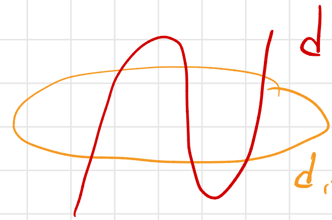
Enriched tropical intersection I

Bézout's theorem for curves over $\mathbb{C}$

$$C_1 = V(F_1) \subseteq \mathbb{P}^2_{\mathbb{C}}$$

$$C_2 = V(F_2) \subseteq \mathbb{P}^2_{\mathbb{C}}$$



$$d_1 := \deg F_1, \quad d_2 := \deg F_2$$

Then
$$\sum_{p \in C_1 \cap C_2} \operatorname{mult}_p(C_1, C_2) = d_1 \cdot d_2$$

Proof: $V := \mathcal{O}_{\mathbb{P}^2}(d_1) \oplus \mathcal{O}_{\mathbb{P}^2}(d_2)$

$$\begin{tikzcd} V \arrow[d] \\ X := \mathbb{P}^2 \arrow[u, bend left, "(F_1, F_2)"'] \end{tikzcd}$$

intersection pts of C_1 and C_2

$=$ # zeros of section (F_1, F_2)

$=$ degree of top chern class ($=$ Euler class)

$= d_1 \cdot d_2$

$\square$

Over $\mathbb{R}$:

$$\begin{tikzcd} V \arrow[r] & X \arrow[l, bend right, "\sigma"'] \end{tikzcd}$$ oriented rank n vector bundle

X smooth closed oriented n-mfld

$\sigma : X \to V$ general section, $p \in \{\sigma = 0\}$

Def: Choose · oriented coordinates around p

· trivialization of V around p comp with orientation

Then locally

$$\sigma : \underset{\mathbb{R}^n}{\overset{p \in}{\underset{\cup\!\shortmid}{U}}} \to \mathbb{R}^n$$

$$\boxed{\mathrm{ind}_p\, \sigma := \deg_p \sigma}$$

where

$$\deg_p \sigma = \deg\left(\underset{S^n}{\underset{\cong}{U/U \setminus \{p\}}} \xrightarrow{\sigma} \underset{S^n}{\underset{\cong}{\mathbb{R}^n/\mathbb{R}^n \setminus \{0\}}} \right)$$

Poincaré-Hopf thm:

$$\boxed{\deg e(V) = \sum_{p \in \{\sigma = 0\}} \mathrm{ind}_p \sigma}$$

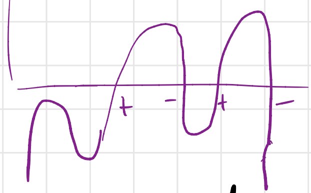
Euler class

Ex:

$\mathcal{O}(d)$

$\downarrow$ d even

$\mathbb{RP}^1$



Morel's $\mathbb{A}^1$-degree: k arbitrary field

$$\deg^{\mathbb{A}^1} : [\mathbb{P}^n_k / \mathbb{P}^{n-1}_k , \mathbb{P}^n_k / \mathbb{P}^{n-1}_k]_{\mathbb{A}^1} \to GW(k)$$

replaces $\deg : [S^n, S^n] \to \mathbb{Z}$

$GW(k)$ is generated by $\langle a \rangle$ where $a \in {k^\times}/{(k^\times)^2}$

$\langle a \rangle = ax^2$

Relations: 1) $\langle a \rangle \langle b \rangle = \langle ab \rangle$

2) $\langle a \rangle + \langle b \rangle = \langle a+b \rangle + \langle ab(a+b) \rangle$

3) $\langle a \rangle + \langle -a \rangle = \langle 1 \rangle + \langle -1 \rangle =: h$

hyperbolic form

Ex:
- $GW(\mathbb{C}) \cong \mathbb{Z}$
- $GW(\mathbb{R}) \cong \mathbb{Z}[C_2]$

<u>Idea</u> (Kass–Wickelgren):

Replace $\deg_p \sigma$ by $\deg_p^{\mathbb{A}^1} \sigma$

in PH theorem.

<u>Def</u>: A vector bundle $V \to X$ is

relatively oriented if

alg k-variety

$\exists$ line bundle $\mathcal{L} \to X$

+ iso ρ: $\mathrm{Hom}(\det TX, \det V) \xrightarrow{\sim} \mathcal{L}^{\otimes 2}$

$$\begin{array}{c} \mathrm{Hom}(\det TX, \det V) \\ \cong \\ \omega_{X/k} \otimes \det V \end{array}$$

- coordinates define a section of $\det TX$
- trivialization defines a section of $\det V$

↳ these are compatible with ρ if the induced section is sent to a square in $\mathcal{L}^{\otimes 2}$ by ρ

<u>Example:</u> $V = \mathcal{O}_{\mathbb{P}^2_k}(d_1) \oplus \mathcal{O}_{\mathbb{P}^2_k}(d_2)$

$$\downarrow$$

$$\mathbb{P}^2_k$$

is relatively orientable $\iff$ $d_1 + d_2$ is odd

$\omega_{\mathbb{P}^2/k} \otimes \det V = \mathcal{O}_{\mathbb{P}^2}(-3 + d_1 + d_2)$

smooth, proper k-variety

Let $V \to X$ be relatively oriented vb of rank $n = \dim X$ and $\sigma : X \to V$ a section with only isolated zeros

<u>Def</u> (Kass-Wickelgren):

$$\mathrm{ind}_p \sigma := \deg_p^{\mathbb{A}^1} \sigma$$

coord + triv compatible with rel or

(Poincaré-Hopf) Euler number

$$n^{PH}(V, \rho) := \sum_{\text{zeros}} \mathrm{ind}_p \sigma \in GW(k)$$

<u>Fact:</u> This is independent of the choice of section (Bachmann-Wickelgren)

Formulas for the local $\mathbb{A}^1$-degree

(Kass-Wickelgren)

Assume $f: \mathbb{A}^n_k \to \mathbb{A}^n_k$ with an isolated zero p, st $\det \mathrm{Jac}(p) \neq 0$.

Then
- $k(p) = k$

$$\deg_p^{\mathbb{A}^1} f = \langle \det \mathrm{Jac}(p) \rangle$$

$\leftarrow GW(k)$

- $k(p)$ separable over k

$$\deg_p^{\mathbb{A}^1} f = Tr_{k(p)/k} \langle \det \mathrm{Jac}(p) \rangle$$

where

$k(p)$-VS

$\leftarrow$ quad form

$$\underbrace{W \xrightarrow{\alpha} k(p) \xrightarrow{Tr_{k(p)/k}} k}_{Tr_{k(p)/k}(\alpha)}$$

Quadratically refined Bézout for curves

(McKean)

$V = \mathcal{O}_{\mathbb{P}^2}(d_1) \oplus \mathcal{O}_{\mathbb{P}^2}(d_2) \to \mathbb{P}^2$ with $d_1 + d_2$ odd (rel or).

$\deg F_1 = d_1$
$\deg F_2 = d_2$

Then

$$n^{PH}(V) = \sum_{\substack{\text{intersection} \\ \text{pts } p}} \operatorname{ind}_p (F_1, F_2)$$

Set $x_0 = 1$

$$= \sum_{p} \operatorname{Tr}_{k(p)/k} \langle \det \operatorname{Jac}^{(F_1, F_2)}(p) \rangle$$

$$= \frac{d_1 \cdot d_2}{2} \cdot h \in GW(k)$$