Enriched tropical intersection II

Bézout for curves over an arbitrary field k:

$$C_1 = V(F_1) \subseteq \mathbb{P}^2 \qquad d_1 = \deg F_1$$

$$C_2 = V(F_2) \subseteq \mathbb{P}^2 \qquad d_2 = \deg F_2$$

$V = \mathcal{O}(d_1) \oplus \mathcal{O}(d_2) \xrightarrow{(F_1, F_2)} \mathbb{P}^2$ is <u>relatively orientable</u> if $d_1 + d_2$ odd.

In this case

$$n^{PH}(V) = \sum_{\substack{\text{intersection} \\ \text{pts } p}} \mathrm{Tr}_{k(p)/k} \langle \det \mathrm{Jac}(F_1, F_2)(p) \rangle$$

$$\underset{\text{McKean}}{=} \frac{d_1 \cdot d_2}{2} \cdot h$$

where $h = \langle 1 \rangle + \langle -1 \rangle$

Equations for tropical curves:

$$F_1 = \sum_{\substack{\text{finitely many} \\ \text{tuples} \\ I=(i_1,i_2) \\ i_1,i_2 \in \mathbb{Z}_{\geq 0}}} \alpha_I \, x_1^{i_1} x_2^{i_2} \, t^{\varphi(I)} \in k[[t]][x_1,x_2]$$

$\alpha_I \in k$

$\varphi : \mathbb{Z}_{\geq 0}^2 \to \mathbb{Q}$

$$F_2 = \sum_J \beta_J \, x_1^{j_1} x_2^{j_2} \, t^{\psi(J)}$$

$\beta_J \in k$

$\psi : \mathbb{Z}_{\geq 0}^2 \to \mathbb{Q}$

$\xrightarrow{-\mathrm{val}}$ tropical polynomials

$\rightsquigarrow$ tropical curves

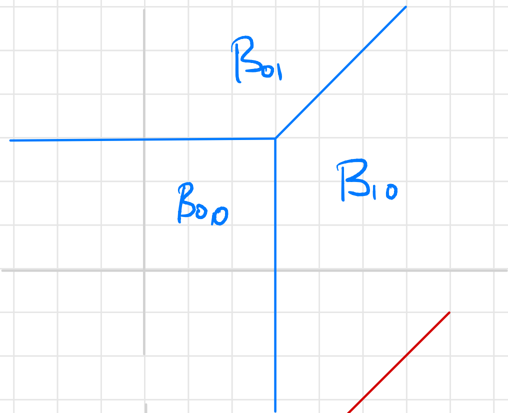
Example:

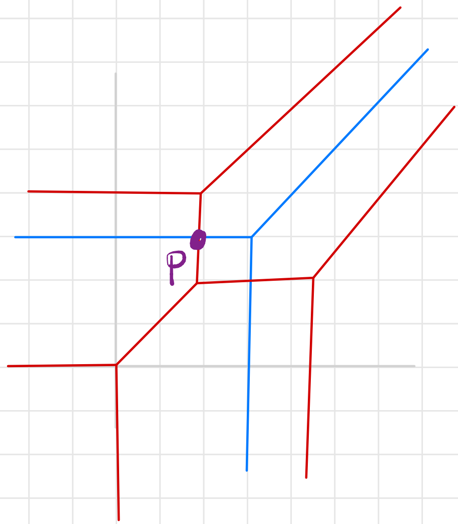
$$F_1 = \alpha_{00} + \alpha_{10} x_1 + \alpha_{01} x_2 + \alpha_{20} x_1^2 t^4 + \alpha_{11} x_1 x_2 t^2 + \alpha_{02} x_2^2 t^4$$

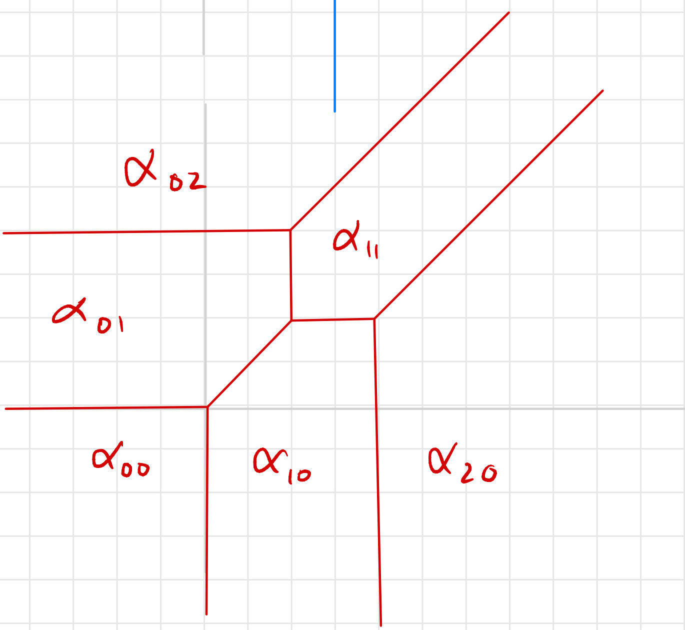
$$F_2 = \beta_{00} + \beta_{10} x_1 t^3 + \beta_{01} x_2 t^3$$

$$\Big\downarrow -\mathrm{val}$$

$$\max(0,\ x_1 - 3,\ x_2 - 3)$$







Idea: Define enriched intersection multiplicity

$$\widetilde{\mathrm{mult}}_p(C_1, C_2) = \mathrm{Tr}_{E/k[\![t]\!]} \langle \det \mathrm{Jac}(F_1, F_2)(p) \rangle$$

something like the field of def of p

Q: What is $GW(k[\![t]\!])$?

generated $\langle a \rangle$ where $a \in k[\![t]\!]^{\times} / (k[\![t]\!]^{\times})^2$

$$a = \sum_{i=i_0}^{\infty} a_i t^{i/n} = a_{i_0} \left(t^{i_0/n} + \sum_{i=i_0+1}^{\infty} b_i t^{i/n} \right)$$

$a_{i_0} \neq 0$

where $b_i = \frac{a_i}{a_{i_0}}$

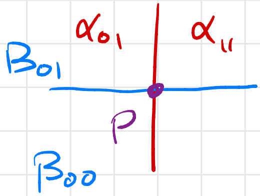
Exercise: $t^{i_0/n} + \sum_{i=i_0+1}^{\infty} b_i t^{i/n}$ is a square in $k[\![t]\!]$

So $GW(k[\![t]\!]) \cong GW(k)$

$$\left\langle \sum_{i=i_0}^{\infty} a_i t^{i/n} \right\rangle \longmapsto \langle a_{i_0} \rangle$$

Back to the example:

Locally



$$\alpha_{01} x_2 + \alpha_{11} x_1 x_2 t^2$$

$$\beta_{00} + \beta_{01} x_2 t^3$$

$$\langle \det \mathrm{Jac}(F_1, F_2)(p) \rangle$$

$$= \langle \det \begin{pmatrix} \alpha_{11} x_2 t^2 & 0 \\ \alpha_{01} + \alpha_{11} x_1 t^2 & \beta_{01} t^3 \end{pmatrix} \rangle$$

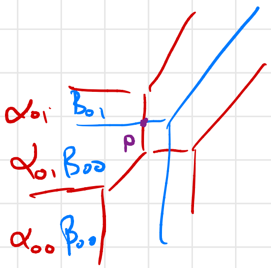
$$= \langle \alpha_{11} \beta_{01} x_2 t^5 \rangle$$

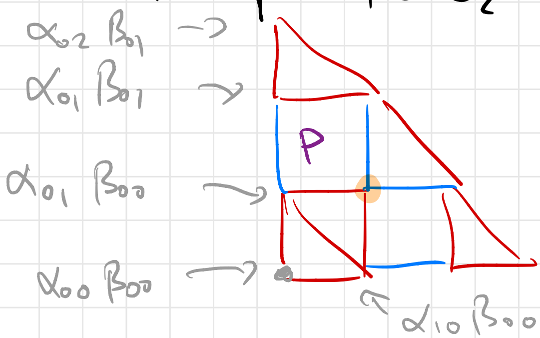
$$= \langle -\alpha_{11} \beta_{00} \rangle \qquad x_2 = -\frac{\beta_{00}}{\beta_{01}} t^3$$

$$\in GW(k)$$

Combinatorial formula for $\widetilde{\mathrm{mult}}_p(C_1, C_2)$

Dual subdivision of $C_1 \cup C_2$





Def: We call a lattice pt $\in \mathbb{Z}\times\mathbb{Z}$ in the dual subdivision **odd** if it equals $(1,1) \in \mathbb{Z}/2 \times \mathbb{Z}/2$.

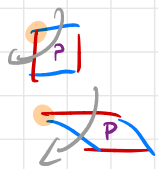
Theorem (Jaramillo Puentes — P.)

$$\widetilde{\mathrm{mult}}_p(C_1, C_2) = \sum_{\substack{\text{odd} \\ \text{vertices}}} \langle \varepsilon\, \alpha_I \beta_J \rangle$$

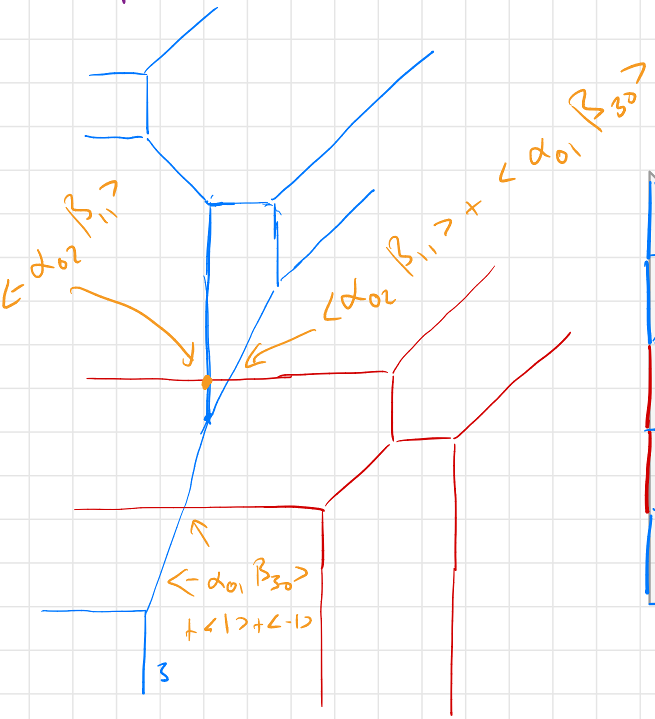
coeff of the vertex

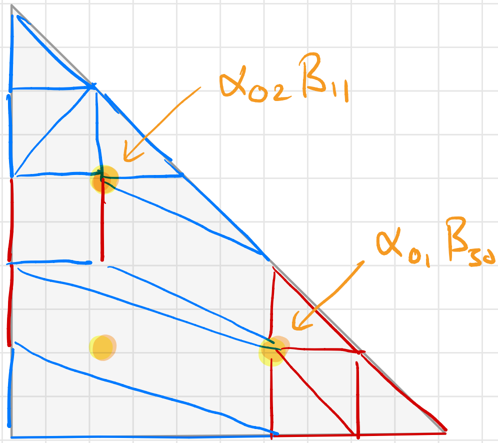
in $W(k) = GW(k)/\mathbb{Z}\cdot h$

$$\varepsilon = \begin{cases} +1 & \text{blue first} \\ -1 & \text{red first} \end{cases}$$



Example





$$\langle a \rangle + \langle -a \rangle = h$$