Enriched tropical intersection I

April 5th, 2022

These are the notes for the tropical part of the talk, corresponding to the introduction of tropical notions used in our resutls, explained in the second part of the talk. For references I suggest many of the introductory books to the subject, in particular the online book *Tropical Geometry* by Mikhalkin and Rau, or to write to any of the speakers, we will be happy to answer your questions.

1 Toric deformations

Given a polynomial $f = \sum_{I\in A} \alpha_I x^{i_1} y^{i_2} \in k[x,y]$, where $A \subset \mathbb{Z}^2_{\geq 0}$ is a finite set and $I = (i_1, i_2)$, we consider the family of polynomials given by

$$f_t(x,y) = \sum_{I\in A} \alpha_I x^{i_1} y^{i_2} t^{\varphi(I)},$$

where $\varphi : A \longrightarrow \mathbb{Q}$ is the restriction of a convex rational function to the set of indices A.

We can think of the variable t as a variable in k^* whose specialization to $t = 1$ is our initial polynomial.

We can consider the family f_t as an element of the polynomial ring $k\{\{t\}\}[x,y]$ with coefficients in the field of Puisseux series $k\{\{t\}\}$, since it has a finite amount of monomials and the exponent in the variable t are rational.

The field of Puisseux series has a valuation given by

$$\nu : \begin{array}{ccc} k\{\{t\}\}^* & \longrightarrow & \mathbb{Q} \\ \sum_{i=i_0}^{\infty} a_i t^{i/N} & \longmapsto & -i_0/N \end{array}.$$

Such valuation satisfy

$$\begin{array}{rcll} \nu(x+y) & = & \max\{\nu(x), \nu(y)\} & \text{if } \nu(x) \neq \nu(y), \\ \nu(xy) & = & \nu(x) + \nu(y). & \end{array}$$

Given two polynomials $f_t, g_t \in k\{\{t\}\}[x,y]$, we have that the system

$$f_t(x,y) = g_t(x,y) = 0$$

has a solution in $\overline{k}\{\{t\}\}^2$ given by

$$\begin{array}{rcl} x(t) & = & x_0 t^{i_0} + \text{ higher order terms in } t, \\ y(t) & = & y_0 t^{j_0} + \text{ higher order terms in } t. \end{array}$$

if and only if the term of lowest power in t appears twice in $f_t(x(t), y(t))$ and $g_t(x(t), y(t))$.

2 Tropical curves and tropicalization maps

2.1 Tropical curves

The tropical semifield is the set $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ endowed with the operations (denoted by "·")

$$\begin{array}{rcl} \text{“}x+y\text{”} & = & \max\{x,y\}, \\ \text{“}x \cdot y\text{”} & = & x+y. \end{array}$$

is a semifield (satisfies all axioms of a field but the existence of additive inverse).

A polynomial $p \in \mathbb{T}[x]$ has the form

$$p(x) = \text{“}\sum_{i=0}^{d} a_i x^i\text{”} = \max_{i=o}^{d}\{a_i + ix\}.$$

A polynomial $p \in \mathbb{T}[x, y]$ has the form

$$p(x, y) = \text{“}\sum_{I \in A} a_I x^{i_1} y^{i_2}\text{”} = \max_{I \in A}\{a_I + i_1 x + i_2 y\}.$$

Hence, a polynomial in n variables with coefficients in $\mathbb{T}$ defines a function in $(\mathbb{T}^*)^n = R^n$ that is piece-wise linear. Its *tropical locus* is defined as the locus of non-differentiability, i.e., the points in $\mathbb{R}^n$ such that the maximum is obtained at least twice. We denoted it by V_{Trop}, and in two variable it is expressed by

$$V_{\text{trop}}(p) = \{(x, y) \in \mathbb{R}^2 \mid \exists (i, j) \neq (i', j') \in A\,(p(x, y) = ix + jy + a_{i,j} = i'x + j'y + a_{i',j'})\}.$$

2.2 Tropicalization maps

Using the aforementionned valuation map

$$\nu \colon k\{\{t\}\}^* \longrightarrow \mathbb{Q}$$

we can *tropicalize* a polynomial by taking the valuation of its coefficients and reinterpreting the addition and multiplication

$$\begin{array}{rccc} & k\{\{t\}\}[x, y] & \longrightarrow & \mathbb{T}[x, y] \\ \text{“}\,.\,\text{”}\colon & \sum_{I \in A} \alpha_I(t) x^{i_1} y^{i_2} & \longmapsto & \text{“}\sum_{I \in A} \nu(\alpha_I(t)) x^{i_1} y^{i_2}\text{”} \;\; = \max_{I \in A}\{\nu(\alpha_I(t)) + i_1 x + i_2 y\} \end{array} .$$

On the other hand, we can *tropicalize* the solutions to a polynomial (in two variables in this example), at the level of sets, by taking the closure in $\mathbb{R}^2$ of the image of the valuation taken point-wise.

$$\text{Trop} \colon X \subset (k\{\{t\}\}^*)^2 \longmapsto \overline{\{(\nu(\alpha), \nu(\beta)) \mid (\alpha, \beta) \in X\}} \subset \mathbb{R}^2.$$

If X is an algebraic curve, its tropicalization is a tropical curve, defined by the tropicalization of a defining polynomial for X, given that $k\{\{t\}\}$ is algebraically closed. In other words, we have the following theorem.

Theorem 2.1 (Kapranov). *If $k\{\{t\}\}$ is algebraically closed, then*

$$V_{Trop}(\text{“}f\text{”}) = \text{Trop}(V(f)).$$

3 Combinatorics of tropical curves

We redefine the concept of tropical curve from a combinatorial point of view. The following definition coincide with a tropical curve defined algebraically as before and it is known in the literature as an embedded abstract tropical curve.

Definition 3.1. A *tropical curve* C is a finite weighted graph (V, E, ω) embedded in $\mathbb{R}^2$, where $E = E^\circ \cup E^\infty$ is the disjoint union of non-directed edges $E^\circ \subset \{e \subset V \mid \text{Card}(e) = 2\}$ and univalent edges $E^\infty \subset V$, such that every edge $e \in E^\circ$ embeds into a segment of the graph of a line given by $a_e x + b_e y = c_e$ with $a_e, b_e \in \mathbb{Z}$, every edge $l \in E^\infty$ embeds into a ray of a line $a_e x + b_e y = c_e$, and every vertex $v \in V$ satisfies

$$\sum_{e \in E, v \in e} \omega(e) \cdot \mathrm{u}_e = 0$$

where $\mathrm{u}_e = \dfrac{\pm 1}{\gcd(a_e, b_e)}(b_e, -a_e)$ is a primitive vector oriented outwards from v, and $\omega : E \longrightarrow \mathbb{Z}$ is a non-negative function.

We call the *degree* of the tropical curve C the multiset of primitives vectors associated to its legs $\{\mathrm{u}_l \mid l \in E^\infty\}$, counted with multiplicities.

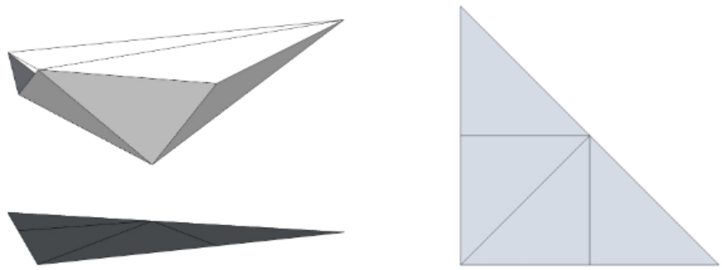

Figure 1: The dual subdivision of “$(-1)x^2 + (-1)y^2 + 1xy + x + y + 0$. Taken from *Tropical Geometry* by Mikhalkin and Rau.

3.1 Dual subdivision

To a polynomial $p \in \mathbb{T}[x, y]$ given by

$$p = \text{“}\sum_{I \in A} a_I x^{i_1} y^{i_2}\text{”}, a_I \neq -\infty$$

we associated a refined integer polygon in $\mathbb{R}^2$ constructed in the following way. First, the *Newton polygon* NP of p is the convex hull

$$\mathrm{NP}(p) = \mathrm{Conv}\left(\{(i_1, i_2) \mid a_{(i_1,i_2)} \neq -\infty\}\right) = \mathrm{Conv}(A),$$

endowed with the refinement given by the projection to $\mathbb{R}^2$ of the edges of the upper faces of the polyhedron

$$\mathrm{Conv}\left(\{(i_1, i_2, a_{(i_1,i_2)}) \mid a_{(i_1,i_2)} \neq -\infty\}\right).$$

This refinement produces a graph which we call DS(p) the dual subdivision of p. There is a one-to-one correspondance of the elements

$V_{Trop}(p)$	DS(p)
vertex v	connected component of $\mathrm{NP}(p) \setminus \mathrm{DS}(p)$
edge e	edge e'
connected component C of $\mathbb{R}^2 \setminus V_{Trop}(p)$	vertex v_C

Moreover, the corresponding edges e and e' are perpendicular, and the degree of a curve corresponds through this duality to the Newton polygon.

3.2 Tropical intersection

We say that two curves intersect *tropically transversely* if they intersect in finitely many points, and every point in this intersection is not a vertex of any of the curves. Locally, this intersections look like the one in Figure 2.

If two such curves intersect transversely at a point p, the multiplicity of the intersection at this point is given by

$$mult_p^{\mathbb{T}}(C_1, C_2) = \omega_1(e_1)\omega_2(e_2)\left|\det(\mathrm{u}_{e_1}, \mathrm{u}_{e_2})\right|,$$

where e_1 and e_2 are the edges of C_1 and C_2 containing p, ω_1 and ω_2 are the weight functions, and u_{e_1} and u_{e_2} are the primitive vectors of e_1 and e_2, respectively.

Theorem 3.2 (Tropical Bézout, Bernstein–Kushnirenko theorem)**.** *Let C_1 and C_2 be tropical curves with Newton polygons Δ_1 and Δ_2, respectively. If C_1 and C_2 intersect tropically transversely, then*

$$C_1 \cdot C_2 = \sum_{p \in C_1 \cap C_2} \omega_1(e_1)\omega_2(e_2)\left|\det(\mathrm{u}_{e_1}, \mathrm{u}_{e_2})\right| = \mathrm{area}(\Delta_1 + \Delta_2) - \mathrm{area}(\Delta_1) - \mathrm{area}(\Delta_2).$$

Here, the polygon $\Delta_1 + \Delta_2$ is the Minkowski sum of the polygons.

The case when $\Delta_1 = \mathrm{Conv}(\{(0,0), (d_1, 0), (0, d_1)\})$ and $\Delta_2 = \mathrm{Conv}(\{(0,0), (d_2, 0), (0, d_2)\})$ correspond to curves in the projective plane. Where

$$\Delta_1 + \Delta_2 = \mathrm{Conv}(\{(0,0), (d_1 + d_2, 0), (0, d_1) + d_2\})$$

and

$$C_1 \cdot C_2 = \mathrm{area}(\Delta_1 + \Delta_2) - \mathrm{area}(\Delta_1) - \mathrm{area}(\Delta_2) = \frac{(d_1 + d_2)^2}{2} - \frac{d_1}{2} - \frac{d_2^2}{2} = d_1 \cdot d_2.$$

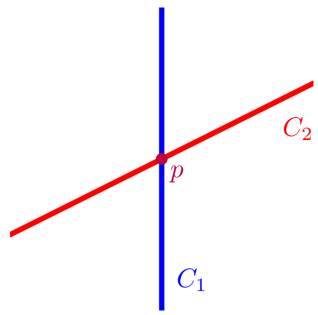

Figure 2: Tropical transversal intersection

3.3 Enriched tropical curves and Viro Polynomials

Viro's patchworking is a combinatorial construction yielding topological properties of real algebraic curves. This is an algorithmic construction whose input is a subdivision of a polygon and a set of signs $\sigma(I)$ (either plus or minus) for every integer point I in the polygon (boundary and interior). The Viro polynomial associated to this data is the polynomial

$$\sum_{I \in \Delta \cap \mathbb{Z}^2} \sigma(I) t^{\varphi(I)}$$

where φ is a convex piece-wise linear function inducing the subdivision.

Based on this ideas, we generalized this concept by changing the notion of signs with elements of $k^*/(k^*)^2$, giving rise to the following definition.

Definition 3.3. An *enriched tropical curve* $\widetilde{C}$ is a tropical curve with an element of $k^\times/(k^\times)^2$ assigned to each connected component or equivalently to each vertex in the dual subdivision. We call such an element of $k^\times/(k^\times)^2$ *coefficient* of the component/vertex of the dual subdivision. We write C for the underlying (classical) tropical curve.