

Virtual fundamental classes in motivic homotopy theory

Marc Levine: The intrinsic stable normal cone

Previously: We defined the **intrinsic normal cone**, the **fundamental class** and **virtual fundamental classes** in several settings.

Today: do the same in motivic homotopy theory

- 1) Notation
- 2) SH^G
- 3) Borel-Moore homology
- 4) The intrinsic stable normal cone
- 5) Specialization to the normal cone $\leadsto$ fundamental class
- 6) p.o.t. $\leadsto$ virtual fundamental classes
- 7) Example

1) Notation

B = base scheme

eg $B = \text{Spec } k$

G = group scheme/ B

$G = \{1/d\}$

G -resolution-property: $\forall$ finitely generated quasi-coherent

G -module M on B $\exists$ locally free G -module $E \twoheadrightarrow M$

$\text{Sch}_B^G = G$ -quasi-projective B -schemes

$\text{Sm}_B^G =$ smooth G -quasi-projective B -schemes

$\pi_X: X \rightarrow B$ = structure map

$\mathcal{H}_*^G(X)$ = equivariant unstable motivic homotopy category

$\text{SH}^G(X)$ = equivariant stable motivic homotopy category

$\mathbb{S}_B \in \text{SH}^G(B)$ sphere spectrum

2) SHG

$X \in \text{Sch}_B^G$

- $\text{SH}^G(X)$ is
- closed symmetric monoidal
 $1_X, \wedge_X, \text{Hom}_X(-, -)$
 - triangulated

Six operations:

Hoyois: The six operations
in equivariant motivic
homotopy theory

$f: Y \rightarrow X$ in Sch_B^G

$$f^*: \text{SH}_B^G(X) \rightleftarrows \text{SH}_B^G(Y): f_*$$

$$f_!: \text{SH}_B^G(Y) \rightleftarrows \text{SH}_B^G(X): f^!$$

For f smooth, f^* admits a left adjoint $f^\#$

Some properties:

- f^* is symmetric monoidal
- natural transformation $f_! \rightarrow f_*$ which is an iso for f proper
- j open immersion $\Rightarrow j^! = j^*$
- $Z \xrightarrow{i} X \xleftarrow{j} U$
closed immersion open complement

$\leadsto$ localization distinguished triangle

$$j_! j^! \rightarrow \text{Id}_{\text{SH}^G(X)} \rightarrow i_* i^*$$

For a G -vector bundle $p: V \rightarrow X$

$$\leadsto \Sigma^V: \text{SH}^G(X) \rightarrow \text{SH}^G(X)$$

$$\alpha \mapsto \text{Th}_X(V) \wedge \alpha$$

$$\leadsto \text{Inverse } \Sigma^{-V}: \text{SH}^G(X) \rightarrow \text{SH}^G(X)$$

For $f: Y \rightarrow X$ smooth

$$f_! \cong f_{\#} \circ \Sigma^{-T_{Y/X}} \quad \text{and} \quad f^! \cong \Sigma^{T_{Y/X}} \circ f^*$$

Autoequivalences:

(Ayoob, Riou for $G=\text{Id}$)
Levine for general G

For a perfect complex E .

$$\leadsto \text{autoequivalence } \Sigma^{E, V}: \text{SH}^G(X) \rightarrow \text{SH}^G(X)$$

st for $f: Y \rightarrow X$

$$\Sigma^{f^* E, V} \circ f^* \cong f^* \circ \Sigma^{E, V}$$

$$\Sigma^{f^* E, V} \circ f_! \cong f_! \circ \Sigma^{E, V}$$

$$f_* \circ \Sigma^{f^* E, V} \cong \Sigma^{E, V} \circ f_*$$

$$f_! \circ \Sigma^{f^* E, V} \cong \Sigma^{E, V} \circ f_!$$

$$E^1 \rightarrow E^0 \rightarrow E^2 \rightarrow \dots$$

$$\leadsto \sum E^{\bullet} \cong \sum E^{1\bullet} \circ \sum E^{2\bullet}$$

$$\text{and } \sum E^{\bullet}[1] \cong (\sum E^{\bullet})^{-1}$$

$$E_x: E = (E_n \rightarrow \dots \rightarrow E_m)$$

$$\Rightarrow \sum E^{\bullet} \cong \sum (-1)^n E_n^{\bullet} \circ \dots \circ \sum (-1)^m E_m^{\bullet}$$

At'-homotopy invariance:

$f: V \rightarrow Z$ affine space bundle $\Rightarrow f_! f^! \rightarrow \text{Id}$
is a natural isomorphism.

$$f_! \circ f^*$$

Proper pushforward and smooth pullback

$$p: Z \rightarrow W \text{ proper}$$

$$\leadsto \begin{matrix} \pi_Z \searrow & & \swarrow \pi_W \\ Z & \xrightarrow{p} & W \end{matrix}$$

proper pullback

$$p^*: \pi_{W!} \rightarrow \pi_{Z!} \circ p^*$$

$$\downarrow \qquad \qquad \qquad \uparrow$$

$$\pi_{W!} p_* p^* \rightarrow \pi_{W!} p_! p^*$$

$$f: Z \rightarrow W \text{ smooth}$$

$$\leadsto \text{Smooth pushforward}$$

$$f_*: \pi_{Z!} \circ \sum T_{Z/W} \circ f^* \rightarrow \pi_{W!}$$

If f is an affine space bundle
 f_* is an isomorphism.

For a section s of f one can define

$$s_*: \pi_{W!} \rightarrow \pi_{Z!} \circ \sum T_{Z/W} \circ f^* \text{ st } f_* \circ s_* = \text{id}$$

3) Borel-Moore $\mathcal{E}$ -homology

$\mathcal{E} \in \mathcal{SH}^q(B)$ commutative monoid

$$\mathcal{E}^{a,b}(-) = \operatorname{Hom}_{\mathcal{SH}(B)}(-, \mathcal{E} \wedge_B (S^1)^{a-b} \wedge_B \mathbb{G}_m^b)$$

Borel-Moore motive $\mathbb{Z}/B_{B.M.} := \pi_{\mathbb{Z}}! \mathbb{1}_{\mathbb{Z}} \in \mathcal{SH}^q(B)$

$$\mathbb{Z} \hookrightarrow M$$

$$\simeq \pi_{M!}(M/M-\mathbb{Z})$$

Borel-Moore homology $\mathcal{E}_{a,b}^{B.M.}(\mathbb{Z}) := \mathcal{E}^{-a, -b}(\mathbb{Z}/B_{B.M.})$

Examples: $G = \{Id\}$, $B = \operatorname{Spec} k$

1) $\mathcal{E} = H\mathbb{Z}$ = ring spectrum representing
motivic cohomology / higher
Chow groups

$H\mathbb{Z}$ is **oriented** that means for

$V \rightarrow \mathbb{Z}$ a r -k r VB / perfect complex E_r
of virtual r -k r

we get **Thom isomorphisms**

$$H\mathbb{Z}^{a,b}(\pi_{\mathbb{Z}}! \Sigma^V \alpha) \simeq H\mathbb{Z}^{a-2r, b-r}(\pi_{\mathbb{Z}}! \alpha)$$

$$\alpha \in \mathcal{SH}(\mathbb{Z})$$

$$H\mathbb{Z}^{a,b}(\pi_{\mathbb{Z}}! \Sigma^{E_r} \alpha) \simeq H\mathbb{Z}^{a-2r, b-r}(\pi_{\mathbb{Z}}! \alpha)$$

The virtual fundamental class will live in $H\mathbb{Z}^{0,0}(\pi_{Z!} \sum^{E_0} 1_Z) \cong H\mathbb{Z}_{2r,r}^{B.M.}(Z)$

$\cong$
 $H\mathbb{Z}$
 $\cong$

$\cong$
 $CH_r(Z)$

$$2) E = H_0(S_k)$$

represents Milnor - Witt K-theory

i.e. for $X \in Sm_k$

$$H_0(S_k)^{a+b,b}(X) = H_{Nis}^a(X, K_b^{MW})$$

$$H_0(S_k)^{a+b,b}(\pi_{X\#} \sum^{E_0} 1_X) = H_{Nis}^a(X, K_b^{MW}(\det E_0))$$

The virtual fundamental class will live

$$\text{in } H(S_k)^{0,0}(\pi_{Z!} \sum^{E_0} 1_Z) \cong \tilde{CH}_r(Z, \det E_0)$$

$\lambda \cong$

$$H(S_k)_{2r,r}^{B.M.}(Z)$$

$\cong$

$$\tilde{CH}_r(Z)$$

$H_0(S_k)$ is **SL-oriented** i.e. an iso

$$\lambda: \det E_0 \rightarrow \mathcal{L}^{\otimes 2} \rightsquigarrow \text{Thom isomorphisms}$$

$\uparrow$ line bundle
on Z

4) The intrinsic stable normal cone

Set up for the rest of the talk

$$Z \in \text{Sch}_B^G$$

$i: Z \hookrightarrow M$ closed immersion, $M \in \text{Sm}_B^G$
with ideal sheaf I_Z

$$\mathcal{C}_i := \text{Spec} \bigoplus_{n \geq 0} I^n / I^{n+1}$$

$$\begin{array}{ccccc} \mathcal{C}_i & \xrightarrow{p_i} & Z & \xhookrightarrow{i} & M \\ & \searrow \pi_{\mathcal{C}_i} & \downarrow \mathcal{G}_i & \nearrow \pi_M & \\ & & B & & \end{array}$$

Intrinsic stable normal cone $\mathcal{C}_Z^{\text{st}} \in \text{SH}^G(B)$

comes with an isomorphism

$$\mathcal{C}_Z^{\text{st}} \xrightarrow[\cong]{\alpha_i} \pi_{\mathcal{C}_i!} \circ \sum \mathcal{G}_i^* T_{M/B} 1_{\mathcal{C}_i}$$

$$\begin{array}{c} \downarrow \cong \text{ Lemma 2.1} \\ \alpha_i \searrow \pi_{\mathcal{C}_i!} \circ \sum \mathcal{G}_i^* T_{M/B} 1_{\mathcal{C}_i} \end{array}$$

Ex: $Z \in \text{Sm}_B^g$, $M = Z$, $i = \text{id}$

Then $(\mathbb{P}_Z^{\text{st}} \simeq \pi_{Z!} \circ \sum \pi_{Z/B}^* 1_Z = \pi_{Z\#} 1_Z$

"
 $Z \in \text{SH}^g(B)$

5) Specialization to the normal cone
and the fundamental class

Goal: define a fundamental class

$$[\mathbb{P}_Z^{\text{st}}]_{\mathcal{E}} \in \mathcal{E}_{\mathcal{F}_B}^{0,0}(\mathbb{P}_Z^{\text{st}})$$

So we need a $\mathbb{P}_Z^{\text{st}} \rightarrow \mathcal{F}_B \rightarrow \mathcal{E}$

commutative
monoid

Ex: $\mathcal{E} = H\mathbb{Z}$

$$[\mathbb{P}_Z^{\text{st}}]_{H\mathbb{Z}} \in H\mathbb{Z}^{0,0}(\mathbb{P}_Z^{\text{st}})$$

$$\parallel \\ \text{CH}_{d_{\mathbb{P}_Z}}(\mathbb{P}_Z)$$

Deformation Space

$$\text{Def}(i) := \text{Bl}_{Z \times 0} M \times A^1 \cap \text{Bl}_Z M \xrightarrow{P} M \times A^1$$

$\swarrow \text{ } i_{C_i}$
 $\nwarrow \text{ } j$

C_i
 $M \times (A^1 - 0)$

Fundamental Class:

Localization distinguished triangle

$$j_! j^! \rightarrow \text{Id}_{\text{Stg}(\text{Def}(i))} \rightarrow i_{C_i!} i_{C_i}^*$$

+ twist by $\sum P^* P_1^* T_{M/B}$

+ pushforward by $\pi_{\text{Def}(i)!}$

$$\begin{array}{ccc} & M \times A^1 - 0 & \\ P_1^* \swarrow & & \searrow P_2^* \\ M & & A^1 - 0 \end{array}$$

→ distinguished triangle

$$\sum_T^{-1} M \times (A^1 - 0)_+$$

$$T = S^1 \wedge G_m$$

$$\pi_{M \times (A^1 - 0)!} \sum P_1^* T_{M/B} \perp M \times (A^1 - 0)$$

$$\rightarrow \pi_{\text{Def}(i)!} \sum P^* P_1^* T_{M/B} \perp_{\text{Def}(i)}$$

$$\rightarrow \pi_{C_i!} \sum C_i^* T_{M/B} \perp_{C_i}$$

$$\pi_{C_i!} \circ \sum 2_i^* \tau_{H/B} 1_{C_i} \xrightarrow{\partial} \cancel{\sum_{S'}} \cancel{\sum_{S'}^{-1}} \sum_{G_m}^{-1} (M \times A^1 - \partial)_+$$

$$\uparrow \alpha_i$$

$$C_Z^{st}$$

$$\downarrow$$

$$\sum_{G_m}^{-1} G_m = S_B$$

$$\leadsto C_Z^{st} \rightarrow S_B \in S_B^{0,0}(C_Z^{st})$$

$=:$ fundamental class $[C_Z^{st}]$

For a commutative monoid E

$$\text{we get } E \circ [C_Z^{st}] \in E^{0,0}(C_Z^{st})$$

$\nwarrow \text{unit}$

$$[C_Z^{st}]_E$$

Remark: Lemma 3.1 shows that this is well-defined

Ex: $E = H\mathbb{Z}$, $B = \text{Spec } k$, $G = \{Id\}$

∂ induces a map

$$H\mathbb{Z}'''(M \times A^1 - \partial) \rightarrow H^{0,0}(\pi_{C_i!} \circ \sum 2_i^* \tau_{H/B} 1_{C_i})$$

$$\Sigma^u[t] \mapsto \partial[t]$$

$$[C_i] \in CH_{d_{C_i}}(C_i)$$

6) Perfect obstruction theories and virtual fundamental class

$[\Phi]: E_0 \rightarrow \mathbb{L}_{Z/B}$ be a p.o.t. on Z of virtual rank r .

Goal: Define a virtual fundamental class

$$[Z, [\Phi]]_{\varepsilon}^{\text{vir}} \in \mathcal{S}_B^{0,0}(\pi_{Z!} \circ \Sigma^{E.v} 1_Z)$$

ε

$$\| E \rightarrow H^{\vee}$$

$$CH_*(Z)$$

There is $\widehat{\Phi}: (F_1 \rightarrow F_0) \rightarrow (\mathbb{L}_{Z/B}^2 \rightarrow i^* \Omega_{M/B})$

• iso on h_0

• surj on h_r

representing $[\Phi]$

Normalized representative

(F_0, Φ) is **normalized** if Φ_0, Φ_1 are surjective

G-resolution property $\Rightarrow$ this exists

Assume (F_0, Φ) is normalized

$$\Phi_1: F_1 \twoheadrightarrow I_2/I_2^2$$

$$\leadsto \text{Sym}^* F_1 \twoheadrightarrow \bigoplus_{n \geq 0} I_2^n / I_2^{n+1}$$

$$i_\Phi: \mathbb{C}_i \hookrightarrow \text{Spec Sym}^* F_1 =: F'$$

Diagram illustrating the relationship between various spaces and maps:

- $\mathbb{C}_i$ maps to $\text{Spec Sym}^* F_1 =: F'$ via i_Φ .
- $\mathbb{C}_i$ maps to Z via p_i .
- F' maps to Z via $p_{F'}$.
- Z maps to F' via s_0 .
- $\mathbb{C}_i$ maps to M via G_i (large curved arrow).
- $\mathbb{C}_i$ maps to M via i (smaller curved arrow).

$$\pi_{\mathbb{Z}!} \circ \sum^{-\mathbb{F}!} \circ \sum^{i^* T_{M/B}} 1_{\mathbb{Z}}$$

$$\begin{array}{c} \searrow S_0^* \\ \pi_{\mathbb{F}!} \circ \sum^{T_{\mathbb{F}/\mathbb{Z}}} p_{\mathbb{F}}^* \sum^{-\mathbb{F}!} \sum^{i^* T_{M/B}} 1_{\mathbb{Z}} \end{array} \quad = p_{\mathbb{F}}^* \mathbb{F}^!$$

$$\pi_{\mathbb{F}!} \circ \sum^{112} p_{\mathbb{F}}^* i^* T_{M/B} 1_{\mathbb{F}}$$

$$\downarrow i_{\Phi}^*$$

$$\pi_{\mathbb{C}!} \circ i_{\Phi}^* \circ \sum p_{\mathbb{F}}^* i^* T_{M/B} 1_{\mathbb{F}}$$

$$112$$

$$\pi_{\mathbb{C}!} \circ \sum \mathcal{G}_i^* T_{M/B} 1_{\mathbb{C}}$$

$$\cong \bigwedge^2 \alpha_i$$

$$\mathbb{C}_{\mathbb{Z}}^{st}$$

Ex: $E = H\mathbb{Z}$, $G = \langle \text{id} \rangle$, $B = \text{Spec } k$

$$s_0^*: CH_{d_i}(\mathbb{F}') \rightarrow CH_r(\mathbb{Z})$$

$$i_{\Phi_*} = CH_{d_i}(\mathbb{C}_i) \rightarrow CH_{d_i}(\mathbb{F}')$$

are the intersection with the zero section and proper pushforward.

Reduced normalized representatives

A normalized representative (F_0, Φ) of $[\Phi]$ is **reduced** if $F_0 = i^* \Omega_{\mu/B}$ and $\Phi_0 = \text{id}$.

Lemma 4.2/4.3 If Z is affine then $\exists$ a reduced normalized representative unique up to a unique chain homotopy.

Reduction to the affine case

From now on assume B is affine

Jouanolou device

$$\begin{array}{ccc} \tilde{Z} & \xrightarrow{i_{\tilde{Z}}} & \tilde{M} \\ p_Z \downarrow & & \downarrow p_M \\ Z & \xrightarrow{i_Z} & M \end{array} \quad \text{affine space bundle}$$

Marc defines an induced p.o.t.

$(\tilde{E}, p_M^*[\phi])$ on $\tilde{Z}$ (Lemma 4.4)

and for a reduced normalized representation

$(\tilde{F}, \tilde{\Phi})$ an isomorphism (4.3)

$$\pi_{Z!} \circ \Sigma^{E.v} 1_Z \xrightarrow{\sim} \pi_{\tilde{Z}!} \circ \Sigma^{-\tilde{F}} \circ \Sigma^{i_{\tilde{Z}}^* T_{\tilde{M}/B}} 1_{\tilde{Z}}$$

$$\Phi^* \int_{E \cdot [\Phi]}$$

$$i_{\tilde{\Phi}}^* \circ s_{00} \downarrow$$

$$\pi_{\tilde{\Phi}!} \circ \Sigma^{G_{i_{\tilde{Z}}}} T_{\tilde{M}/B} 1_{\tilde{\Phi}!}$$

$$112$$

$$\subset \tilde{Z}^{st}$$

virtual fundamental class

$$\downarrow \cong \text{Lemma 2.4}$$

$$\begin{aligned}
 [Z, [\Phi]]^{\text{vir}} &:= \Phi_{E, [\Phi]}^{\bullet}([\mathcal{O}_Z^{\text{st}}]) \xrightarrow{\quad} \mathcal{O}_Z^{\text{st}} \\
 &\in \mathbb{S}_B^{0,0} / (\pi_{Z!} \circ \Sigma^{E, \vee} 1_Z) \downarrow \\
 &\quad \quad \quad \mathbb{S}_B \\
 &\quad \quad \quad \downarrow \\
 &\quad \quad \quad E
 \end{aligned}$$

Rmk: Marc proves this is well-defined.

Comparison with Behrend-Fantechi

Z affine, $B = \text{Spec } k$, $G = \{id\}$

Let $[\Phi]$ be a p.o.t. on Z
with reduced normalized
representative

$$\begin{aligned}
 (F_1 \rightarrow F_0) &\xrightarrow{\Phi} \mathcal{I}_Z / \mathcal{I}_Z^2 \rightarrow i^* \Omega_{M/B} \\
 &\quad \quad \quad \parallel \\
 &\quad \quad \quad i^* \Omega_{M/B}
 \end{aligned}$$

$$\mathcal{O}_Z = [\mathcal{O}_i / i^* \mathcal{T}_{M/B}] \hookrightarrow \mathcal{N}_Z = [\mathcal{N}_i / i^* \mathcal{T}_{M/B}] \hookrightarrow [F' / F_0]$$

pullback along $F' \rightarrow [F'/F^0]$

$i^* T_{M/B}$

$$\mathcal{O}(F^0) \subset \mathcal{N}(F^0) \subset F'$$

$$\parallel$$

$$\mathcal{O}_i$$

$$\parallel$$

$$\mathcal{N}_i$$

$$[Z, [\phi]]_{BF}^{vir} = s_0^*([\mathcal{O}(F^0)])$$

$$\parallel$$

$$s_0^*([\mathcal{O}_i]) \in CH_r(Z)$$

It follows that

$$[Z, [\phi]]_{BF}^{vir} = [Z, [\phi]]_{HZL}^{vir}$$

because 1) $[\mathcal{O}_i] \in CH_{d_{\mathcal{O}_i}}(\mathcal{O}_i)$



$$\parallel$$

$$H\mathcal{L}^{0,0}(\pi_{\mathcal{O}_i}^* \circ \sum \mathcal{O}_i^* T_{M/B} \downarrow_{\mathcal{O}_i})$$

$$\parallel \alpha_i$$

$$[\mathcal{O}_Z^{st}]_{HZL} \in H\mathcal{L}^{0,0}(\mathcal{O}_Z^{st})$$

2) S_{0*} and $i_{\mathcal{D}}^*$ induced the intersection with the zero section and proper pushforward in Chow groups, respectively.

7) Example $G = \{Id\}$, $B = \text{Spec } k$
 $\uparrow$
 perfect

$g = (g_1, \dots, g_n) : A^n_k \rightarrow A^n_k$
 st $Z = (g^{-1}(0))_{\text{red}}$ is 0-dim

$I = (g_1, \dots, g_n) \quad i: Z \hookrightarrow A^n$

$i^* \Omega_{A^n/k}^\vee \twoheadrightarrow I/I^2$

$\frac{\partial}{\partial x_i} \mapsto g_i$

$\leadsto$ reduced and normalized p.o.t.

$$\Phi : (i^* \Omega_{A^n/k}^{\vee} \xrightarrow{\deg} i^* \Omega_{A^n/k})$$

$$\rightarrow (\bar{I}/\bar{I}^2 \xrightarrow{d} i^* \Omega_{A^n/k})$$

$$\hookrightarrow [Z, g]^{vir} \in \mathcal{S}_k^{0,0}(\pi_{Z!} \circ \underbrace{\sum i^* \Omega_{A^n/k}^{\vee} - i^* \Omega_{A^n/k}^{\vee}}_{id} 1_Z)$$

$T_{A^n/k}$ has basis $\frac{\partial}{\partial x_i}$

$T_{A^n/k}^{\vee}$ has basis dx_i

$id \quad //$

$$\mathcal{S}_k^{0,0}(\pi_{Z!} 1_Z)$$

$//$

$$\mathcal{S}_{k,0,0}^{BM}(Z)$$

$$\downarrow \pi_{Z*}$$

$$\mathcal{S}_{k,0,0}^{BM}(\text{Spec } k)$$

$//$

$$\hookrightarrow \deg [Z, g]^{vir} \in GL(k)$$

Claim: $\deg [Z, g]^{\text{vir}} = S_{A'}(g, z) = \sum S_{A'}(g, x_i)$
 $z = \{x_1, \dots, x_r\}$

where $S_{A'}(g, z)$ is the local A' -degree

↑

see Kass-
Wickelgren

EKL paper

$S^{A'}(g, x) = \text{Morel's } A'\text{-degree}$
of stabilization of

$$S_{u, n}^{2n, n} \simeq \frac{P_u^n}{P_u^{n-1}} \rightarrow \frac{P_u^n}{P_u^n - x} \simeq \frac{u}{u-x}$$

small
neighbor
of x

$$\int g(u, u-x)$$

$$S_{u, n}^{2n, n} \simeq \frac{P_u^n}{P_u^{n-1}} \simeq P_u^n / P_u^n - 0$$

$$\text{So } S_{A^1}(g, x) \in \text{End}_{SH(k)}(S_k) = G_W(k)$$

Why should this claim be true?

Let's assume g is étale in a nhhd of $\mathbb{Z}$.

$$\mathbb{C}_i = \mathbb{C}_{\mathbb{Z} \subset A^n} \longrightarrow i^* T_{A^n/k}^{\vee} = \overline{T}^1$$

$// \mathbb{Z}$

$// \mathbb{Z}$

$$A^n_{\mathbb{Z}} \longrightarrow A^n_{\mathbb{Z}}$$

$\uparrow$

Jacobian of g
 $J(g)$

$$\begin{aligned} \Rightarrow \deg [\mathbb{Z}, g]^{\text{vir}} &= \text{Tr}_{\mathbb{Z}/k} \langle \det J(g) \rangle \\ &= S_{A^1}(g, \mathbb{Z}). \end{aligned}$$