Talk 11 - Mot. Seminar WS 20

Comparison of Khan's & BF's Fund. Classes:

Recall: Last time $\rightsquigarrow$ we saw given a q-smooth morphism $\mathcal{X} \to \mathcal{Y}$ of dStk we have a $[\mathcal{X}/\mathcal{Y}] \in E^{BM}(\mathcal{X}/\mathcal{Y}, \mathbb{L}_{\mathcal{X}/\mathcal{Y}})$

we used $SH_{ét}$

We can restrict to dSch $\rightsquigarrow$ then we can use

SH_{Nis} $\qquad$ (q-proj /S)

Given $Z \in Sch^{cl}/_{w}$ w/ P.O.T. $\mathcal{E}_\cdot \xrightarrow{\varphi} L_{Z/S}$ in the sense of BF $\Rightarrow$ compare $[Z, \mathcal{E}_\cdot]_{BF}$ w/ some class $[Z_\varphi] \rightsquigarrow$ Khan's technique

Prop.: Given $X \in dSch$ q-sm / S, then the truncation of X, $\tau_0 X \in Sch^{cl}$ ($\iota : Sch \rightleftarrows dSch : \tau_0$) induce a P.O.T. on $X_0 := \tau_0 X$:

$$X_0 \overset{t_0}{\hookrightarrow} X$$

$$t_0^* \mathbb{L}_{X/S} \to \mathbb{L}_{X_0/S}$$

Given $\mathcal{E}_\cdot$ P.O.T. on $Z \in Sch \Rightarrow$ it's not always coming from a q-sm d scheme!

But $\rightsquigarrow$ it's true up to $\mathbb{A}^1$-hpty

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Z q-proj, $\mathcal{E}_\cdot$ P.O.T.

$Z \subseteq \underset{sm.}{M} \Rightarrow$ we can take a Jouanolou cover $\tilde{M}$ of $M \Rightarrow$

$$\begin{array}{c} \tilde{Z} \subseteq \tilde{M} \\ \tilde{p} \downarrow \\ Z \end{array}$$

are affine $\rightsquigarrow \tilde{\mathcal{E}}_\cdot = \tilde{p}^* \mathcal{E}_\cdot \oplus \iota^* \Omega_{\tilde{M}/M}[1]$

We can suppose $\underset{\mathrm{Spec}(N_I)}{Z} \subseteq \underset{\mathrm{Spec}(A)}{M}$ affine and that our P.O.T. $\varphi : \mathcal{E}_\cdot \to \mathbb{L}_{Z/S}$ is of the form

$$[\mathcal{E}_1 \to \iota^* \Omega_{M/S}] \xrightarrow{\varphi} [I/I^2 \to \iota^* \Omega_M]$$

We want to find a loc. free sheaf $\mathcal{G}_1$ on M s.t. $\iota^* \mathcal{G}_1 \simeq \mathcal{E}_1$

Since Z is affine, $\mathcal{E}_1$ is glob. gen $\Rightarrow$

$\mathcal{O}_Z^N \overset{f_1}{\twoheadrightarrow} \mathcal{E}_1 \Rightarrow Z \to Gr(N, r)$

$$\rightsquigarrow \begin{tikzcd} E_1 \arrow[r, "g"] \arrow[d] & U_{N,r} \arrow[d] \\ Z \arrow[r] & Gr(N,r) \end{tikzcd} \qquad g^* \gamma_{N,r} = \mathcal{E}_1$$

Instead of taking M $\rightsquigarrow$ we take

$$\begin{tikzcd} M' \arrow[r, phantom, "="] & M \times Gr(N,r) \arrow[d, "pr_2"] \\ & Gr(N,r) \end{tikzcd}$$

and we chose:

$$\mathcal{G}_1 := pr_2^* \gamma_{N,r}$$

$$Z \xhookrightarrow{\iota} M \rightsquigarrow \iota^* \mathcal{G}_1 = \mathcal{E}_1$$

We reduce to the affine case again by a Jouanolou trick

$\rightsquigarrow$ We can suppose to have $Z \hookrightarrow M$ both affines

w/ $\mathcal{G}_1$ on M s.t. $\iota^* \mathcal{G}_1 \simeq \mathcal{E}_1$

$$\begin{tikzcd} \mathcal{G}_1 \arrow[r] \arrow[d] & J \subset I \subset A \arrow[d] \\ \mathcal{E}_1 \arrow[r, two heads, "\varphi_1"'] & I/I^2 \end{tikzcd}$$

$$\begin{tikzcd}
(\mathcal{G}_1)_p \arrow[r, two heads] \arrow[d] & \mathcal{I}_p = \mathcal{J}_p \arrow[d] \\
(\mathcal{E}_1)_p \arrow[r, two heads] & (\mathcal{I}/\mathcal{I}^2)_p
\end{tikzcd}$$

n) We can shrink M to an open neighborhood of Z s.t. $\mathcal{G}_1 \twoheadrightarrow \mathcal{I}$

$$\begin{tikzcd}
M \arrow[r, "s_e"'] & \mathbb{V}_M(\mathcal{G}_1) =: G_1
\end{tikzcd}$$

We take the derived vanishing locus:

$$\begin{tikzcd}
Z_\varphi \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner^h", very near start] & M \arrow[d, "s_0"] \\
M \arrow[r, "s_\varphi"'] & G_1
\end{tikzcd}$$

Locally where G_1 trivializes:

$$\begin{tikzcd} A[t_1 \ldots t_n] \arrow[r] \arrow[d] & A[t_1 \ldots t_n]/(t_1 \ldots t_n) \arrow[d] \\ A \arrow[r] & A/\!/(a_j) \end{tikzcd}$$

$t_j \mapsto a_j$

$\rightsquigarrow \quad \pi_0(A/\!/(a_j)) = A/(a_j)$

but the a_j are generators of I

$\Rightarrow \quad \tau_0 Z_\varphi = Z_1$

Notice that Z_φ is q-smooth / M

Derived Deformation Space:

Given $Z_\varphi \to M$ q-sm

we saw:

$$\mathrm{Def}^{der}_{Z/M} := \mathrm{Bl}^{der}_{Z\times\{0\}}(M \times \mathbb{A}^1) \setminus \mathrm{Bl}^{der}_{Z}(M)$$

$$\mathrm{Bl}^{der}_{\mathcal{Z}\times\{0\}}(M\times\mathbb{A}^1) := (M\times\mathbb{A}^1)\times^h \mathrm{Bl}^{cl}_{\{0\}}(G_1\times\mathbb{A}^1)$$

$$\mathrm{Bl}^{der}_{\mathcal{Z}}(M) := M\times^h \mathrm{Bl}^{cl}_{\{0\}}(G_1)$$

$\Rightarrow$ Fiber over 0 is $\mathbb{V}_{\mathcal{Z}}(\mathbb{L}_{\mathcal{Z}/M}[-1]) =$

$= N_{\mathcal{Z}/M}$

$t_0 N_{\mathcal{Z}/M}$

$$\mathbb{V}_{\mathcal{Z}}(\mathcal{E}) : \quad dAff \to Spc$$

$$(Spec(R) \xrightarrow{f} \mathcal{Z}) \mapsto Map_{D(R)}(f^*\mathcal{E}, R)$$

if $Spec(R) \in Sch^{cl}$

$$\Rightarrow \quad \begin{tikzcd} Spec(R) \arrow[r, "g"] \arrow[rrr, bend left, "f"] & t_0\mathcal{Z} = Z \arrow[r, "t_0"] & \mathcal{Z} \end{tikzcd}$$

So :

$$t_0 N_{\mathcal{Z}/M} = Map_{D(R)}(f^*\mathcal{E}, R)$$

$$= Map_{D(R)}(g^* t_0^* \mathbb{L}_{\mathcal{Z}/M}[-1], R)$$

$$\begin{tikzcd} Z_\psi \arrow[r] \arrow[d, "\iota_Z"'] & M \arrow[d] \\ M \arrow[r] & G_1 \end{tikzcd}$$

$\forall$ section $M \to G_1$

$$\mathbb{L}_{M/G_1} \simeq G_1[+1]$$

and the cotg. complex is stable under p.b.

$$\Rightarrow \quad \mathbb{L}_{Z/M} \simeq \iota_Z^* \mathbb{L}_{M/G_1} \simeq \iota_Z^* G_1[+1]$$

$$\Rightarrow \quad \tau_0 N_{Z/M} = \mathbb{V}_Z(\mathcal{E}_1) =: E_1$$

$$\begin{tikzcd} \mathfrak{C}_{Z/M} \arrow[r, hook] \arrow[d, hook] & \mathrm{Def}^{cl} \arrow[d, dotted] \\ E_1 = \tau_0 N_{Z/M} \arrow[r, hook] & \tau_0 \mathrm{Def}^{der}_{Z/M} \end{tikzcd}$$

$$\tau_0\left(Bl_Z(M)\right) = \tau_0\left(M \times^h Bl_{\{0\}}(G_1)\right) =$$

$$= M \times^{cl} Bl_{\{0\}}(G_1)$$

$$\begin{tikzcd} M \times^{cl} Bl_{\{0\}}(G_1) \arrow[d, hook] \\ Bl_Z(M) \end{tikzcd}$$

$$\Rightarrow \quad Def^{cl}_{Z/M} \hookrightarrow t_0\, Def^{der}_{Z/M}$$

$$\begin{tikzcd} \hat{E} \arrow[r, hook] \arrow[d, "f"] & Def^{cl} \arrow[d, hook] & M \times \mathbb{G}_m \arrow[l, hook'] \arrow[d, equal] \\ E_1 \arrow[r, hook] & t_0\, Def^{der} & M \times \mathbb{G}_m \arrow[l, hook'] \end{tikzcd}$$

Specialization Map:

$$[Z_\varphi]^{Khan}$$

$$M \times \{0\} \hookrightarrow \mathbb{A}^1 \times M \hookleftarrow \mathbb{G}_m \times M$$

$$\gamma_t : \mathbb{E}^{BM}(M/S, v) \xrightarrow{\sim} \mathbb{E}^{BM}(\mathbb{G}_m M/S, v)[-1]$$

Last time:

$$(d) \qquad \mathbb{G}_m M \hookrightarrow \mathrm{Def}^{\mathrm{der}}_{Z/M} \hookleftarrow N_{Z/M}$$

But we also saw!

Thm (Khan): $\forall \; X \in d\,Sch$ we get an equiv.

$$SH(X) \simeq SH(\underset{\tau_0 X}{\underset{\shortparallel}{X_0}})$$

$$(d') \qquad \mathbb{G}_m M \hookrightarrow \tau_0 \mathrm{Def}^{\mathrm{der}}_{Z/M} \hookleftarrow E_1 = \tau_0 N_{Z/M}$$

$$sp^d : \begin{tikzcd} \mathbb{E}^{BM}(S/S, 0) \arrow[r] & \mathbb{E}^{BM}(M/S, [T_M]) \arrow[r, "\gamma_t"] & \mathbb{E}^{BM}(\mathbb{G}_m M/S, [T_M])(-1) \arrow[d, "\partial_{E_1}"] \\ \mathbb{1} \arrow[ddrr] & & \mathbb{E}^{BM}(E_1/S, [T_M]) \\ & & sp^d(\mathbb{1}) \end{tikzcd}$$

(d) $\mathbb{G}_m M \hookrightarrow \mathrm{Def}^{cl}_{Z/M} \hookleftarrow \mathfrak{C}_{Z/M}$

$$\begin{tikzcd}
sp^{cl} : \mathbb{E}^{BM}(S/S, 0) \arrow[r] & \mathbb{E}^{BM}(\mathbb{G}_m M/S, [T_M])(-1) \arrow[d] \\
 & \mathbb{E}^{BM}(\mathfrak{C}/S, [T_M])
\end{tikzcd}$$

$$1 \longmapsto sp^{cl}(1) = [\mathfrak{C}^{st}]$$

$$\mathfrak{C} \xhookrightarrow{\phi} E_1$$

$$\Rightarrow \mathbb{E}^{BM}(\mathfrak{C}/S, [T_M]) \xrightarrow{\phi_*} \mathbb{E}^{BM}(E_1/S, [T_M])$$

Prop.: Given,

$$\begin{tikzcd}
W \arrow[r, hook] \arrow[d, "f"'] \arrow[dr, phantom, "\ulcorner", very near start] & Z \arrow[d, "f"] & U \arrow[l, hook'] \arrow[d] \\
X \arrow[r, hook] & Y & V \arrow[l]
\end{tikzcd}$$

then we get comm. squares:

$$\begin{tikzcd}
\mathbb{E}^{BM}(W, v) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{E}^{BM}(Z, v) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{E}^{BM}(U, v) \arrow[d] \\
\mathbb{E}^{BM}(X, v) \arrow[r] & \mathbb{E}^{BM}(Y, v) \arrow[r] & \mathbb{E}^{BM}(V, v)
\end{tikzcd}$$

Applying this to:

$$\begin{tikzcd} \mathcal{E} \arrow[r, hook] \arrow[d, "\phi"'] & \mathrm{Def}^{d'} \arrow[d] \\ E_1 \arrow[r, hook] & \mathrm{Def}^{d} \end{tikzcd}$$

We get that $sp^{d}(\mathbb{1}) = \phi_* \, sp^{d'}(\mathbb{1})$

Comparison w/ BF:

The class $[Z_\varphi)$ was constructed taking $sp^{d}(\mathbb{1})$ to $\mathbb{E}^{BM}(Z_\varphi/S, v) = \mathbb{E}^{BM}(Z/S, v)$

via:

$$\mathbb{E}^{BM}(E_1/S, v) \xrightarrow{s_0^*} \mathbb{E}^{BM}(Z/S, v - E_1)$$

$$[Z_\varphi] = s_0^* \, sp^{d}(\mathbb{1})$$

In the special case of $\mathbb{E} = H\mathbb{Z}$, we get that,

$$[Z_\varphi) \in H\mathbb{Z}^{BM}(Z/S, T_M - E_1) = CH_{\mathrm{rnk}(\mathcal{E}_\cdot)}(Z)$$

$$Z \subseteq M$$

$$sp^{cl}(\mathbb{1}) = [\mathcal{C}^{st}] \in CH_{d_0}(\mathcal{C})$$

$$\| $$

$$[\mathcal{C}] \qquad\qquad \mathcal{C} \xrightarrow{\phi} \mathcal{E},$$

And we have,

$$[Z_\varphi] = s_0^* \, sp^d(\mathbb{1}) = s_0^* \, \phi_* \, sp^{cl}(\mathbb{1})$$

$$= s_0^* \, \phi_* [\mathcal{C}] = [Z, \mathcal{E}_0]_{BF}$$