \beta$

$$I\alpha' + I\beta' = d-1$$

$$\delta' = \delta - I\alpha' - I\beta' - (I\beta'' - |\beta''|)$$
$$= \delta - (d-1) + |\beta' - \beta|$$

<u>Prop 4.8</u>: In a nbhd of $[X_0]$ $V^{d,\delta}(\alpha,\beta)$ has

$$\binom{\beta'}{\beta} \cdot \frac{I^{\beta''}}{\operatorname{lcm} \beta''} \leftarrow \operatorname{lcm}\{\beta_i'' \neq 0\}$$

branches each of which has intersection multiplicity $\operatorname{lcm} \beta''$ with H_p along V'.

Deformation spaces of tacnodes

ith order tacnode

= curve singularity equivalent to origin in

$$y(y + x^i) = 0$$

deformation space $\pi : \mathcal{G} \to \Delta$

$\Delta = \mathbb{A}^{2i-1}$ with coord $(a_0, \dots, a_{i-2}, b_0, \dots, b_{i-1})$

$$\mathcal{G} = \{ f(x,y) = 0 \} \subseteq \Delta \times \mathbb{A}^2$$

$$f(x,y) = y^2 + (x^i + a_{i-2}x^{i-2} + \dots + a_0) \cdot y + b_{i-1}x^{i-1} + \dots + b_0$$

$\Delta_i :=$ closure of locus of $(a,b) \in \Delta$ s.t $S_{a,b}$ is reducible

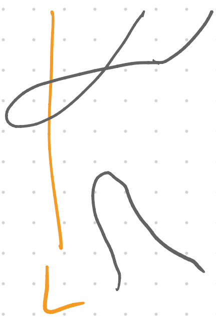
$\Updownarrow$

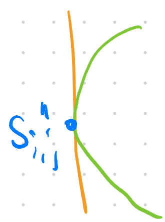
$S_{a,b}$ has i nodes

$= \{ b_{i-1} = \dots = b_0 = 0 \}$

$\Delta_{i-1} :=$ closure of locus of $(a,b) \in \Delta$ st $S_{a,b}$ has $i-1$ nodes

4





$\Delta_{i-1} \setminus \Delta_i$ Δ_i

Products of deformation spaces of tacnodes

new tacnodes

Redefine:

$$\Delta := \prod_{S''_{i,j}} \Delta$$

$S''_{i,j}$ in (4)

ith order tacnodes

$$\Delta_i := \prod_{S''_{i,j}} \Delta_i$$

$$\Delta_{i-1} := \prod_{S''_{i,j}} \Delta_{i-1}$$

Proof idea for Prop 4.8:

$\Lambda := \{r_{i,j}\}_{1 \le j \le B_i}$ subset of

$\{r'_{i,j}\}_{1 \le j \le B'_j}$

unassigned pts in normalization of X

s.t. $\forall i$

$$\{r_{i,1}, \dots, r_{i,B_i}\}$$

$$\subseteq \{r'_{i,1}, \dots, r'_{i,B'_i}\}$$

Rmk: There are $\binom{B'}{B}$ choices for Λ

relaxed local Severi variety W_Λ

:= closure of the locus of curves

X_t s.t. ① - ③ not ④

There is $\Phi_\Lambda : W_\Lambda \to \Delta$ ← product of deformation spaces

and locally around $[X_0]$

$$V^{d,\delta}(\alpha, \beta) = \bigcup_\Lambda \overline{\Phi_\Lambda^{-1}(\Delta_{i-1} \setminus \Delta_i)}$$

C-H show that $W = \operatorname{im} \Phi_\Lambda$ satisfies the assumptions in a technical lemma about products of deformation spaces of tacnodes (Lemma 4.3) and conclude that

$$W \cap \Delta_{i-1} = \Delta_i \cup \Gamma_1 \cup \dots \cup \Gamma_K$$

where $K = \frac{I^{\beta''}}{\operatorname{lcm} \beta''}$

and Γ_j are distinct reduced unibranch curves having intersection multiplicity lcm β" with Δ_i at the origin.

How does the Recursion formula follow?

Caporasso-Harris recursion formula

assign one more pt

$$N^{d,\delta}(\alpha,\beta) = \sum_{k:\,\beta_k>0} k\cdot N^{d,\delta}(\alpha+e_k,\ \beta-e_k)$$

$$+ \sum_{\substack{\alpha'\le\alpha\\ \beta'\ge\beta}} \underbrace{I^{\beta'-\beta}\binom{\beta'}{\beta}}_{\text{Prop 4.8}}\binom{\alpha}{\alpha'} N^{d-1,\delta'}(\alpha',\beta')$$

$X_0 = L \cup X$

$I\alpha' + I\beta' = d-1$

$\delta-\delta' + |\beta'-\beta| = d-1$

$0\le\delta'\le\delta$

choices for $\Omega'\subset\Omega$

assigned pts of X

Count of irred rational curves of degree d through $3d-1$ general pts $=: N_{d,\delta}$

$$N^{d,\delta}(\alpha,\beta) = \sum_{k:\,\beta_k>0} k\cdot N^{d,\delta}(\alpha+e_k,\beta-e_k)$$

$$+ \sum_{\substack{\alpha'\le\alpha \\ \beta'\geq\beta \\ \delta-\delta'+|\beta'-\beta|=d-1 \\ I\beta'+I\alpha'=d-1}} I^{\beta'-\beta}\binom{\beta'}{\beta}\binom{\alpha}{\alpha'} N^{d-1,\delta'}(\alpha',\beta')$$

Write n for $(n,0,0,\dots)$

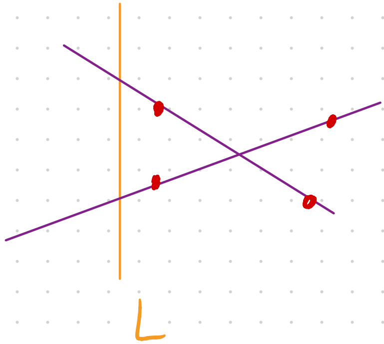
$d=1$: $N_{1,0} = N^{1,0}(0,1) = N^{1,0}(1,0) = 1$

$d=2$: $N_{2,0} = N^{2,0}(0,2) = N^{2,0}(1,1) = N^{2,0}(2,0)$
$= N^{1,0}(0,1) = 1$

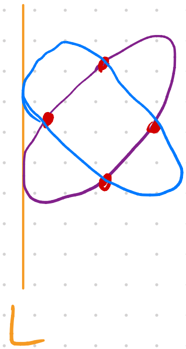
$d=3$: $N_{3,1} = N^{3,1}(0,3) = N^{3,1}(1,2) = N^{3,1}(2,1)$
$= N^{3,1}(3,0) + 2\cdot \underbrace{N^{2,0}(0,2)}_{=1}$

$$N^{3,1}(3,0) = \binom{3}{1}\cdot \underbrace{N^{2,0}(1,1)}_{=1}$$
$$+ N^{2,1}(0,2)$$
$$+ 2\cdot N^{2,0}(0,(0,1,0,\dots))$$

$N^{2,1}(0,2) = 3$



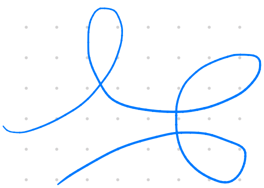
$N^{2,0}(0,(0,1,0,\dots)) = 2$



So $N_{3,1} = 2 + 3 + 3 + 2 \cdot 2 = 12$

$d = 4$: ⚠ $N_{4,3} \neq N^{4,3}(0,4)$

but $N_{4,3} = N^{4,3}(0,4) - \text{degree } W$



C-H compute $N^{4,3}(0,4) = 675$

$\text{degree } W = 55 = \binom{11}{2}$

$\Rightarrow N_{4,3} = 620$

C-H also write down a recursion formula for $N_{d,\delta}(\alpha,\beta)$ ← ired curves

which follows from the same arguments

$$N_{d,\delta}(\alpha,\beta) = \sum_{k:\beta_k>0} k \cdot N_{d,\delta}(\alpha+e_k, \beta-e_k)$$

$$+ \sum \frac{1}{\sigma} \binom{2d+g-2+|\beta|}{2d_1+g_1-1+|\beta^1|, \dots, 2d_k+g_k-1+|\beta^k|} \cdot \binom{\alpha}{\alpha^1, \dots, \alpha^k} \cdot \prod_{j=1}^{k} \binom{\beta^j}{\beta^j - \beta''^j} \cdot \prod I^{\beta''^j} \cdot \prod_{j=1}^{k} N_{d_j,\delta_j}(\alpha^j, \beta^j)$$

$X_0 = X \cup L$

$X = X_1 \cup \dots \cup X_k$

$\alpha' = \alpha^1 + \dots + \alpha^k \leq \alpha$

$\beta' = \beta^1 + \dots + \beta^k = \beta + \sum_{j=1}^{k} \beta''^j$ $\qquad |\beta''^j| > 0$

$d_1 + \dots + d_k = d-1$

$g - (g_1 + \dots + g_k) = |\beta''^1 + \dots + \beta''^k| + k$

Formula for σ:

$i, j \in \{1, .., k\}$

$i \sim j :\Leftrightarrow d_i = d_j, \; g_i = g_j$

$\alpha^i = \alpha^j, \; \beta^i = \beta^j$

$\beta''^i = \beta''^j$

$\sigma = \prod$ cardinalities of the equivalence classes