Tropical Gromov-Witten Invariants

May 25,2021

Problem (Enumerative problem in $\mathbb{C}P^2$)

Compute the number

$$N^{\mathrm{irr}}(g,d) \text{ (resp. } N(g,d))$$

of irreducible (resp. all) curves in $\mathbb{C}P^2$ *of degree* d *and genus* g *passing through a collection* $\mathcal{Z} = \{z_1, \cdots, z_{3d-1+g}\}$ *of* $(3d-1+g)$ *points in* $\mathbb{C}P^2$ *in general position.*

Remark

The number $N^{\mathrm{irr}}(g,d)$ (resp. $N(g,d)$) is finite and does not depend on the choice of collection of points $\mathcal{Z}$ as long as the choice is generic.

Definition

The number $N^{\mathrm{irr}}(g,d)$ (resp. $N(g,d)$) is known as the *Gromov-Witten invariant* (resp. *multicomponent Gromov-Witten invariant*) of $\mathbb{C}P^2$.

Remark

- The number $N^{\mathrm{irr}}(0,d)$ was given by Kontsevich.
- The number $N(g,d)$ was given by Caporaso-Harris.
- The number $N(g,d)$ determines $N^{\mathrm{irr}}(g,d)$, and vice versa (cf. Caporaso-Harris).

The following table lists some numbers

$N^{\mathrm{irr}}(g,d)$					$N(g,d)$				
	$d=1$	$d=2$	$d=3$	$d=4$		$d=1$	$d=2$	$d=3$	$d=4$
$g=0$	1	1	12	620	$g=-1$	0	3	21	666
$g=1$	0	0	1	225	$g=0$	1	1	12	675
$g=2$	0	0	0	27	$g=1$	0	0	1	225
$g=3$	0	0	0	1	$g=2$	0	0	0	27

Definition

Define

$$N^{\text{irr}}_{\text{trop}}(g,d) \text{ (resp. } N_{\text{trop}}(g,d))$$

to be the number of irreducible (resp. all) tropical curves of genus g and degree d passing through $\mathcal{P} = \{p_1, \cdots, p_{3d-1+g}\}$ of $(3d-1+g)$ points in $\mathbb{R}^2$ in general position (counted with the multiplicity).

The number $N^{\text{irr}}_{\text{trop}}(g,d)$ (resp. $N_{\text{trop}}(g,d)$) is called the *tropical Gromov-Witten invariants* (resp. *multi-component tropical Gromov-Witten invariants*) of $\mathbb{R}^2$.

Theorem (Mikhalkin Correspondence Theorem)

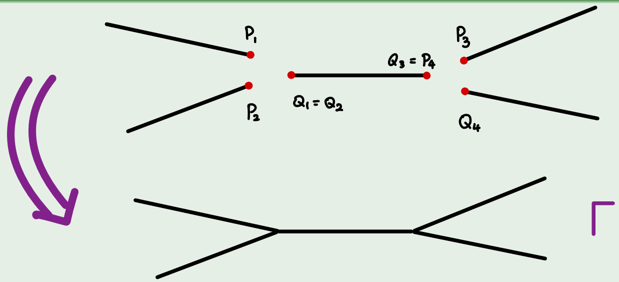
The number $N^{\text{irr}}(g,d)$ (resp. $N(g,d)$) equals the number $N^{\text{irr}}_{\text{trop}}(g,d)$ (resp. $N_{\text{trop}}(g,d)$).

The first goal of this talk is to study the number $N^{\text{irr}}_{\text{trop}}(g,d)$ (resp. $N_{\text{trop}}(g,d)$) and its combinatorial structure.

Definition (Graph)

Let $I_1, \ldots, I_k$ be closed (bounded or half bounded) real intervals. Choose some (not necessarily distinct) boundary points $P_1, \ldots, P_r$ and $Q_1, \ldots, Q_r$ of the intervals $I_1 \coprod \ldots \coprod I_k$. The topological space Γ that is obtained by identifying P_i and Q_i for all $i = 1, \ldots, r$ in $I_1 \coprod \ldots \coprod I_k$ is called a *graph*. A *graph* is called *connected* if it is connected as a topological space.

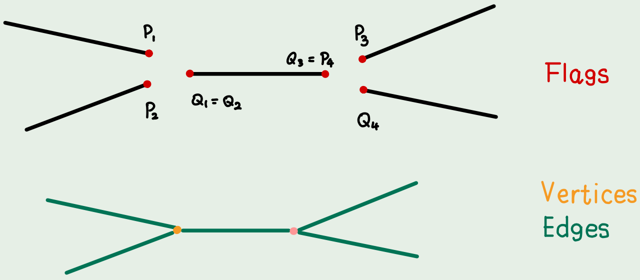
Example



Definition

1. The boundary points of the closed intervals $I_1, \ldots, I_k$ are called the *flags* of Γ.
2. The images of the flags in Γ are called the *vertices* of Γ. If F is a flag, its image in Γ (a vertex of Γ) will be denoted by ∂F.
3. Let V be a vertex. Define the *valence* of V, $\mathrm{valence}(V)$, as the number of flags F such that $\partial F = V$.
4. The open intervals $\mathrm{Int}(I_1), \ldots, \mathrm{Int}(I_k)$ are called the *edges* of Γ. A flag F belongs to exactly one edge of Γ which shall be denoted by $[F]$.
5. An edge is called *bounded* if its corresponding open interval is bounded, and *unbounded* if otherwise. The unbounded edges will also be called *ends* of Γ.

Example



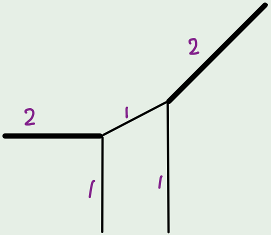
Convention

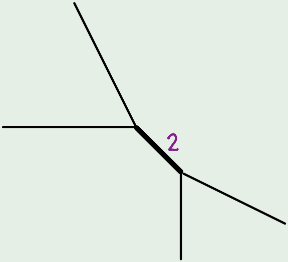
- Γ' denotes the set of flags.
- Γ^0 denotes the set of vertices.
- Γ^1_0 denotes the set of bounded edges.
- Γ^1_∞ denotes the set of unbounded edges.

Definition

A *weighted graph* is a graph Γ together with *weights*, i.e. natural numbers prescribed to the edges. That is to say, if $E_1, \ldots, E_k$ are edges of Γ, the weights are natural numbers $w_1, \ldots, w_k$ associated to the edges $E_1, \ldots, E_k$ respectively.

Example (Weighted graph)





Definition

A *parametrized tropical curve* is a pair (Γ, h) where Γ is a weighted graph and $h : \Gamma \to \mathbb{R}^2$ is a continuous map such that:

1. Γ is an *abstract tropical curve*, i.e. a graph such that all vertices have valence at least 3.
2. If e is an edge of Γ, then the map $h : e \hookrightarrow \Gamma \to \mathbb{R}^2$ takes the form:

$$h(t) = a + t \cdot v$$

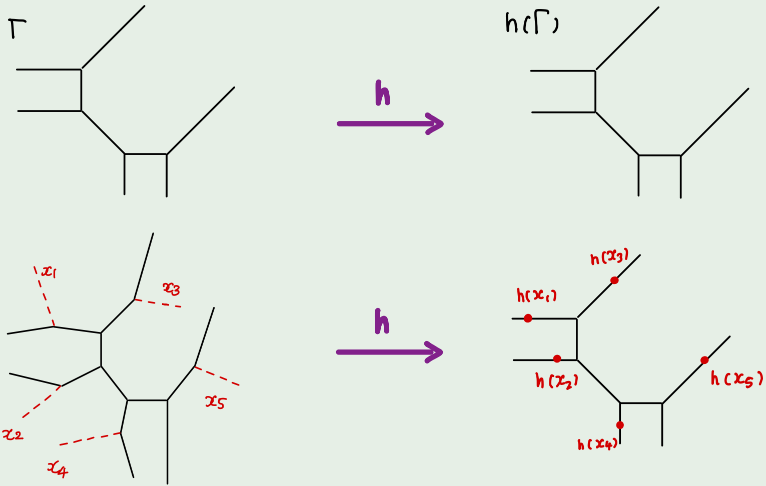
   where $a \in \mathbb{R}^2$ and $v \in \mathbb{Z}^2$. That is to say h is "affine linear with rational slop".
3. At every vertex $V \in \Gamma^0$, the following *balancing condition* is satisfied. Let $e_1, \ldots, e_k$ be edges adjacent to V, and let $w_1, \ldots, w_k$ be their weights. Let $v_1, \ldots, v_m \in \mathbb{Z}^2$ be the primitive integer vectors at the point $h(V)$ in the direction of the edges $h(e_j)$ (we take $v_j = 0$ if $h(e_j)$ is a point). We have

$$\sum_{j=1}^{k} w_j v_j = 0.$$

Definition

The image $h(\Gamma)$ shall be called the *tropical curve* of (Γ, h).

Example (Parametrized tropical curve)



Definition

If

$$f(x,y) = \text{“}\sum_{i,j} a_{ij}x^i y^j\text{”} := \max_{i,j}(a_{ij} + ix + iy)$$

is a tropical polynomial, we let $V_f \subset \mathbb{R}^2$ be the corresponding *tropical hypersurface*, i.e.

$$V_f := \left\{(x_0, y_0) \in \mathbb{R}^2 \mid \exists (i,j) \neq (k,l), f(x_0,y_0) = \text{“}a_{ij}x_0^i y_0^j\text{”} = \text{“}a_{kl}x_0^k y_0^l\text{”}\right\}$$

Theorem (Mikhalkin)

Any tropical curve can be identified with a tropical hypersurface V_f for some polynomial f. Conversely, any tropical hypersurface V_f in $\mathbb{R}^2$ can be parametrized by a tropical curve.

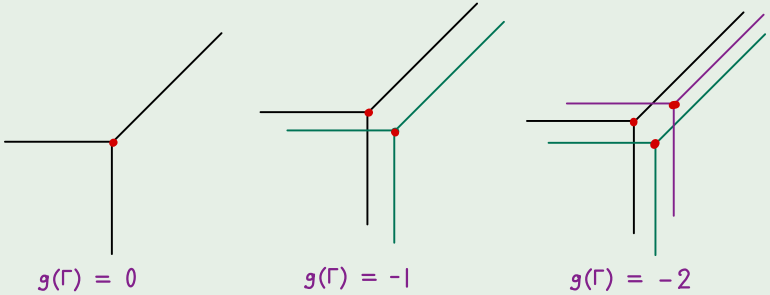
Definition

The *genus* of a graph Γ is defined to be

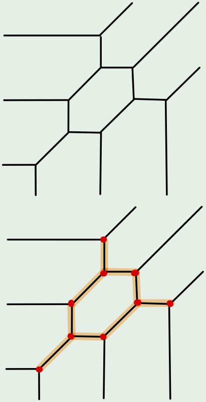
$$g(\Gamma) := 1 - \#\Gamma^0 + \#\Gamma^1_0.$$

The *genus* of a parametrized tropical curve (Γ, h) is defined to be the genus of the graph Γ. The *genus* of a tropical curve $h(\Gamma)$ is the minimum genus among all parameterizations of C.

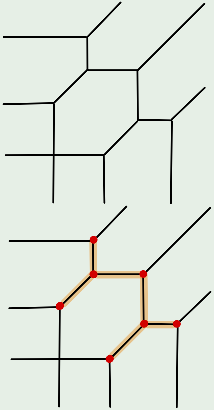
Example (Genus)



Example (Genus)



$$g(\Gamma) = 1 - \#\Gamma^0 + \#\Gamma^1_0 = 1 - 9 + 9 = 1$$



$$g(\Gamma) = 1 - \#\Gamma^0 + \#\Gamma^1_0 = 1 - 7 + 6 = 0$$

Let $\mu_1, \ldots, \mu_m \in \mathbb{Z}^2$ be primitive integer vectors pointing out the direction of unbounded edges $h(e_1), \ldots, h(e_m)$ of $h(\Gamma)$. Assume that $w_1, \ldots, w_m$ are the weights of $e_1, \ldots, e_m$. The primitive integer vectors $\mu_1, \ldots, \mu_m \in \mathbb{Z}^2$ can be reordered as

$$\mu_{(1,1)}, \ldots, \mu_{(1,s_1)}, \quad \mu_{(2,1)}, \ldots, \mu_{(2,s_2)}, \quad \ldots, \quad \mu_{(q,1)}, \ldots, \mu_{(q,s_q)}$$

such that $\sum_i s_i = m$ and

$$\begin{cases} \mu_{(i,s)} = \mu_{(j,t)} & \text{if } i = j \text{ for any } s, t \\ \mu_{(i,s)} \neq \mu_{(j,t)} & \text{if } i \neq j \text{ for any } s, t \end{cases}.$$

The weights also inherit this ordering. One defines that

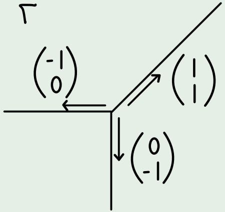
$$\tau_i = \sum_{t=1}^{s_i} w_{(i,t)} \mu_{(i,t)}$$

By the balancing condition, one sees that $\sum_i \tau_i = 0$.

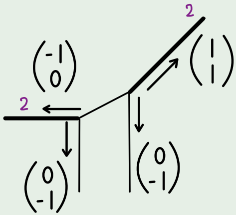
Definition (Degree)

The *degree* of a parametrized tropical curve (Γ, h) is the set $\mathcal{T} = \{\tau_1, \ldots, \tau_q\} \subset \mathbb{Z}^2$. If the degree is the set $\Delta_d := \left\{ \begin{pmatrix} -d \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -d \end{pmatrix}, \begin{pmatrix} d \\ d \end{pmatrix} \right\}$, we call the parametrized tropical curve (Γ, h) is of degree d.

Example (Degree)

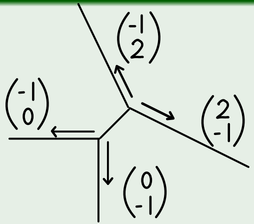


$$\text{degree} = \left\{ \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

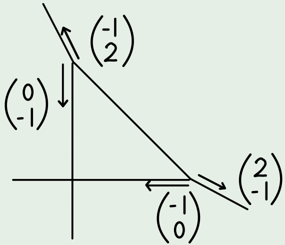


$$\text{degree} = \left\{ \begin{pmatrix} -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\}$$

Example (Degree)



$$\text{degree} = \left\{ \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}0\\-1\end{pmatrix}, \begin{pmatrix}-1\\2\end{pmatrix}, \begin{pmatrix}2\\-1\end{pmatrix} \right\}$$



$$\text{degree} = \left\{ \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}0\\-1\end{pmatrix}, \begin{pmatrix}-1\\2\end{pmatrix}, \begin{pmatrix}2\\-1\end{pmatrix} \right\}$$

Lemma

Let C be a tropical curve, and let V be a 3-valent vertex of C. Let w_1, w_2, w_3 be the weights of the edges adjacent to V and let v_1, v_2, v_3 be the primitive integer vectors in the direction of the edges. Then, the following holds

$$w_1 w_2 |\det[v_1, v_2]| = w_2 w_3 |\det[v_2, v_3]| = w_3 w_1 |\det[v_3, v_1]|.$$

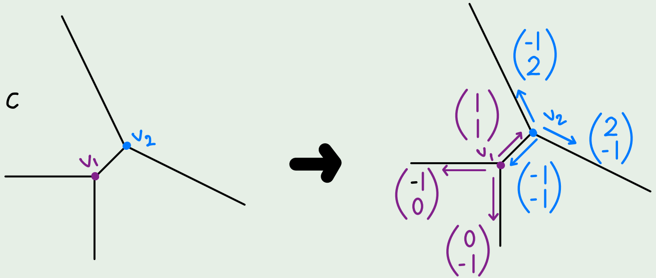
Proof.

Note that the determinant $|\det[v_1, v_2]|$ is the area of the parallelogram spanned by v_1 and v_2. The balancing condition tells us that $w_1 v_1 + w_2 v_2 + w_3 v_3 = 0$. Say $v_i = \begin{pmatrix} a_i \\ b_i \end{pmatrix}$. Then, $\det[v_1, v_2] = a_1 b_2 - a_2 b_1$ and $\det[v_2, v_3] = a_2 b_3 - a_3 b_2$. Moreover, the balancing condition says $w_1 a_1 + w_2 a_2 + w_3 a_3 = w_1 b_1 + w_2 b_2 + w_3 b_3 = 0$. So, $w_1 w_2 a_1 b_2 - w_1 w_2 a_2 b_1 = w_2 w_3 a_2 b_3 + w_3 w_1 a_3 b_1 + w_3^2 a_3 b_3 = w_2 w_3 a_2 b_3 - w_2 w_3 a_3 b_2$. □

Definition (Multiplicity)

Let C be a tropical curve. The *multiplicity* of C at its 3-valent vertex V is the positive integer $w_1 w_2 |\det[v_1, v_2]|$, denoted by $\mathrm{mult}_V(C)$.

Example



$$\mathrm{mult}_{v_1}(C) = \left| \det \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right| = 1$$

$$\mathrm{mult}_{v_2}(C) = \left| \det \begin{bmatrix} -1 & -1 \\ 2 & -1 \end{bmatrix} \right| = 3$$

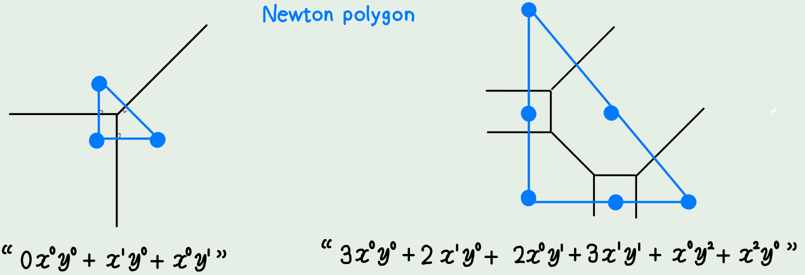
Definition

Let

$$f(x,y) = \text{“}\sum_{(i,j)\in A} a_{ij}x^i y^j\text{”} := \max_{(i,j)\in A}(a_{ij} + ix + iy)$$

be a tropical polynomial where $A \subset \mathbb{Z}^2_{\geq 0}$ is a finite subset. The *Newton polygon* of f is defined to be the convex hull $\Delta := \mathrm{ConvexHull(A)}$.

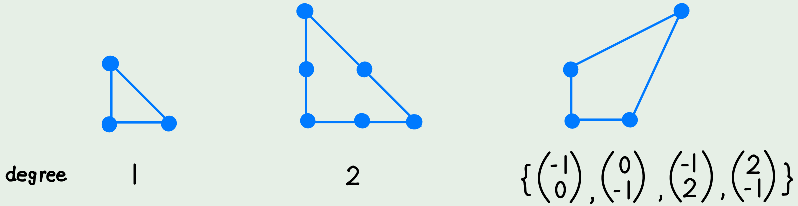
Example



Remark

The degree of the graph of V_f is determined by the Newton polygon Δ of f. For each side $\Delta' \subset \partial\Delta$ we take the primitive integer outward normal vector and multiply it by the lattice length of Δ' to get the degree of C.

Example



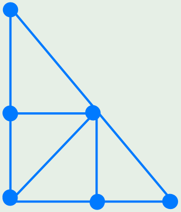
Definition

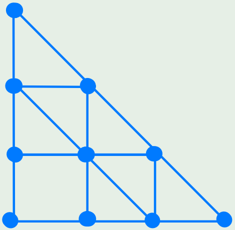
Let Δ be a Newton polygon in $\mathbb{R}^2$. Let $\Delta_1, \ldots, \Delta_k$ be a collection of convex lattice polygons (given as convex hulls of their vertices in $\mathbb{Z}^2$), such that their interiors do not intersect, and such that their union is equal to Δ. Then the set

$$\mathrm{Sub}(\Delta) = \{\Delta_1, \ldots, \Delta_k\}$$

is called a *subdivision* of Δ. It is called *regular* if it is dual to a tropical curve. It is called *simple* if it contains only triangles and parallelograms.

Example

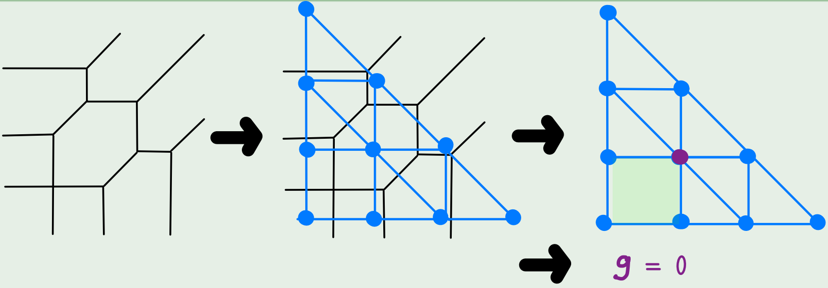




Lemma

Every graph of a tropical hypersurface V_f has a subdivision dual to it. The number of unbounded edges counted with multiplicities equals $\#(\partial\Delta \cap \mathbb{Z}^2)$. The genus of V_f equals the number of interior vertices of this subdivision minus the number of parallelograms if the subdivision is simple.

Example



Definition

A parameterized tropical curve $h : \Gamma \to \mathbb{R}^2$ is called *simple* if it satisfies to all of the following conditions.

- The graph Γ is 3-valent.
- The map h is an immersion.
- For any $y \in \mathbb{R}^2$ the inverse image $h^{-1}(y)$ consists of at most two points.
- If $a, b \in \Gamma$, $a \neq b$, are such that $h(a) = h(b)$ then neither a nor b can be a vertex of Γ.

A tropical curve is called *simple* if it admits a simple parameterization.

Lemma

A tropical curve is simple if and only if its subdivision consists of triangles and parallelograms only.

Definition

Points $p_1, \ldots, p_k \in \mathbb{R}^2$ are said to be *in general position tropically* if for any tropical curve $h : \Gamma \to \mathbb{R}^2$ of genus g and with s ends such that $k \geq g + s - 1$ and $p_1, \ldots, p_k \in h(\Gamma)$ we have the following conditions.

1. The tropical curve $h : \Gamma \to \mathbb{R}^2$ is simple.
2. Inverse images $h^{-1}(p_1), \ldots, h^{-1}(p_k)$ are disjoint from the vertices of Γ.
3. $k = g + s - 1$.

Lemma

Two distinct points $p_1, p_2 \in \mathbb{R}^2$ are in general position tropically if and only if the slope of the line in $\mathbb{R}^2$ passing through p_1 and p_2 is irrational.

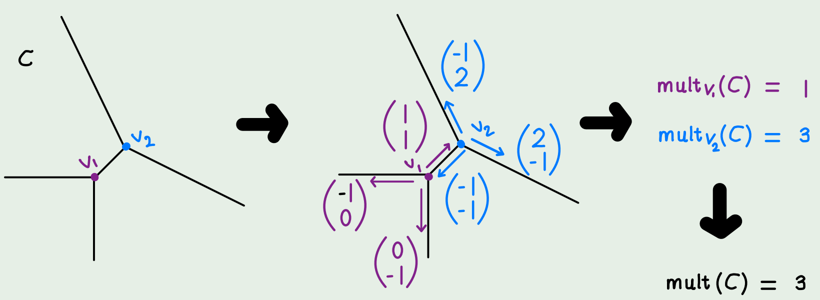
Proof.

If the slope of the line in $\mathbb{R}^2$ passing through p_1 and p_2 is rational, we can find a tropical line $h : \Gamma \to \mathbb{R}^2$ of genus 0 and of three ends with p_1 as its vertex, and $p_2 \in h(\Gamma)$. This contradicts condition (2). □

Definition

The multiplicity of a tropical curve $C \subset \mathbb{R}^2$ of degree Δ and genus g, denoted by $\mathrm{mult}(C)$, equals to the product of the multiplicities of all the 3-valent vertices of C.

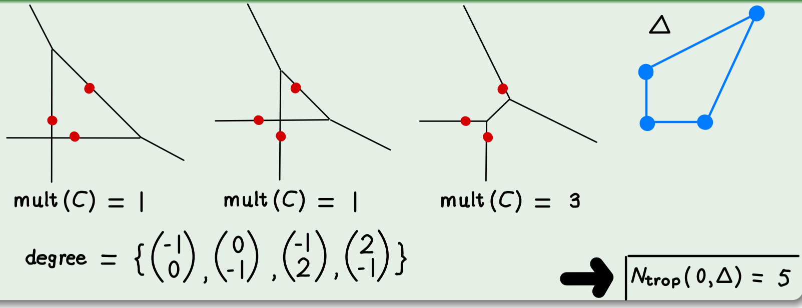
Example



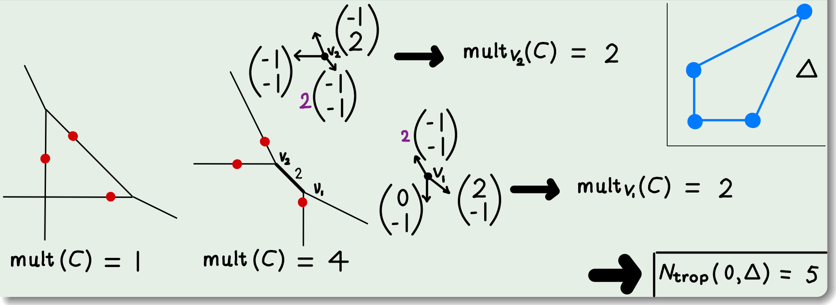
Definition

Let $\mathcal{P}$ be a configuration of points in tropical general position. Define the number $N^{\text{irr}}_{\text{trop}}(g, \Delta)$ to be the number of irreducible tropical curves of genus g and degree Δ passing via $\mathcal{P}$ where each such curve is counted with the multiplicity. Similarly we define the number $N_{\text{trop}}(g, \Delta)$ to be the number of all tropical curves of genus g and degree Δ passing via $\mathcal{P}$.

Example



Example



Theorem (Mikhalkin; Markwig)

The numbers $N^{\mathrm{irr}}_{\mathrm{trop}}(g,\Delta)$ and $N_{\mathrm{trop}}(g,\Delta)$ are finite and do not depend on the choice of $\mathcal{P}$.

Next, we want to understand the λ-increasing lattice paths and their "multiplicities", because they appear in the following interesting theorem.

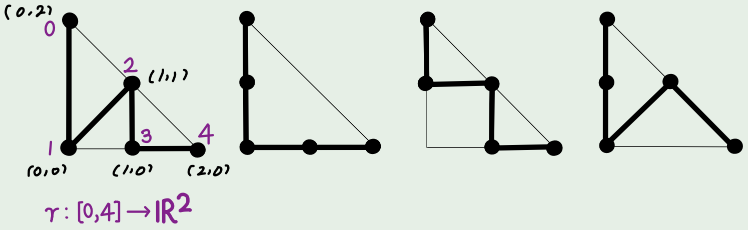
Theorem

The number $N_{\mathrm{trop}}(g, \Delta)$ is equal to the number of λ-increasing lattice paths $\gamma : [0, s+g-1] \to \Delta$ with $\gamma(0) = p$ and $\gamma(s+g-1) = q$ (counted with multiplicities).

Definition

A path $\gamma : [0, n] \to \mathbb{R}^2$, $n \in \mathbb{N}$, is called a *lattice path* if $\gamma|_{[j-1,j]}$, $j = 1, \ldots, n$ is an affine-linear map and $\gamma(j) \in \mathbb{Z}^2$, $j \in 0, \ldots, n$.

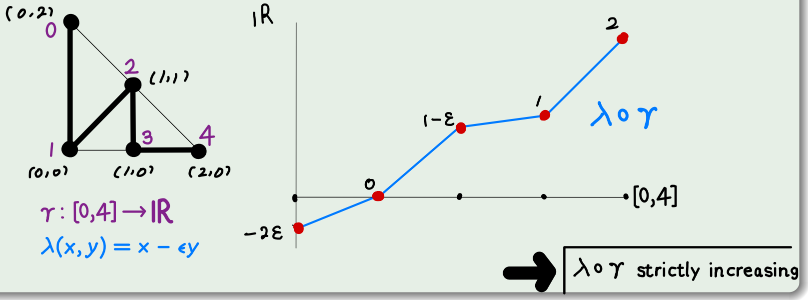
Example



Definition

Let $\lambda : \mathbb{R}^2 \to \mathbb{R}$ be the map $\lambda(x, y) = x - \epsilon y$ where ϵ is a small irrational number. A lattice path $\gamma : [0, n] \to \mathbb{R}^2$ is called *λ-increasing* if $\lambda \circ \gamma$ is strictly increasing.

Example



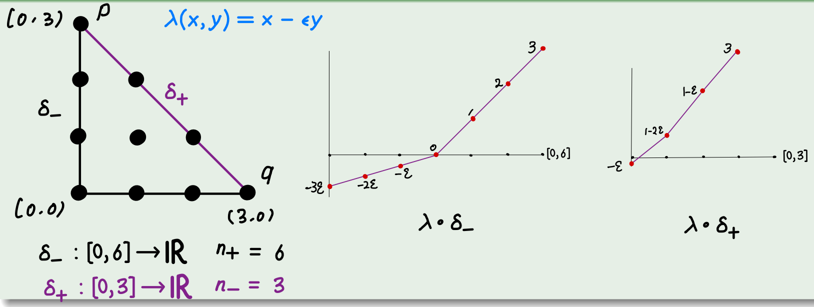
Remark

Let p and q be the points in Δ where $\lambda|_\Delta$ reaches its minimum (resp. maximum). Then p and q divide the boundary $\partial\Delta$ into two λ-increasing lattice paths

$$\begin{cases} \delta_+ : [0, n_+] \to \partial\Delta & \text{going clockwise around } \partial\Delta \\ \delta_- : [0, n_-] \to \partial\Delta & \text{going counterclockwise around } \partial\Delta \end{cases}$$

where $n_\pm$ denotes the number of integer points in the $\pm$-part of the boundary.

Example



Definition

Let $\gamma : [0, n] \to \Delta$ be a λ-increasing path from p to q,that is, $\gamma(0) = p$ and $\gamma(n) = q$. The *(positive and negative) multiplicities* $\mu_+(\gamma)$ and $\mu_-(\gamma)$ are defined recursively as follows:

- $\mu_\pm(\delta_\pm) := 1$.
- If $\gamma \neq \delta_\pm$ let $k_\pm \in [0, n]$ be the smallest number such that γ makes a left turn (respectively a right turn) at $\gamma(k_\pm)$. (If no such $k_\pm$ exists we set $\mu_\pm(\gamma) := 0$). Define λ-increasing lattice paths $\gamma'_\pm$ and $\gamma''_\pm$ as follows:
  - $\gamma'_\pm : [0, n-1] \to \Delta$ is the path that cuts the corner of $\gamma(k_\pm)$, i.e.

$$\begin{cases} \gamma'_\pm(j) := \gamma(j) & \text{for } j < k_\pm \\ \gamma'_\pm(j) := \gamma(j+1) & \text{for } j \geq k_\pm \end{cases}$$

  - $\gamma''_\pm : [0, n] \to \Delta$ is the path that completes the corner of $\gamma(k_\pm)$ to a parallelogram, i.e.

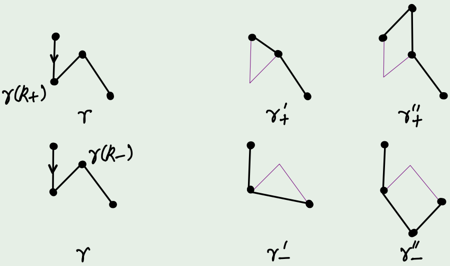
$$\begin{cases} \gamma''_\pm(j) := \gamma(j) & \text{if } j \neq k_\pm \\ \gamma''_\pm(j) := \gamma(j+1) + \gamma(j-1) - \gamma(j) & \text{if } j = k_\pm \end{cases}$$

Set

$$\mu_\pm(\gamma) := 2 \cdot \mathrm{Area}\, T \cdot \mu_\pm(\gamma'_\pm) + \mu_\pm(\gamma''_\pm)$$

where T is the triangle with vertices $\gamma(k_\pm - 1), \gamma(k_\pm), \gamma(k_\pm + 1)$. If $\gamma''_\pm$ is not inside Δ, $\mu_\pm(\gamma''_\pm) := 0$.

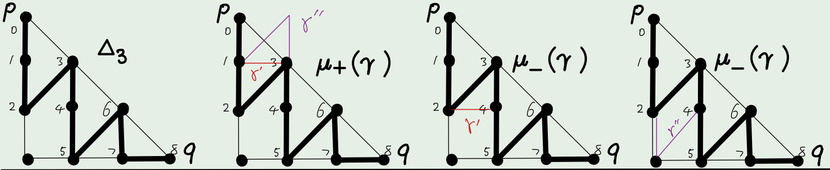
Example



Definition

The *multiplicity* $\mu(\gamma)$ of a λ-increasing lattice path γ is defined to be $\mu(\gamma) := \mu_+(\gamma)\mu_-(\gamma)$.

Example(multiplicity of $\gamma : [0,8] \to \Delta_3$)



$\mu_+(\gamma)$ $k_+ = 2$, $\gamma(2) = (0,1)$. γ' and γ'' are described in the second diagram. $\Rightarrow$ γ'' is not inside Δ_3 .
Note that the area of T is $\frac{1}{2}$.

$\Rightarrow$ $\mu_+(\gamma) = 2\,\text{Area}(T)\mu_+(\gamma'_+) + \mu_+(\gamma''_+) = \mu_+(\gamma'_+)$

Proceeding further, we get $\mu_+(\gamma) = \mu_+(\gamma'_+) = \cdots = \mu_+(\delta_+) = 1$.

$\mu_-(\gamma)$ $k_- = 3$, $\gamma(3) = (1,2)$. $\Rightarrow$ $\gamma''(3) = (0,0)$

$$\begin{aligned}\mu_-(\gamma) &= 2\,\text{Area}(T)\mu_-(\gamma'_-) + \mu_-(\gamma''_-)\\ &= \mu_-(\gamma'_-) + \mu_-(\gamma''_-) = 2\end{aligned}$$

Both $\mu_-(\gamma'_-)$, $\mu_-(\gamma''_-)$ can be computed inductively .

$\Rightarrow$ $\mu(\gamma) = \mu_+(\gamma)\cdot\mu_-(\gamma) = 2$.

Definition

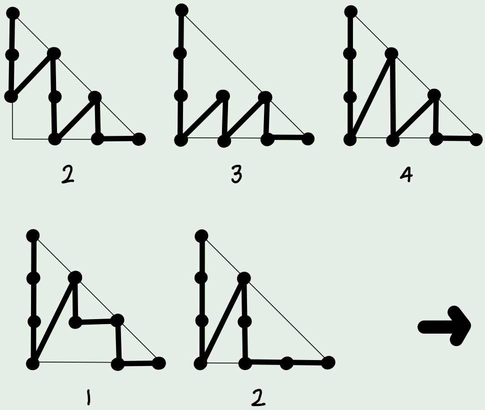
Let p and q be the points in Δ where $\lambda|_\Delta$ reaches its minimum (resp. maximum). Define $N_{\mathrm{path}}(g, \Delta)$ to be the number of λ-increasing lattice paths $\gamma : [0, s + g - 1] \to \Delta$ with $\gamma(0) = p$ and $\gamma(s + g - 1) = q$ (counted with multiplicities).

Theorem

The number $N_{\mathrm{trop}}(g, \Delta)$ *is equal to* $N_{\mathrm{path}}(g, \Delta)$.

Example

Recall $N(0,3) = 12$



$$N_{\text{path}}(0,\triangle_3) = 12$$

Let $\mathcal{P}$ be a collection of $3d - 1 + g$ points in general position in the real projective plane $\mathbb{R}P^2$.

Definition

Define the number

$$N_{\mathbb{R}}^{\mathrm{irr}}(g, d, \mathcal{P})(\text{resp. } N_{\mathbb{R}}(g, d, \mathcal{P}))$$

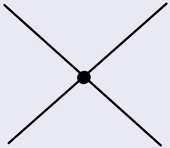
to be the number of irreducible (resp. all) real curves of degree d and genus g which pass through the points of $\mathcal{P}$.

The number $N_{\mathbb{R}}^{\mathrm{irr}}(g, d, \mathcal{P})$ does depend on the choice of $\mathcal{P}$. For example, the number $N(0, 3, \mathcal{P})$ can take values 8, 10, and 12 by a theorem of Degtyarev and Kharlamov.

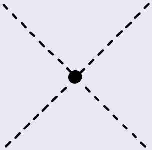
Definition

A real non-degenerate double point Q of a nodal real curve C is called

- *hyperbolic*, if Q is the intersection of two real branches of the curve,
- *elliptic*, if Q is the intersection of two imaginary conjugated branches.



Hyperbolic



Elliptic

Definition

Let $s(C)$ denote the number of elliptic double points of C. Define the *sign* of C to be $(-1)^{s(C)}$, and set

$$N_W^{\mathrm{irr}}(g, d, \mathcal{P})(\text{resp. } N_W(g, d, \mathcal{P}))$$

to be the number of irreducible (resp. all) real curves of degree d and genus g which pass through the configuration $\mathcal{P}$ counted with signs.

Theorem (Welschinger)

The number $N_W^{\mathrm{irr}}(0, d, \mathcal{P})$ is invariant and does not depend on the choice of the configuration $\mathcal{P}$.

Remark

Therefore, the number $N_W^{\mathrm{irr}}(0, d, \mathcal{P})$ can be denoted by W_d, and called *Welschinger invariants*. The absolute value of W_d provides a lower bound the numbers $N_{\mathbb{R}}^{\mathrm{irr}}(0, d, \mathcal{P})$.

Recall that for a tropical curve C. For every 3-valent vertex V of C, we have defined its associated multiplicity $\mathrm{mult}_V(C)$.

Definition

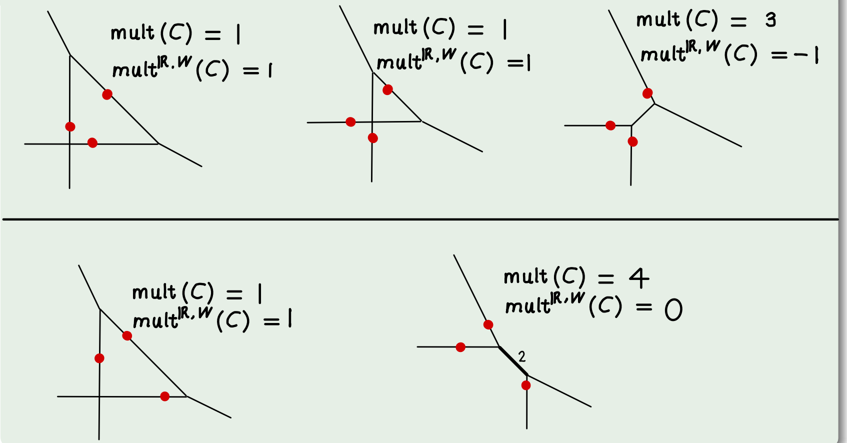
Define the *tropical Welschinger sign* by

$$\mathrm{mult}^{\mathbb{R},W}(C) := \prod_V \mathrm{mult}_V^{\mathbb{R},W}(C)$$

where the sum runs through all the 3-valent vertices V of C, and

$$\mathrm{mult}_V^{\mathbb{R},W}(C) := \begin{cases} 0 & \text{if } \mathrm{mult}_V(C) \text{ is even} \\ (-1)^{\frac{\mathrm{mult}_V(C)-1}{2}} & \text{if } \mathrm{mult}_V(C) \text{ is odd} \end{cases}$$

Example


mult (C) = 1
mult^{ℝ,W}(C) = 1
mult (C) = 1
mult^{ℝ,W}(C) = 1
mult (C) = 3
mult^{ℝ,W}(C) = −1
mult (C) = 1
mult^{ℝ,W}(C) = 1
mult (C) = 4
mult^{ℝ,W}(C) = 0
2

Definition

Let $\mathcal{P} \subset \mathbb{R}^2$ be a configuration of $s + g - 1$ points in tropically general position where $s := \#(\partial\Delta \cap \mathbb{Z}^2)$. Define the number

$$N^{\mathrm{irr}}_{W,\mathrm{trop}}(g, \Delta, \mathcal{P})(\text{resp. } N_{W,\mathrm{trop}}(g, \Delta, \mathcal{P}))$$

to be the number of irreducible (resp. all) tropical curves of degree Δ and genus g which pass through the configuration $\mathcal{P}$ counted with tropical Welschinger signs.

Theorem (Mikhalkin Correspondence Theorem for $\mathbb{R}P^2$)

Suppose that $\mathcal{P} \subset \mathbb{R}^2$ is a configuration of $3d + g - 1$ points in tropically general position. Then there exists a configuration $\mathcal{Q} \subset \mathbb{R}P^2$ of $3d + g - 1$ real points in general position such that

$$N^{\mathrm{irr}}_{W,\mathrm{trop}}(g, \Delta_d, \mathcal{P}) = N^{\mathrm{irr}}_{W}(g, d, \mathcal{Q}) \text{ and } N_{W,\mathrm{trop}}(g, \Delta_d, \mathcal{P}) = N_{W}(g, d, \mathcal{Q}).$$

In particular, the number W_d equals $N^{\mathrm{irr}}_{W,\mathrm{trop}}(0, \Delta_d, \mathcal{P})$.

Recall the definitions in Lattice paths in the complex case.

Definition

Let $\gamma : [0, n] \to \Delta$ be a lattice path connecting $p, q \in \Delta$. Define the *Mikhalkin-Welschinger multiplicity* of γ by

$$\nu(\gamma) := \nu_+(\gamma) \cdot \nu_-(\gamma)$$

where $\nu_\pm(\gamma)$ is defined (analogously as $\mu_\pm(\gamma)$ but replace

$$\mu_\pm(\gamma) := 2 \cdot \mathrm{Area}\, T \cdot \mu_\pm(\gamma'_\pm) + \mu_\pm(\gamma''_\pm)$$

with)

$$\nu_\pm(\gamma) := b(T) \cdot \nu_\pm(\gamma'_\pm) + \nu_\pm(\gamma''_\pm)$$

where

$$b(T) := \begin{cases} 0 & \text{if at least one side of } T \text{ is even} \\ (-1)^{\#(\mathrm{Int}(T) \cap \mathbb{Z}^2)} & \text{if otherwise} \end{cases}$$

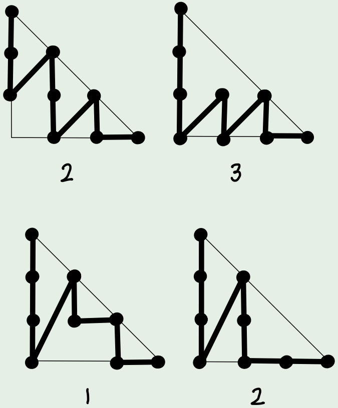
Theorem

There exists a configuration $\mathcal{P}$ of $s+g-1$ generic points in $\mathbb{R}P^2$ such that the number $N_{W,\mathrm{trop}}(g,\Delta,\mathcal{P})$ is equal to the number of λ-increasing lattice paths $\gamma : [0, s+g-1] \to \Delta$ connecting p and q counted with Mikhalkin-Welschinger multiplicities.

Remark

When $g=0$, this theorem helps to compute the Welschinger invariant W_d.

Example



$$\Rightarrow \quad N_{W,trop}(0,\Delta_3,\mathcal{P}) = 8 = W_3$$