Motives Seminar SS21

Intro to:
Tropical Geometry

§0. Intro

Tropical Algebraic Geometry is sometimes called the "combinatorial shadow" of Algebraic Geometry

The name "tropical" is in honor of Imre Simon, a Brazilian mathematician

<u>Key Idea</u>: In Tropical Geometry we want to geometry over the max-plus (or min-plus) algebra $(\mathbb{R}, \oplus, \odot)$ $\rightsquigarrow$ we recover in this way a sort of linearised geometry

§1. Tropical Algebra:

Def.: $(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$ is called the tropical semiring and it is given as a set by $\mathbb{R}$ plus an extra element $\{-\infty\}$ with operations:

$$x \oplus y := \max\{x, y\} \quad \& \quad x \odot y := x + y$$

Rmk.: $(\mathbb{R} \cup \{-\infty\}, \oplus, \odot) =: \mathbb{T}$ is a semigroup w.r.t. $\oplus$, that is all the axioms are satisfied except the existence of an inverse for $\oplus$

Indeed $x \oplus x = x$ is idempotent.

In $\mathbb{T}$, the distributive law still holds:

$$x \odot (y \oplus z) = x \odot y \oplus x \odot z$$

Ex.: • $3 \odot (7 \oplus 11) = 3 \odot 11 = 14$

$3 \odot 7 \oplus 3 \odot 11 = 10 \oplus 14 = 14$

N.B.: $-\infty$ is the identity element for $\oplus$

0 is the identity element for $\odot$

Notice that for example:

$$(x \oplus y)^3 = x^3 \oplus y^3$$

Def.: A tropical polynomial is a PL function of the form:

$$p(\underline{x}) = p(x_1 \_ x_n) = a_1 \odot \underline{x}^{i_1} \oplus a_2 \odot \underline{x}^{i_2} \oplus \dots$$

where $i_j \in \mathbb{Z}^n$ and $a_j \in \mathbb{R}$ (where only fin. many are $\neq -\infty$)

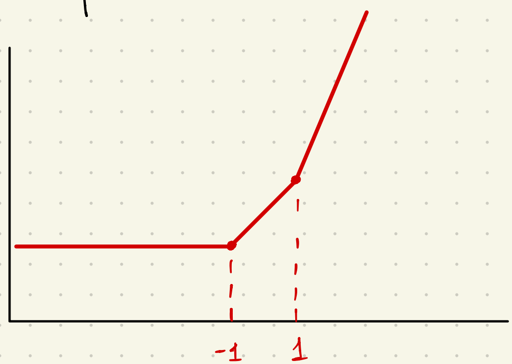
Every tropical polynomial represents a function $\mathbb{R}^n \xrightarrow{p} \mathbb{R}$

- p is continuous
- p is PL
- p is convex

Tropical Polynomial $\equiv$ PL convex functions on $\mathbb{R}^n$ w/ integer coefficients

Ex.:

$p(x) = 0 \oplus 1 \odot x \oplus x^2$



Spoiler: To a tropical polynomial in 2 var. we will associate some closed subset of $\mathbb{R}^2$:

$$V_{tr}(p) \subseteq \mathbb{R}^2$$

that will be our (very affine) tropical curve

§2. (Embedded) Tropical Varieties:

§2.1. Hypersurfaces:

We want to mime what we usually do for $P \in \mathbb{K}[x_1 \dots x_n] \rightsquigarrow V(P)$

$I \subseteq \mathbb{K}[x_1 \dots x_n] \rightsquigarrow V(I) = \bigcap_{P \in I} V(P)$

Def.: $[n=2]$ Given $p: \mathbb{R}^2 \longrightarrow \mathbb{R}$ tropical polynomial:

$$p(x,y) = \bigoplus_{i} a_{ij} \odot x^i \odot y^j$$

$$= \max\{ a_{ij} + i \cdot x + j \cdot y \}$$

Corner Locus

where $a_{ij} \in \mathbb{T}$ and only fin. many are $\neq -\infty$

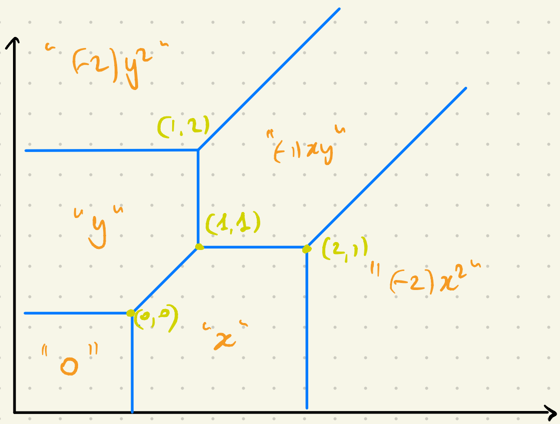
$$V_{tr}(p) \overset{\text{def.}}{:=} \{ (x,y) \in \mathbb{R}^2 \mid \exists (i,j) \neq (k,h) \text{ s.t. } p(x,y) = \text{"}a_{ij} x^i y^j\text{"} = \text{"}a_{kh} x^k y^h\text{"} \}$$

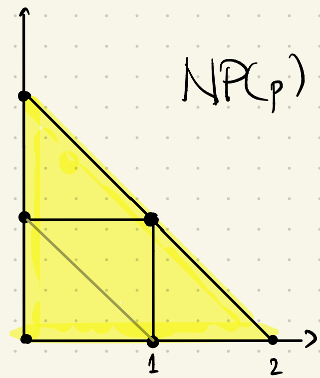
Rmk: $V_{tr}(p)$ is a PL graph in $\mathbb{R}^2$

Def.: Given a tropical polynomial $p: \mathbb{R}^2 \to \mathbb{R}$ we can define the Newton Polytope:

$$NP(p) := \text{ConvexHull}\{ j \in \mathbb{Z}^2 \mid a_j \neq -\infty \}$$

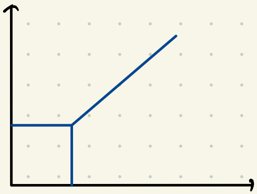
Ex.: $p(x,y) = 0 \oplus x \oplus y \oplus (-1)xy \oplus (-2)y^2 \oplus (-2)x^2$

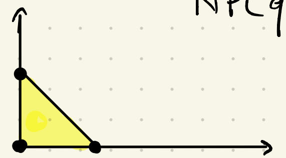




Ex.:

$q(x,y) = 0 \oplus x \oplus y$

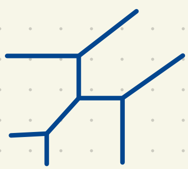




$NP(q)$

Def.: Let $\Delta_d := \text{Convex Hull}\big((0,0), (d,0), (0,d)\big)$ the d-fold standard simplex. The degree of a curve $V_{tr}(p) \subseteq \mathbb{R}^2$ is the smallest d s.t. $NP(d) \subseteq \Delta_d$

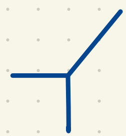
Ex.:



$= V(p)$

$NP(p) \subseteq \Delta_2$

$\Rightarrow \deg = 2$



$= V(q)$

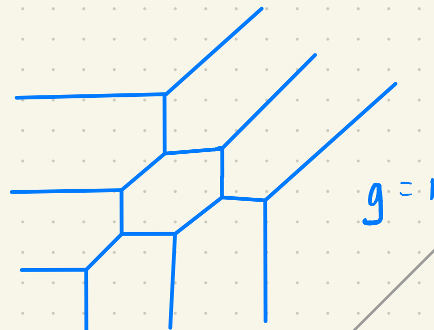
$NP(q) \subseteq \Delta_1$

$\Rightarrow \deg = 1$

Def.: Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ tropical polynomial w/ $C := V_{tr}(f)$ the associated planar tropical curve. The (embedded) genus is:

$$g(C) := b_1(C) = \dim H_1(C; \mathbb{Z})$$

Ex.:



"equivalently": C is smooth when $NP(p)$ is maximal, i.e. consists of d^2 triangles w/ area $= \frac{1}{2}$

Rmk: If C is "smooth" (or 3-valent + weight 1 +...) then

$$g = \frac{(d-1)(d-2)}{2}$$

as in the usual case in alg. geom.

If it's singular g will be smaller.

§2.2. Balancing Condition:

We already said that planar tropical curves have a structure of a graph embedded piecewise linearly in $\mathbb{R}^2$. Can we characterize them among all such graphs? Yes, we can! They are the "balanced" ones!

Def.: • A polyhedron in $\mathbb{R}^2$ n is an intersection of finitely many half spaces in $\mathbb{R}^2$ n.

• A polyhedral complex $\mathcal{P}$ in $\mathbb{R}^2$ n is a collection of polyhedra s.t.:

(a) $Q \in \mathcal{P} \Rightarrow \mathcal{P}$ contains all the faces of Q

(b) $Q, Q' \in \mathcal{P} \Rightarrow Q \cap Q'$ is a common face

For any $f \in \mathbb{T}[x,y]$, the associated $C = V_{tr}(f)$ is a polyhedral complex of pure dim $=1$ in $\mathbb{R}^2$.

every max. polyhedron in $\mathcal{P}$ has dim $= 1$

All the edges of C have rational slope ($\Leftarrow \exists$ dual subdivision $\subseteq NP(f) \rightsquigarrow$ Next Talk), more precisely:

$$\mathbb{R}E = \langle v_E \rangle_{\mathbb{R}} \quad \text{for some } v_E \in \mathbb{Z}^2$$

edge of C

We say that C is a rational polyhedral complex.

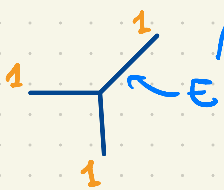
The edges of C carry naturally a weight.

Def.: Let E be an edge in $C = V_{tr}(f)$, then there will be at least two monomials $(i,j) \neq (k,h)$ where f reach the max on E

$$w_E := \underset{(i,j),(k,h)}{GCD}(|i-k|, |j-h|)$$

among all pairs

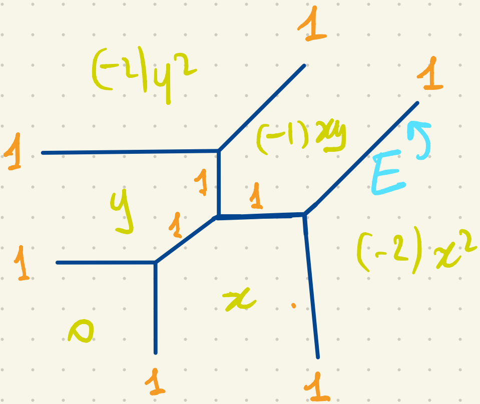
Ex.: $f = 0 + x + y$



$E = \{x = y \geq 0\}$

$w_E = GCD(|1-0|, |0-1|) = 1$

Ex.:

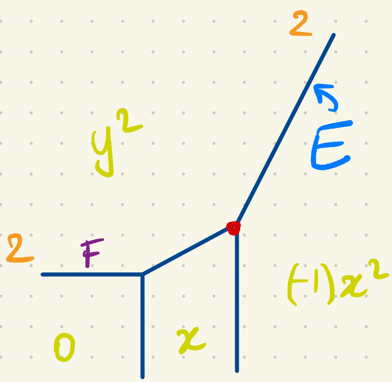


$$g = 0 \oplus x \oplus y \oplus (-1)xy \oplus (-2)x^2 \oplus (-2)y^2$$

$$E = \{ \text{"}(-1)xy\text{"} = \text{"}(-2)x^2\text{"} \}$$

$$w_E = GCD(|2-1|, |0-1|) = 1$$

Ex.:



$$q = 0 \oplus x \oplus y^2 \oplus (-1)x^2$$

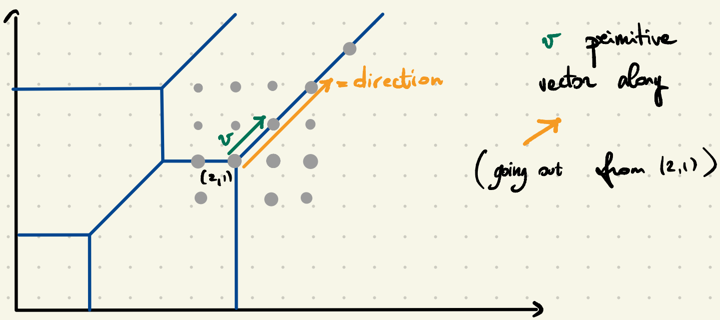
$$E \rightsquigarrow a_{02}, a_{20}$$

$$w_E = GCD(|0-2|, |2-0|) = 2$$

Def.: An integer vector $v \in \mathbb{Z}^2$ is called primitive if it's minimal among all integer vectors in that direction:

$$\forall w = \lambda v, \ \lambda \in \mathbb{R}, \ w \in \mathbb{Z}^2 \Rightarrow \lambda \in \mathbb{Z}$$

Ex.:

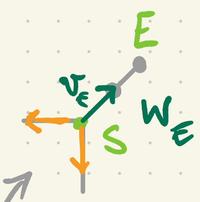


Prop: Let $C = V_{tr}(f)$ be a tropical curve. Then for every vertex s in C we have:

$$\boxed{\sum_{E \ni s} w_E \cdot v_{E/s} = 0}$$

Balancing Condition

edges

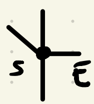


where w_E = weight of E and $v_{E/s}$ is the primitive vector along E, going in the outer direction w.r.t. s.

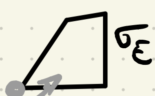
the "tension" is 0

Pf/Spoiler: the idea is that at each vertex

(see next week)



we can associate a dual polygon



where $l(\sigma_E) = w_E \Rightarrow \sum w_E \cdot v_E = 0$ □

Prop: Let $C \subset \mathbb{R}^2$ be a weighted rational polyhedral complex of pure dim $= 1$ s.t. the balancing condition holds.

i.e. we have a weight function on edges $w: \mathcal{E} \to \mathbb{N}$

Then there exists a tropical polynomial $f \in \mathbb{T}[x,y]$ s.t. $C = V_{tr}(f)$.

(Moreover f is unique up to rescaling "λf" $= \lambda + f$, $\lambda \in \mathbb{R}$)

Rmk: We cheated a bit here: with $C = V_{tr}(f)$ we really meant that the supports are the same and the weights matches.

§. 3. Tropicalization:

How can we relate objects from classical alg. geom. to tropical objects? $\rightsquigarrow$ via tropicalization

Amoebas:

Using analytic geometry, via a limiting process, we can construct a tropical var. associated to a (very affine) var. $X \subseteq (\mathbb{C}^*)^n$

We set: $$\mathrm{Log}: (\mathbb{C}^*)^n \longrightarrow \mathbb{R}^n$$
$$(z_1, \dots, z_n) \longmapsto (\log|z_1|, \dots, \log|z_n|)$$

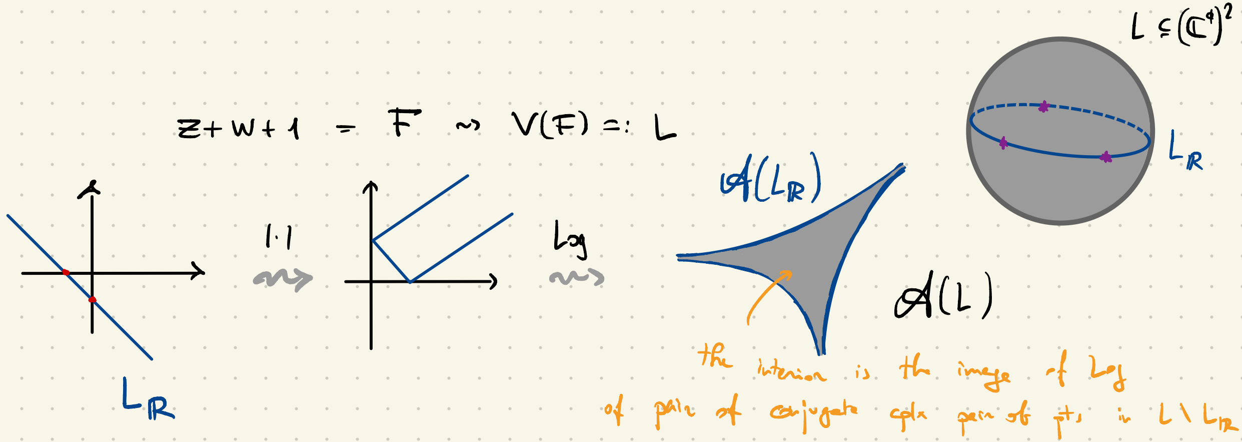
The image of $X \subseteq (\mathbb{C}^*)^n$ under the Log map is called the <u>amoeba</u> $\mathcal{A}(X)$ of X.

Let's consider an hypersurface $\rightsquigarrow F \in \mathbb{C}[z_1, \ldots, z_n]$,

$$\mathcal{A}(F) := \mathrm{Log}\left(V(F) \cap (\mathbb{C}^*)^n\right) \subseteq \mathbb{R}^n$$

Ex.:

$z + w + 1 = F \rightsquigarrow V(F) =: L$



Notice that the Log map could be taken w.r.t. any base t. So if we start w/ a family of curves / polynomials depending on some parameter $t \in \mathbb{R}$:

$$F_t = \sum \alpha_I(t) z^I$$

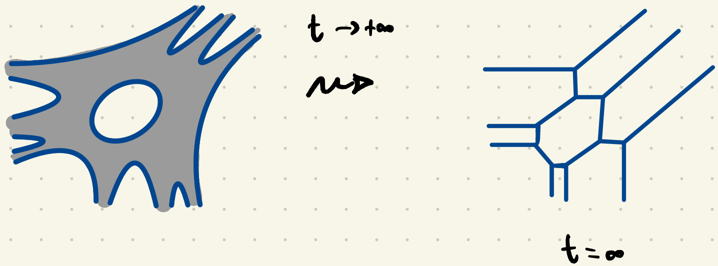
some function in t

then we can take Log_t coordinatewise (we assume $t > 1$):

$$\mathrm{Log}_t = \frac{1}{\log t} \mathrm{Log}$$

$$\rightsquigarrow \quad \mathcal{A}_t(F_t) := \mathrm{Log}_t\left(V(F_t) \cap (\mathbb{C}^*)^n \right) \subseteq \mathbb{R}^n$$

For $t \to \infty$ the amoebas get shrunk and collapse to their "spine":



Thm (Kapranov): $F_t = \sum \alpha_I(t) z^I$ family of complex polynomial.

We assume:

- $t \gg 1 \implies \underset{\{\alpha_I(t) \neq 0\}}{\text{Supp}}(F_t)$ stabilizes (say to some $S \subset \mathbb{Z}^n$)
- $\forall\, I \in S \ \exists\, b_I \in \mathbb{C}^*$ and $a_I \in \mathbb{R}$ s.t. $\alpha_I(t) \underset{t \to \infty}{\sim} b_I t^{a_I}$

Then $\lim_{t \to \infty} \mathcal{A}_t(F_t) = V_{\text{trop}}(f)$ w/ $f = \sum a_I x^I$

Rmk.: The limit is ambiguous $\Rightarrow$ we actually need the notion of metric Hausdorff spaces to make it precise.

Rmk: Why do we need families F_t? In the case $F_t \equiv F$ constant $\Rightarrow$ $\mathrm{Log}_t = \frac{1}{\log t} \mathrm{Log}$ is a rescaling $\Rightarrow$ it shrinks the amoeba to the origin w/ only asymptotic direction of $\mathcal{A}(F)$ surviving:

§ 3.2. Tropicalization: The Algebraic Side of the Story

§. 3.2.1. Valuations:

Def.: A valuation on a field $\mathbb{K}$ is a function

$$\nu : \mathbb{K} \longrightarrow \mathbb{R} \cup \{\infty\} \quad \text{s.t.:}$$

1. $\nu(a) = \infty$ iff $a = 0$
2. $\nu(a \cdot b) = \nu(a) + \nu(b)$
3. $\nu(a+b) \geq \min(\nu(a), \nu(b))$ if $\nu(a) \neq \nu(b)$ $\Rightarrow \nu(a+b) = \min(\nu(a), \nu(b))$

We denote $\mathrm{Im}(\nu) =: \Gamma_\nu$ the value group.

N.B.: given ν valuation $\Rightarrow \lambda\nu$ is a valuation too

$\Rightarrow$ We will assume $\boxed{1 \in \Gamma_\nu}$

Consider:

$$R_{\mathbb{K}} = \{ b \in \mathbb{K} \mid \nu(b) \geq 0 \} \qquad \mathfrak{m}_{\mathbb{K}} = \{ b \in \mathbb{K} \mid \nu(b) > 0 \}$$

We denote w/ $k_\nu := R/\mathfrak{m}$ the residue field of $(\mathbb{K}, \nu)$

Ex.: $\mathbb{Q}$ w/ p-adic evaluation

$$\nu_2(4/7) = 2 \qquad \nu_2(3/16) = -4$$

Ex.: $\boxed{\text{Puiseux Series}}$ The field of Puiseux Series w/ coefficients in $\mathbb{C}$ is:

$$\mathbb{C}\{\{t\}\} := \bigcup_{n \geq 1} \mathbb{C}((t^{\frac{1}{n}})) \rightsquigarrow \text{Alg closed}$$

Laurent series in var $t^{\frac{1}{n}}$

(actually $\mathbb{C}\{\{t\}\} = \overline{\mathbb{C}(t)}$)

The field $\mathbb{C}\{\{t\}\}$ has a natural valuation:

$$\nu : \mathbb{C}\{\{t\}\} \longrightarrow \mathbb{R}$$

$$c(t) = c_1 t^{a_1} + c_2 t^{a_2} + \dots \longmapsto a_1 = \text{lowest (non zero) exponent}$$

w/ $a_1 < a_2 < \dots$

<u>Rmk</u>: We can consider $\mathbb{C}(t) \hookrightarrow \mathbb{C}\{\{t\}\}$ since every rat. function $c(t)$ can be uniquely expanded as a Laurent series in t $\Rightarrow$ $\nu(c(t)) > 0$ if $c(0) = 0$

$\nu(c(t)) < 0$ if $c(0) =$ pole

Ex.:

- $c(t) = \dfrac{4t^2 - 7t^2 + 9t^5}{6 + 11t^4} = \dfrac{2}{3} t^2 - \dfrac{7}{6} t^3 + \dfrac{3}{2} t^5 \dots$

$\Rightarrow$ $\nu = 2$

- $c(t) = \dfrac{14t + 3t^2}{7t^4 + 3t^7 + 8t^8} = 2t^{-3} + \dfrac{3}{7} t^{-2} + \dots \Rightarrow \nu = -3$

Maclegan Strumfels

Lemma ([MS, 2.1.15]): $\mathbb{K}$ is alg. closed w/ non-trivial ν. Then $\mathbb{K}^* \twoheadrightarrow \Gamma_\nu$ (we still use Γ_ν even if we consider $\nu|_{\mathbb{K}^*}$) splits:

it's not always $\nu \equiv 0$

$$\exists\ \psi : \Gamma_\nu \longrightarrow \mathbb{K}^* \text{ s.t. } \nu(\psi(w)) = w$$

NOTATION: We denote $t^w := \psi(w) \in \mathbb{K}^*$

coherent w/ the splitting for $\mathbb{C}\{\{t\}\}$

Rmk: A field $\mathbb{K}$ w/ ν has an induced norm setting

$|a| := \exp(-\nu(a))$ for $a \neq 0$ & $|0| = 0$. We have:

exp in any base

- $|a| = 0$ iff $a = 0$
- $|ab| = |a|\,|b|$
- $|a+b| \leq |a| + |b|$ ~> a stronger statement holds: $|a+b| \leq \max\{|a|, |b|\}$

"$\forall x\ \exists n$ s.t. $|n \cdot x| > 1$"

ultrametric / non-archimedean

§. 3.2.2. Back to Tropicalization:

Let $\mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ w/ $(\mathbb{K}, \nu)$ non-trivial!

Given $f = \sum_{u \in \mathbb{Z}^n} c_u x^u$ we define:

max-plus vs min-plus

Def.: • $\mathrm{Trop}(f) := \max\left(-\nu(c_u) + u \cdot x \right)$ "$= \nu(c_u) \odot x^u$"

• Fix $\underline{w} \in (T_\nu)^n$ and let $W := \mathrm{Trop}(f)(\underline{w}) =$

$= \max\left(-\nu(c_u) + u \cdot \underline{w} \mid c_u \neq 0 \right)$

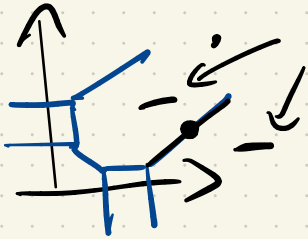
for $a \in \mathbb{K}$

$\bar{a}$ = class of a in $\kappa = \mathbb{K}/\mathfrak{m}_\nu$

$$\mathrm{in}_{\underline{w}}(f) \overset{\text{def.}}{:=} \overline{t^{+W} f(t^{w_1} x_1, \dots, t^{-w_n} x_n)}$$

$$= \overline{t^{+W} \sum_u c_u t^{-\underline{w} \cdot u} x^u}$$

$$= \sum_{\nu(c_u) - \underline{w} \cdot u = W}' \overline{c_u t^{-\nu(c_u)}} x^u \in \kappa_\nu[x_1^{\pm 1} \_ x_n^{\pm 1}]$$



$\mathrm{in}_w(f) =$

Rmk: $\mathrm{in}_{\underline{w}}(f)$ = the terms/monomials in f that reach the max in $\underline{w}$ for $\mathrm{Trop}(f)$

Ex.: $f = (t+t^2)x_0 + (2t^2)x_1 + (3t^4)x_2 \in \mathbb{C}\{\{t\}\}[x_0^{\pm 1}, x_1^{\pm 1}, x_2^{\pm 1}]$

$\nu = 1$ $\quad \nu = 2$ $\quad \nu = 4$

$\underline{w} = (0,0,0)$

$\mathrm{Trop}(f) = (-1)x_0 \oplus (-2)x_1 \oplus (-4)x_2$ $\qquad W = -1$

$\mathrm{in}_{\underline{w}}(f) = \overline{t^{-1} f(x_0, x_1, x_2)} = \overline{t^{-1}(t+t^2)}\, x_0 + \overline{t^{-1}(2t^2)} + \overline{t^{-1}(3t^4)}\, x_2 = x_0$

$\nu \Rightarrow \overline{(\ )} = 0$

$\nu \Rightarrow \overline{(\ )} = 0$

$W := \mathrm{Trop}(f)(\underline{w})$

For $w = (4,2,0)$ $\quad W = 4$

$\mathrm{in}_{\underline{w}}(f) = 2x_1 + 3x_2$

For $w = (2,1,0)$ $\quad W = 3$

$\mathrm{in}_w(f) = x_0 + 2x_1$

Thm (Kapranov, Again) [MS, 3.1.3]: $\mathbb{K}$ alg. closed w/ ν (non-triv). Fix a Laurent polynomial $F = \sum_{u \in \mathbb{Z}^n} c_u x^u$. The following are equal:

1. $V_{tr}(\mathrm{Trop}(F)) \subseteq \mathbb{R}^n$

2. $\{ w \in \mathbb{R}^n \mid \mathrm{in}_w(F) \text{ is not a monomial} \}$

3. $\overline{\{ (-\nu(y_1), \dots, -\nu(y_n)) \mid (y_1, \dots, y_n) \in V(F) \}}^{\mathbb{R}^n}$

closure in $\mathbb{R}^n$

↳ Pf (Kapranov):

$\boxed{(1)=(2)}$ $w \in V_+(\mathrm{Trop}(F))$ then by def the min $W := \mathrm{Trop}(F)(w)$ is reached twice $\Rightarrow in_{\underline{w}}(F)$ is not a monomial $\Rightarrow (1) \subseteq (2)$.

If $in_{\underline{w}}(F)$ is not a monomial $\Rightarrow$ the min is reached at least twice $\Rightarrow (2) \subseteq (1)$

$\boxed{(1) \supseteq (3)}$ (1) is closed $\Rightarrow$ it's enough to show that $(-\nu(y_1), \dots, -\nu(y_n)) \in (1)$ for y s.t. $F(y)=0$

But $F(y)=0 \Rightarrow \nu(0) = \nu(F(y)) = \nu(\sum c_u y^u) = \infty$

$\Rightarrow \nu(F(y)) > \nu(c_{u'} y^{u'}) \quad \forall u'$ s.t. $c_{u'} \neq 0$

$a = F(y)$

$b = -c_{u'} y^{u'}$

$\Rightarrow \neq \Rightarrow \nu(a+b) = \min(\nu(a), \nu(b))$

$\min(\nu(c_{u'} y^{u'}))$ among u' s.t. $c_{u'} \neq 0$ reached twice $\Rightarrow -\nu(y) \in (1)$

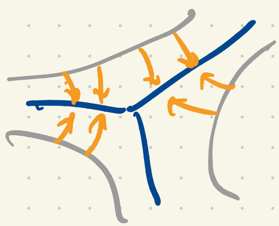
$\boxed{(3) \Rightarrow (1)}$ $\rightsquigarrow$ The hard part

Prop.: Let $F = \sum c_u x^u$, $\underline{w} \in \Gamma_\nu^n$.

Suppose $\mathrm{in}_{\underline{w}}(F)$ not a monomial and let $\alpha \in (\Bbbk^*)^n$ satisfying $\mathrm{in}_{\underline{w}}(F)(\alpha) = 0$.

Then $\exists\ y \in (\mathbb{K}^*)^n$ s.t. $F(y) = 0$, $-\nu(y) = \underline{w}$,

and $\overline{t^{-w} y} = \alpha$



$\square$

There is an analogue of Kapranov Thm for var.:

Def.: Let $I \subseteq \mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \rightsquigarrow V(I) \subseteq T^n$

Let $X := V(I) \subset T^n$, then:

$$\mathrm{Trop}(X) := \bigcap_{F \in I} V_{trop}(\mathrm{Trop}(F))$$

Rmk: If $\langle F_1 \_ F_d \rangle = I$ then in general

$$\mathrm{Trop}(X) \neq \bigcap_{i=1}^{d} V_{trop}(\mathrm{Trop}(F_i))$$

When it is "=" we say that $F_1, \dots, F_d$ form a tropical basis $\rightsquigarrow$ Thm: A trop. basis always $\exists$

Def.: A tropical variety ($\subseteq \mathbb{R}^n$) is any subset of $\mathbb{R}^n$ of the form $\mathrm{Trop}(X)$ for some $X \subseteq T^n$.

Thm (Fundamental Thm of Trop. Geom.):

$\mathbb{K}$ alg. closed w/ non-triv. ν, $I \subseteq \mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$

TFAE:

1) $\mathrm{Trop}(X) \subseteq \mathbb{R}^n$

2) $\{ \underline{w} \in \mathbb{R}^n \mid \mathrm{in}_{\underline{w}}(I)$ is not invertible / not a mon.$\}$

3) $\overline{\{ (-\nu(y_1), \dots, -\nu(y_n)) \mid \underline{y} \in X \}}^{\mathbb{R}^n}$

If X irred. $\Rightarrow \forall w \in T_\nu^n \cap \mathrm{Trop}(X) \Rightarrow \{ y \in X \mid -\nu(y) = w \}$ Zar. dense

§4. A Glimpse of Affine & Proj Tropical Varieties:

We worked until now w/ $X \subset T^n$ and w/ associated $\mathrm{Trop}(X) \subseteq \mathbb{R}^n$.

Idea:

$$\mathbb{G}_m \rightsquigarrow \mathbb{R}$$

$$\mathbb{A}^1 \rightsquigarrow \mathbb{T} = \mathbb{R} \cup \{-\infty\}$$

$$\mathbb{P}^1 \rightsquigarrow \overline{\mathbb{T}} \cong [0,1]$$

We search for tropicalizations of $\mathbb{A}^n$ & $\mathbb{P}^n$ some sort of trop. compactification of our very affine var. T^n.

§4.1. $\mathbb{A}^1$:

$\mathbb{K}$ alg. closed, ν non-triv. $\rightsquigarrow$ $\mathbb{A}^1_{\mathbb{K}}$

Def.: $\mathrm{Trop}(\mathbb{A}^1_{\mathbb{K}}) := \mathbb{T} = \mathbb{R} \cup \{-\infty\}$

$$\begin{array}{ccc} \mathbb{A}^1 & \xrightarrow{\mathrm{Trop}} & \mathbb{T} \\ \{0\} & \longmapsto & \{-\infty\} \end{array}$$

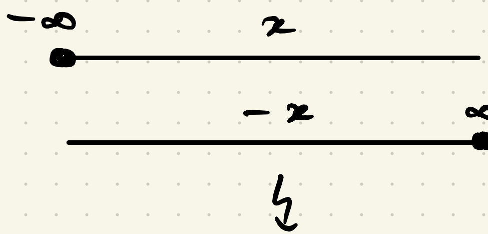
$$\mathrm{Trop}(\mathbb{A}^n_{\mathbb{K}}) = \mathbb{T}^n$$

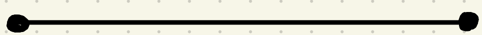
Rmk: We extend the topology on $\mathbb{R}$ to $\mathbb{T}$ taking (a,b) & $[-\infty, a)$ as open basis for the topology on $\mathbb{T}$.

§. 4.2. $\mathbb{TP}^1$:

Classically $\mathbb{P}^1$ is obtain by gluing two copies of $\mathbb{A}^1$ where we identify x w/ x^{-1}

↳ This suggest that we could do the same to get a model for $\mathrm{Trop}(\mathbb{P}^1)$: we glue together two copies of $\mathbb{T}$, identifying x w/ $-x$

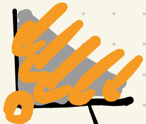


$\mathrm{Trop}(\mathbb{P}^1) :=$  $\simeq [0,1]$

Giving it the quot. topology inherited from $\mathbb{T}$

Or to get $\mathrm{Trop}(\mathbb{P}^1)$ we could have considered

$$\mathbb{P}^1 = \mathbb{A}^2 \setminus \{0\} / \mathbb{G}_m \rightsquigarrow \mathrm{Trop}(\mathbb{P}^1) := \mathbb{T}^2 \setminus \{\infty;\infty\} \Big/ \underset{\mathrm{Trop}(\mathbb{G}_m)}{\mathbb{R}}$$

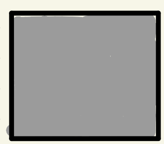


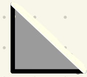
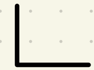
where the action is given by the diagonal action:

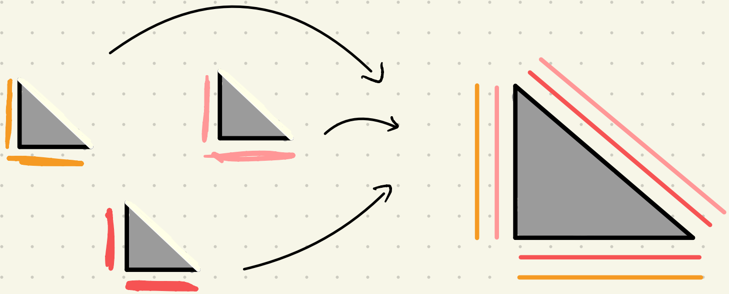
$$\lambda \odot (x,y) = (\lambda \odot x, \lambda \odot y) = (\lambda + x, \lambda + y)$$

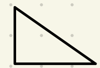
$\rightsquigarrow$ so we are modding out by $\mathbb{R}(1,1)$

Ex.: We use the fact that $\mathrm{Trop}(X \times Y) = \mathrm{Trop}(X) \times \mathrm{Trop}(Y)$ (we won't justify it now)

$\Longrightarrow \mathrm{Trop}(\mathbb{P}^1 \times \mathbb{P}^1) =$  $\cong [0,1]^2$

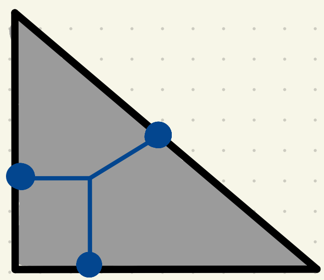
Ex.: $\mathbb{P}^2 \rightsquigarrow 3$ copies of $\mathbb{A}^2 =$  "$\partial \mathbb{A}^2$" $=$ 



"$\partial \mathbb{P}^2$" $=$ 

Ex (Line $\subseteq \mathbb{P}^2$): $\overline{Y} = V(x+y+z) \subseteq \mathbb{P}^2$

$\mathrm{Trop}(\overline{Y}) \rightsquigarrow$ Standard tropical line (coming from $\overline{Y} \cap T^2$) plus 3 points one in each copy of $\mathbb{R}$ in the boundary of $\mathbb{P}^2$:



Rmk: What we very roughly sketched is just a toy example of tropicalization of Toric varieties $\rightsquigarrow$ [MS, Chap. 6] $\rightsquigarrow$ $\exists$ Fund Thm for Trop. Toric Var.

§. 4.3. Affine Trop. Varieties:

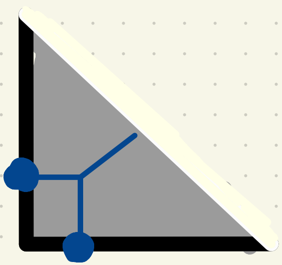
Def.: $f = \sum c_u x^u \in \mathbb{K}[x_1 \ldots x_n]$

$$\mathrm{Trop}(f)(w) := \max\left(-\gamma(c_u) + w \cdot u\right)$$

$$\mathrm{Trop}(f) : \mathbb{T}^n \longrightarrow \mathbb{T}$$

$$V_{tr}(\mathrm{Trop}(f)) = \{ w \in \mathbb{T}^n \mid \text{max reached at least twice} \}$$

Ex.: $t^{-1}x + (t^{-1}+t^2)y + (1+t^2)$ $\rightsquigarrow$ standard trop line w/ 2 more points on "$\partial \mathbb{A}^2$"



Def.: $w \in \mathbb{T}^n$, $f = \sum c_u x^u$, $W = \mathrm{Trop}(f)(w)$

$$\mathrm{in}_{\underline{w}}(f) = \sum_{\nu(c_u) + w\cdot u = W} \overline{t^{-\nu(c_u)} c_u}\, x^u \in k[x_1 \dots x_n]$$

(the sum is over u s.t. $c_u \neq 0$ if $\underset{W}{\underset{\shortparallel}{\mathrm{Trop}(f)(w)}} > -\infty$,
otherwise $\mathrm{in}_{\underline{w}}(f) := 0$)

$$\mathrm{In}_w(I) := \langle \mathrm{In}_{\underline{w}}(f) \mid f \in I \rangle$$

Thm 6.2.13: $Y \subseteq \mathbb{A}^n \rightsquigarrow I_Y =: I$. TFAE:

1) $\bigcap_{f \in I} V_{tr}(\mathrm{Trop}(f))$

2) $w \in \mathbb{T}^n$ s.t. $\mathrm{in}_w(I)$ does not contain monomials

3) the set: $\bigcup_{\sigma \subseteq \{1 \dots n\}} \mathrm{Trop}(Y \cap O_\sigma) \times \{-\infty^\sigma\}$

where $O_\sigma = \{x \in \mathbb{A}^n \mid x_i = 0$ for $i \in \sigma$ & $x_j \neq 0$ for $j \notin \sigma\}$