An Introduction to Virtual Fundamental Classes

November 4, 2020

Construction (Cycle class of a section)

$\mathcal{E}$: a locally free sheaf of rank r on some smooth $X \rightsquigarrow E := \mathbb{V}(\mathcal{E}) \to X$ the vector bundle

$$\mathbb{V}(\mathcal{E}) = \mathrm{Spec}_{\mathcal{O}_X}(\mathsf{Sym}^*\mathcal{E}).$$

A section $s : X \to E$ with zero-subscheme $Z \leftrightarrow$ a surjection $p : \mathcal{E} \to \mathcal{I}_Z$.
This gives the cartesian diagram ($s_0 :=$ the zero-section)

$$\begin{array}{ccc} Z & \longrightarrow & X \\ \big\downarrow & & \big\downarrow{\scriptstyle s} \\ X & \underset{s_0}{\longrightarrow} & E \end{array}$$

Fulton's intersection theory tells us how to associate to this: a class $[s] := s_0^!([X]) \in \mathrm{CH}_{d-r}(Z)$, $d = \dim X$.

Remark

The class $[s] := s_0^!([X]) \in \mathrm{CH}_{d-r}(Z)$ pushes forward to the top Chern class $c_r(E) \in \mathrm{CH}_{d-r}(X) = \mathrm{CH}^r(X)$.

We can view the surjection $p : \mathcal{E} \to \mathcal{I}_Z$ as giving r generators for $\mathcal{I}_Z$, locally on X: Take $U \subset X$ with an isomorphism $\mathcal{E}_{|U} \cong \mathcal{O}_U^r = \oplus_{i=1}^r \mathcal{O}_U e_i$ then $\mathcal{I}_{Z|U} = (p(e_1), \dots, p(e_r))\mathcal{O}_U$.

Thus: Z has pure codimension $r \Rightarrow Z$ is a local complete intersection on X and p induces an isomorphism

$$\bar{p} : \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_Z \xrightarrow{\sim} \mathcal{I}_Z / \mathcal{I}_Z^2$$

Example (local complete intersections)

Suppose Z has pure codimension r on X, with irreducible components $Z_1, \dots, Z_m$. $|Z| = \sum_{i=1}^m n_i Z_i$, the associated cycle,

$$n_i := \text{length}_{\mathcal{O}_{X,Z_i}} \mathcal{O}_{X,Z_i} / \mathcal{I}_Z$$

Then

$$s_0^!([X]) = |Z|.$$

In general $s_0^!([X]) \in \mathrm{CH}_{d-r}(Z)$ is defined by the *deformation to the normal cone*:

Construction (Deformation to the normal cone)

Take the blowup $\mu : \mathrm{Bl}_{s_0(X)\times 0} E \times \mathbb{A}^1 \to E \times \mathbb{A}^1$. Form the deformation space

$$Def(s_0) = \mathrm{Bl}_{s_0(X)\times 0} E \times \mathbb{A}^1 \setminus \mu^{-1}[E \times 0] \subset \mathrm{Bl}_{s_0(X)\times 0} E \times \mathbb{A}^1.$$

We have $\pi : Def(s_0) \to \mathbb{A}^1$ with

$$\pi^{-1}(\mathbb{A}^1 \setminus \{0\}) = E \times (\mathbb{A}^1 \setminus \{0\}); \quad \pi^{-1}(0) = N_{s_0(X)} E$$

Here $N_{s_0(X)} E$ is the normal bundle. $N_{s_0(X)} E \subset Def(s_0)$ is a Cartier divisor.

Note that $N_{s_0(X)} E = E$; let $E_Z = E_{|Z}$ with 0-section $s_{0,E_Z} : Z \to E_Z$.

Let $\tilde{C}$ = closure of $s(X) \times \mathbb{A}^1 \setminus \{0\}$ in $Def(s_0)$. Form the intersection product

$$(N_{s_0(X)} E) \cdot [\tilde{C}] \in \mathrm{CH}_d(N_{s_0(X)} E \cap \tilde{C})$$

Since $N_{s_0(X)} E \cap \tilde{C} \subset E_Z \subset E$, we have

$$(N_{s_0(X)} E) \cdot [\tilde{C}] \in \mathrm{CH}_d(E_Z) \xrightarrow[\sim]{s_{0,E_Z}^*} \mathrm{CH}_{d-r}(Z)$$

and

$$s_0^!([X]) := s_{0,E_Z}^*((N_{s_0(X)} E) \cdot [\tilde{C}]).$$

Construction (The normal cone and its embedding in E_Z)

$(N_{s_0(X)}E) \cap \tilde{C} = C_i := \mathrm{Spec}_{\mathcal{O}_Z}(\oplus_{n\geq 0}\mathcal{I}_Z^n/\mathcal{I}_Z^{n+1})$, the *normal cone* of $i : Z \to X$.

Let $\mathcal{E}_Z := \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_Z$. The surjection $p : \mathcal{E} \to \mathcal{I}_Z \rightsquigarrow$ a surjection $\bar{p} : \mathcal{E}_Z \to \mathcal{I}_Z/\mathcal{I}_Z^2$, inducing the surjection of graded $\mathcal{O}_Z$-algebras

$$\mathrm{Sym}^*\bar{p} : \mathrm{Sym}^*_{\mathcal{O}_Z}\mathcal{E}_Z \to \oplus_{n\geq 0}\mathcal{I}_Z^n/\mathcal{I}_Z^{n+1}$$

$\mathrm{Sym}^*\bar{p}$ induces the closed immersion $i : C_i \to E_Z$ which is exactly the closed immersion $N_{s_0(X)}E \cap \tilde{C} \subset E_Z$. Thus:

$$s_0^!([X]) = s_{0,E_Z}^*(|C_i|).$$

An important fact: $s_{0,E_Z}^*(|C_i|)$ depends only on

i. The embedding $i : Z \to X$

ii. The vector bundle $E_Z = \mathbb{V}(\mathcal{E}_Z) = \mathrm{Spec}\,\mathrm{Sym}^*(\mathcal{E}_Z)$

iii. The surjection $\bar{p} : \mathcal{E}_Z \to \mathcal{I}_Z/\mathcal{I}_Z^2$

(ii) and (iii) require only information on Z itself!

How does $s^*_{0,E_Z}(|C_i|)$ depend on $i : Z \to X$?

A simple case: Take Y smooth over k. Take a morphism $j : Z \to Y$ and replace $i : Z \to X$ with $(i, j) : Z \to X \times Y$. Then

$$\mathcal{I}_{Z\subset X\times Y}/\mathcal{I}^2_{Z\subset X\times Y} \cong \mathcal{I}_Z/\mathcal{I}^2_Z \oplus j^*\Omega_{Y/k}$$

and $p_X : X \times Y \to X$ induces

$$p_X : C_{(i,j)} \to C_i$$

making $C_{(i,j)} \to C_i$ isomorphic to the pullback of T_Y by $C_i \to Z \to Y$.

We replace the surjection $\bar{p} : \mathcal{E}_Z \to \mathcal{I}_Z/\mathcal{I}^2_Z$ with

$$\bar{p}' : \mathcal{E}_Z \oplus j^*\Omega_{Y/k} \to \mathcal{I}_{Z\subset X\times Y}/\mathcal{I}^2_{Z\subset X\times Y}.$$

To encode this dependence:

$i : Z \hookrightarrow X$ induces the exact sequence

$$\mathcal{I}_Z/\mathcal{I}_Z^2 \xrightarrow{d} i^*\Omega_{X/k} \to \Omega_{Z/k} \to 0$$

The surjection $\bar{p} : \mathcal{E}_Z \to \mathcal{I}_Z/\mathcal{I}_Z^2$ gives

$$d \circ \bar{p} : \mathcal{E}_Z \to i^*\Omega_{X/k}$$

and the map of complexes

$$(\bar{p}, \mathrm{Id}) : (\mathcal{E}_Z \xrightarrow{d\circ\bar{p}} i^*\Omega_{X/k}) \to (\mathcal{I}_Z/\mathcal{I}_Z^2 \xrightarrow{d} i^*\Omega_{X/k}). \qquad (*)$$

Since $\bar{p}$ is surjective, $(\bar{p}, \mathrm{Id})$ satisfies:

$h_1((\bar{p}, \mathrm{Id}))$ is surjective and $h_0((\bar{p}, \mathrm{Id}))$ is an isomorphism.

We have the map of complexes for (i,j):

$$(\mathcal{E}_Z \oplus j^*\Omega_{Y/k} \xrightarrow{d\circ\bar{p}'} (i,j)^*\Omega_{X\times Y/k}) \xrightarrow{(\bar{p}',\mathrm{Id})} (\mathcal{I}_{Z\subset X\times Y}/\mathcal{I}^2_{Z\subset X\times Y} \xrightarrow{d} i^*\Omega_{X\times Y/k}). \quad (**)$$

The projection $p_X : X \times Y \to X$ induces a map of $(*)$ to $(**)$. Noting that

$$(i,j)^*\Omega_{X\times Y} = i^*\Omega_{X/k} \oplus j^*\Omega_{Y/k}$$

and

$$\mathcal{I}_{Z\subset X\times Y}/\mathcal{I}^2_{Z\subset X\times Y} \cong \mathcal{I}_Z/\mathcal{I}^2_Z \oplus j^*\Omega_{Y/k}$$

we see that in the commutative diagram

$$\begin{array}{ccc}
(\mathcal{E}_Z \xrightarrow{d\circ\bar{p}} i^*\Omega_{X/k}) & \xrightarrow{(\bar{p},\mathrm{Id})} & (\mathcal{I}_Z/\mathcal{I}^2_Z \xrightarrow{d} i^*\Omega_{X/k}) \\
\downarrow p_X^* & & \downarrow p_X^* \\
(\mathcal{E}_Z \oplus j^*\Omega_{Y/k} \xrightarrow{d\circ\bar{p}'} i^*\Omega_{X/k} \oplus j^*\Omega_{Y/k}) & \xrightarrow{(\bar{p}',\mathrm{Id})} & (\mathcal{I}_Z/\mathcal{I}^2_Z \oplus j^*\Omega_{Y/k} \xrightarrow{d} i^*\Omega_{X/k} \oplus j^*\Omega_{Y/k}).
\end{array}$$

the vertical arrows p_X^* are both quasi-isomorphisms, that is

$$(\bar{p},\mathrm{Id}) = (\bar{p}',\mathrm{Id})$$

as maps in $D^b(\mathrm{Coh}(Z))$.

What about the cones C_i and $C_{(i,j)}$?

The map $(*)$ gives the commutative diagram

$$\begin{array}{ccc} i^*T_X & \xrightarrow{q'} & C_i \\ \| & & \downarrow \iota \\ i^*T_X & \xrightarrow[q]{} & E_Z \end{array}$$

Via q, i^*T_X acts by translation on E_Z and via q' i^*T_X acts on C_i.

Given $(i,j) : Z \to X \times Y$, $T_Y \subset T_{X\times Y}$ acts on $C_{(i,j)}$ and

$$C_i \cong T_Y \backslash C_{(i,j)}, \quad E_Z \cong T_Y \backslash (E_Z \oplus j^*T_Y)$$

which gives

$$s^*_{0,E_Z}([C_{Z/X}]) = s^*_{0,T_Y\backslash(E_Z\oplus j^*T_Y)}([T_Y\backslash C_{Z/X\times Y}]) = s^*_{0,E_Z\oplus j^*T_Y}([C_{Z/X\times Y}]) \in \mathrm{CH}_{d-r}(Z).$$

Suppose $q' : i^* T_X \to C_i$ is injective, giving the "nice" quotient scheme $i^* T_X \backslash C_i$, the vector bundle $i^* T_X \backslash E_Z$ on Z and the cartesian diagram

$$\begin{array}{ccc} C_i & \hookrightarrow & E_Z \\ \downarrow & & \downarrow \\ i^* T_X \backslash C_i & \hookrightarrow & i^* T_X \backslash E_Z \end{array}$$

which gives

$$s^*_{0,E_Z}([C_{Z/X}]) = s^*_{0,i^* T_X \backslash E_Z}([i^* T_X \backslash C_i]) \in \mathrm{CH}_{d-r}(Z).$$

We thus have

$$i^* T_X \backslash C_i \cong i^* T_{X \times Y} \backslash C_{(i,j)}$$

and

$$i^* T_X \backslash E_Z \cong i^* T_{X \times Y} \backslash E_Z \oplus j^* T_Y$$

Using the language of stacks and the category of perfect complexes, Behrend-Fantechi give a “coordinate free” theory of virtual fundamental classes.

Definition

Let $i : Z \to X$ be a closed immersion of a scheme Z in a smooth k-scheme X. Let

$$q : i^* T_{X/k} \to C_i$$

be the map induced by $d : \mathcal{I}_Z/\mathcal{I}_Z^2 \to i^*\Omega_{X/k}$. The *intrinsic normal cone* of Z is the stack quotient $\mathfrak{C}_Z := [i^* T_{X/k} \backslash C_i]$.

Theorem (Behrend-Fantechi)

$\mathfrak{C}_Z$ is independent (up to canonical isomorphism) of the choice of closed immersion $Z \to X$. $\mathfrak{C}_Z$ has a fundamental class $[\mathfrak{C}_Z] \in \mathrm{CH}_0(\mathfrak{C}_Z)$.

This takes care of the cone. Now for the vector bundle stack.

Definition

For a morphism of schemes $f : X \to Y$ we have the *cotangent complex* $L_{X/Y}$ in $D^{perf}(X)$.

Using homological notation, $L_{X/Y}$ is supported in degrees $[0, n]$ for some integer $n \geq 0$. If f is a smooth morphism, then $L_{X/Y} = \Omega_{X/Y}$.

Proposition

Suppose X is a smooth k-scheme and $i : Z \to X$ is a closed immersion. Then $\tau_{\leq 1} L_{Z/k} \cong (\mathcal{I}_Z/\mathcal{I}_Z^2 \xrightarrow{d} i^*\Omega_{X/k})$.

The map $(*)$ gives us a two-term complex of locally free coherent sheaves on Z, $\mathcal{E}_1 \to \mathcal{E}_0$ and a map $(\phi_1, \phi_0) : (\mathcal{E}_1 \to \mathcal{E}_0) \to \tau_{\leq 1} L_{Z/k}$ in $D^b(\mathrm{Coh}(Z))$ such that $h_1(\phi_1, \phi_0)$ is surjective and $h_0(\phi_1, \phi_0)$ is an isomorphism.
If Z is affine, this lifts to $\phi : (\mathcal{E}_1 \to \mathcal{E}_0) \to L_{Z/k}$ in $D^{perf}(Z)$ with the same properties.

We want to express the map $(*)$ used to define the closed immersion $C_i \hookrightarrow E_Z$ in invariant terms. We have already seen by example that by replacing $\mathcal{E}_Z$ with the complex $\mathcal{E}_Z \to i^*\Omega_{X/k}$, replacing $\mathcal{I}_Z/\mathcal{I}_Z^2$ with the complex $\tau_{\leq 1}L_{Z/k}$ and passing to $D^b(\mathrm{Coh}(Z))$, we achieve a (partial) independence of the choice of embedding. Generalizing this is the follow definition.

Definition

Let Z be a k-scheme. An *obstruction theory on Z* is morphism $\phi : \mathcal{E} \to L_{Z/k}$ in $D^{perf}(Z)$ such that

i. $h_1(\phi)$ is surjective and $h_0(\phi)$ is an isomorphism

If in addition
ii. $h_i(\mathcal{E}) = 0$ for $i > 0$ or $i < 1$ ($\mathcal{E}$ *has Tor-amplitude* $[0, 1]$).

ϕ is a *perfect* obstruction theory.

Remark

1. Behrend-Fantechi work in the setting of Deligne-Mumford stacks over a base-scheme S. All the notions described above extend to this setting.

2. The association of the vector bundle stack $[i^*T_X \backslash E_Z]$ to the complex $\mathcal{E}_Z \to i^*\Omega_{X/k}$ can be defined for an arbitrary perfect complex $\mathcal{E}$ in $D^{perf}(Z)$, with the associated vector bundle stack denoted by $h^1/h^0(\mathcal{E}^\vee)$ (or simply $\mathbb{V}(\mathcal{E})$). This vector bundle stack has virtual rank equal to minus the virtual rank $h_0 - h_1$ of $\mathcal{E}$.

3. Some authors use $\mathcal{E} \to \tau_{\leq 1}L_Z$ in $D^b(\mathrm{Coh}_Z)$ instead of $\mathcal{E} \to L_Z$ in $D^{perf}(Z)$.

Theorem (Behrend-Fantechi)

Let Z be a "nice" Deligne-Mumford stack over some base-scheme S. A perfect obstruction theory $\phi : \mathcal{E} \to L_{Z/S}$ induces a canonical closed immersions of stacks $i_\phi : \mathfrak{C}_{Z/S} \to \mathbb{V}(\mathcal{E})$. The virtual fundamental class *$[Z]^{vir}_\phi$ is defined as*

$$[Z]^{vir}_\phi := s^*_{0,\mathbb{V}(\mathcal{E})}(i_{\phi *}([\mathfrak{C}_{Z/S}])) \in \mathrm{CH}_{rank(\mathcal{E})}(Z).$$

Remark

1. In our naive setting of a surjection $\mathcal{E}_Z \to \mathcal{I}_Z/\mathcal{I}_Z^2$, with $Z \subset X$ an embedded scheme, and map $(\mathcal{E}_Z \to i^*\Omega_{X/k}) \to (\mathcal{I}_Z/\mathcal{I}_Z^2 \to i^*\Omega_{X/k})$, the virtual rank is $d - r := \dim X - \mathrm{rank}(E_Z)$ and the class $[Z]_\phi^{vir}$ is $s^*_{0,E_Z}(|C_i|) \in \mathrm{CH}_{d-r}(Z)$.

2. In more detail: $\mathbb{V}(\mathcal{E}) = [i^*T_X \backslash E_Z]$, $\mathfrak{C}_Z = [i^*T_X \backslash C_i]$ and we have the cartesian diagram

$$\begin{array}{ccc} C_i & \hookrightarrow & E_Z \\ \downarrow \pi & & \downarrow \pi \\ \mathfrak{C}_Z & \hookrightarrow & \mathbb{V}(\mathcal{E}) \end{array}$$

with the vertical arrows smooth morphisms (of stacks) of relative dimension $\dim X$. The usual base-change results give

$$\begin{aligned} [Z]_\phi^{vir} &:= s^*_{0,\mathbb{V}(\mathcal{E})}(|\mathfrak{C}_Z|) \\ &= s^*_{0,E_Z}(\pi^*|\mathfrak{C}_Z|) \\ &= s^*_{0,E_Z}(|C_i|) \end{aligned}$$

Example

Take X smooth over k of dimension d, so $L_{X/k} = \Omega_{X/k}$. Take $\mathcal{E} := \Omega_{X/k}$ mapping to $L_{X/k}$ by the identity. We compute the virtual class by taking the identity closed immersion $i : X \to X$. Then $C_i = X$, $[C_i] = [X]$, E_X is the 0-vector bundle $X = X$, so

$$[X]^{vir}_{\mathrm{Id}} = \mathrm{Id}^*_X([X]) = [X]$$

the usual fundamental class of X.

Example

Let $i : Z \to X$ a local complete intersection codimension r closed subscheme of a smooth X of dimension d. Then $L_{Z/k} = (\mathcal{I}_Z/\mathcal{I}_Z^2 \to i^*\Omega_{X/k})$ and $\mathcal{I}_Z/\mathcal{I}_Z^2$ is a rank r locally free sheaf on Z. Take the perfect obstruction theory $\mathrm{Id}_{L_{Z/k}}$. Then $C_i = \mathbb{V}(\mathcal{I}_Z/\mathcal{I}_Z^2) = E_Z$, so

$$[Z]^{vir}_{\mathrm{Id}} = s^*_{0,\mathbb{V}(\mathcal{I}_Z/\mathcal{I}_Z^2)}(|\mathbb{V}(\mathcal{I}_Z/\mathcal{I}_Z^2)|) = [Z] \in \mathrm{CH}_{d-r}(Z)$$

the cycle associated to the dimension $d - r$ scheme Z.

Example

Take X smooth over k of dimension d. Instead of the identity obstruction theory, let $\mathcal{F}$ be an arbitrary locally free sheaf of rank r on X. This gives the obstruction theory

$$\mathcal{E} = (\mathcal{F} \xrightarrow{0} \Omega_{X/k}) \xrightarrow{(0_{\mathcal{F}},\mathrm{Id})} (0 \to \Omega_{X/k})$$

Again $[C_i] = [X]$, but now $E_X = \mathbb{V}(\mathcal{F})$ and $\phi : C_i \to \mathbb{V}(\mathcal{F})$ is the 0-section, so

$$[X]^{vir}_{(0_{\mathcal{F}},\mathrm{Id})} = s^*_{0,\mathbb{V}(\mathcal{F})}(s_{0,\mathbb{V}(\mathcal{F})*}([X]) = c_r(\mathbb{V}(\mathcal{F})) \in \mathrm{CH}_{d-r}(X).$$

If we take $\mathcal{F} = \Omega^\vee_{X/k}$, then $\mathbb{V}(\mathcal{F}) = T^\vee_{X/k}$ and we get

$$[X]^{vir}_{(0_{\Omega^\vee_{X/k}},\mathrm{Id})} = c_d(T^\vee_{X/k}) = (-1)^d c_d(T_{X/k}) \in \mathrm{CH}_0(X)$$

If X is proper over k, then $\deg_k(c_d(T_{X/k}))$ is the Euler characteristic of X, and $(-1)^d[X]^{vir}_{(0_{\Omega^\vee_{X/k}},\mathrm{Id})}$ is the *Euler class* of X.

Remark

This last example

$$\Omega^\vee_{X/k} \xrightarrow{0} \Omega_{X/k}$$

is a (-1-shifted) *symmetric* perfect obstruction theory: $\mathcal{E}^\vee \cong \mathcal{E}[-1]$. Behrend showed that if Z admits a symmetric perfect obstruction theory, then the associated virtual fundamental class is independent of the choice of symmetric perfect obstruction theory and is (in vague terms) a weighted Euler class associated to a constructible function (the *Behrend function*) on Z.

Example (The critical locus)

Let X be smooth and take $f : X \to \mathbb{A}^1$ a function. Let $i : Z \hookrightarrow X$ be the subscheme defined by the vanishing of the section df of Ω_X. We have the surjection $i_{df} : \Omega_X^\vee \to \mathcal{I}_Z$ sending a vector field v to the evaluation $\langle v, df \rangle$, giving $\bar{i}_{df} : i^*\Omega_X^\vee \to \mathcal{I}_Z/\mathcal{I}_Z^2$. In local coordinates $(x_1, \ldots, x_n)$, i_{df} sends $\partial/\partial x_i$ to $\partial f/\partial x_i$. The composition $d \circ \bar{i}_f : i^*\Omega_X^\vee \to i^*\Omega_X$ is represented by the Hessian matrix $(\partial^2/\partial x_i \partial x_j)$ restricted to Z, so

$$(i^*\Omega_X^\vee \to i^*\Omega_X) \to (\mathcal{I}_Z/\mathcal{I}_Z^2 \to i^*\Omega_X)$$

gives a symmetric perfect obstruction theory (at least for Z affine).

The basic problem of classical obstruction theory in algebraic geometry is to describe the germ of a scheme M at some point x by giving a cohomological description of the extensions of a morphism of a pointed Artin scheme $f : (T, t) \to (M, x)$ to a morphism $(\tilde{T}, t) \to (M, x)$ where $T \subset \tilde{T}$ is a closed subscheme defined by a square 0 ideal $\mathcal{J}$.

For example, let M be a "moduli space/stack" for some moduli problem, with flat universal family $p : X \to M$. We have the problem of extending a cartesian square

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \big\downarrow{\scriptstyle q} & & \big\downarrow{\scriptstyle p} \\ T & \xrightarrow[f]{} & M \end{array}$$

to a square over $\tilde{T}$; since p is universal, this is the same as the extension problem for f. Here the obstruction lives in $\mathrm{Ext}^2(q^* L_{X/M}, q^* \mathcal{J})$ and if the obstruction vanishes, the set of extensions is a principal homogeneous space for $\mathrm{Ext}^1(q^* L_{X/M}, q^* \mathcal{J})$ (Illusie, Complexe cotangent, III Thm. 2.1.7).

Example

Take $T = \operatorname{Spec} k$, Y a smooth proper scheme over k, $M = Def(Y)$ the universal deformation space of Y and $X \to M$ the universal family with f the map corresponding to $Y \in Def(Y)$. Let $\tilde{T} = \operatorname{Spec} k[\epsilon]/(\epsilon^2) \Rightarrow$ the set of extensions $= T_Y(Def(Y))$.

$$L_{Y/k} = \Omega_{Y/k}, \operatorname{Ext}^i(L_{Y/k}, (\epsilon)) \cong H^i(Y, T_{Y/k})$$

Since we have the constant extension, the first obstruction vanishes and

$$T_Y(Def(Y)) \cong H^1(Y, T_{Y/k})$$

(proven by Kodaira-Spencer (1958)). For higher-order deformations, there may be obstructions, this was studied by Kuranishi (1962-4), who showed that $Def(Y)$ can be given as an analytic subset of a polydisk in $H^1(Y, T_{Y/k})$, with equations given by the system of higher-order obstructions in $H^2(Y, T_{Y/k})$.

To globalize the Kodaira-Spencer/Kuranishi construction: Assume p is Gorenstein, with dualizing invertible sheaf ω concentrated in degree $-\dim_{X/M}$ and let

$$\mathcal{E}_* := Rp_*(L_{X/M} \otimes \omega)[-1]$$

We have the *Kodaira-Spencer map* $L_{X/M} \to p^* L_M[1]$ as part of the exact triangle

$$p^* L_M \to L_X \to L_{X/M} \to p^* L_M[1]$$

which gives

$$\mathcal{E}_* \to Rp_*(p^* L_M \otimes \omega) = L_M \otimes Rp_*(\omega) \xrightarrow{\mathrm{Id} \otimes \mathrm{Tr}} L_M \otimes \mathcal{O}_M = L_M.$$

where Tr is the canonical trace given by Grothendieck-Verdier-Serre duality. This gives the obstruction theory $\mathcal{E}_* \to L_M$, which is a perfect obstruction theory if

i. No continuous automorphisms: $H^0(f^{-1}(m), L_{X/M} \otimes \omega \otimes k(m)) = 0$ for all $m \in M$

ii. $\dim_{X/M} \leq 2$

Remark (Obstruction theories and moduli spaces)

The whole machinery of perfect obstruction theories and virtual fundamental classes arose because people wanted to do intersection theory on moduli spaces. The tangent space and obstruction space at a point $[X]$ in the moduli space have a description in terms Ext-groups on X, and this often leads to a (perfect) obstruction theory on the moduli stack.

We give some examples of this to illustrate.

Example (The moduli space of stable curves)

Let $\mathcal{M}_{g,n}$ be the moduli (Artin) stack of n-pointed curves of genus g. The construction outlined above gives a perfect obstruction theory with $h_1 = 0$. We are usually interested on the DM stack of stable curves $\bar{M}_{g,n}$ (an open substack of $\mathcal{M}_{g,n}$); the restricted perfect obstruction theory gives us the usual fundamental class

Example (The moduli space of stable maps)

Fix a smooth k-scheme X, an integer n and a genus g. There is a moduli stack of stable maps of n-pointed genus g curves to X, $\bar{M}_{g,n}(X)$, with a universal family $\bar{\pi} : \bar{C}_{g,n}(X) \to \bar{M}_{g,n}(X)$ and universal map $F : \bar{C}_{g,n}(X) \to X$.The fiber of $\bar{\pi}$ over a "map" $f \in \bar{M}_{g,n}(X)$ is the corresponding semi-stable genus g curve and the restriction of F to $\bar{\pi}^{-1}(f)$ is the corresponding morphism.

$\bar{\pi}$ is a projective flat relatively Gorenstein morphism: there is a locally free relative dualizing sheaf ω (supported in degree -1).

Behrend constructs the virtual fundamental class via a *relative* perfect obstruction theory. There is a morphism $q : \mathcal{M}_{g,n}(X) \to \mathcal{M}_{g,n}$ "forget the map to X" (these are the Artin stacks: $\mathcal{M}_{g,n}(X) := \mathrm{Maps}(\mathcal{C}_{g,n}, X)$). The relative version is a map $\mathcal{E}_* \to L_{\mathcal{M}_{g,n}(X)/\mathcal{M}_{g,n}}$ with the same properties as before.

Example (Continued)

Let $\mathcal{C}_{g,n} \to \mathcal{M}_{g,n}$, $\mathcal{C}_{g,n}(X) \to \mathcal{M}_{g,n}(X)$ be the universal curves, we have the cartesian diagram

$$\begin{array}{ccccc}
\bar{C}_{g,n}(X) & \hookrightarrow & \mathcal{C}_{g,n}(X) & \longrightarrow & \mathcal{C}_{g,n} \\
\downarrow{\scriptstyle \bar{\pi}} & & \downarrow{\scriptstyle \pi} & & \downarrow \\
\bar{M}_{g,n}(X) & \hookrightarrow & \mathcal{M}_{g,n}(X) & \xrightarrow[q]{} & \mathcal{M}_{g,n}
\end{array}$$

which gives an isomorphism $\pi^* L_{\mathcal{M}_{g,n}(X)/\mathcal{M}_{g,n}} \cong L_{\mathcal{C}_{g,n}(X)/\mathcal{C}_{g,n}}$. Restricting to the open substack $\bar{M}_{g,n}(X)$ gives $\bar{\pi}^* L_{\bar{M}_{g,n}(X)/\mathcal{M}_{g,n}} \cong L_{\bar{C}_{g,n}(X)/\mathcal{C}_{g,n}}$

The universal map $F : \bar{C}_{g,n} \to X$ induces $dF : F^*\Omega_X \to L_{\bar{C}_{g,n}(X)}$, which maps to $L_{\bar{C}_{g,n}(X)/\mathcal{C}_{g,n}} = \bar{\pi}^* L_{\bar{M}_{g,n}(X)/\mathcal{M}_{g,n}}$. Taking

$$\mathcal{E}_* := R\bar{\pi}_*(F^*\Omega_X \otimes \omega)$$

we have maps

$$\begin{aligned}
\mathcal{E}_* \xrightarrow{R\bar{\pi}_*(dF \otimes \mathrm{Id})} R\bar{\pi}_*(L_{\bar{C}_{g,n}(X)/\mathcal{C}_{g,n}} \otimes \omega) &= R\bar{\pi}_*(\pi^* L_{\bar{M}_{g,n}(X)/\mathcal{M}_{g,n}} \otimes \omega) \\
&= L_{\bar{M}_{g,n}(X)/\mathcal{M}_g} \otimes R\bar{\pi}_*(\omega) \xrightarrow{\mathrm{Id} \otimes \mathrm{Tr}} L_{\bar{M}_{g,n}(X)/\mathcal{M}_{g,n}}
\end{aligned}$$

giving the relative perfect obstruction theory $\phi_{rel} : \mathcal{E}_* \to L_{\bar{M}_{g,n}(X)/\mathcal{M}_{g,n}}$.

Example (Continued)

One can promote this to a perfect obstruction theory: We have the distinguished triangle

$$L_{\mathcal{M}_{g,n}} \to L_{\mathcal{M}_{g,n}(X)} \to L_{\mathcal{M}_{g,n}(X)/\mathcal{M}_{g,n}} \xrightarrow{\delta} L_{\mathcal{M}_{g,n}}[1]$$

and the commutative square

$$\begin{array}{ccc} \mathcal{E}_* & \xrightarrow{\delta \circ \phi_{rel}} & L_{\mathcal{M}_{g,n}}[1] \\ \downarrow{\scriptstyle \phi_{rel}} & & \| \\ L_{\mathcal{M}_{g,n}(X)/\mathcal{M}_{g,n}} & \xrightarrow{\delta} & L_{\mathcal{M}_{g,n}}[1] \end{array}$$

which we can complete to a map of distinguished triangles

$$\begin{array}{ccccccc} L_{\mathcal{M}_{g,n}} & \longrightarrow & \mathcal{E}'_* & \longrightarrow & \mathcal{E}_* & \xrightarrow{\delta \circ \phi_{rel}} & L_{\mathcal{M}_{g,n}}[1] \\ \| & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi_{rel}} & & \| \\ L_{\mathcal{M}_{g,n}} & \longrightarrow & L_{\mathcal{M}_{g,n}(X)} & \longrightarrow & L_{\mathcal{M}_{g,n}(X)/\mathcal{M}_{g,n}} & \xrightarrow{\delta} & L_{\mathcal{M}_{g,n}}[1] \end{array}$$

Then $\phi : \mathcal{E}'_* \to L_{\mathcal{M}_{g,n}(X)}$ is a perfect obstruction theory.

Example (The Hilbert scheme of ideal sheaves on a CY threefold)

Let X be a Calabi-Yau threefold: smooth, projective, $\omega_X \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$. Form the moduli stack $\mathcal{M}$ with $\mathcal{M}(Y) = \{(\mathcal{F}, \phi)\}$ with $\mathcal{F} \in \mathrm{Coh}(Y \times X)$, $\phi : \det(\mathcal{F}) \xrightarrow{\sim} \mathcal{O}_{Y \times X}$ such that

i. $\mathcal{F}$ is flat over Y
ii. $\mathcal{F}$ is perfect
iii. For each $y \in Y$, $\mathrm{End}_{\mathcal{O}_{y \times X}}(\mathcal{F}_y) \cong k(y)$

(we also assume $\mathcal{F}$ is stable with a fixed Hilbert polynomial, but ignore this). $\mathcal{M}$ is represented by an open subscheme M of a Hilbert scheme of sheaves on X. Let $\mathcal{F}_1$ be the universal sheaf on $M \times X$. Let $\mathcal{F} = Cone(R\mathcal{H}om(\mathcal{F}_1, \mathcal{F}_1) \xrightarrow{\mathrm{Tr}} \mathcal{O})[-1]$ and let

$$\mathcal{E} := Rp_{1*} R\mathcal{H}om(\mathcal{F}, \mathcal{O})$$

$\mathcal{E}$ defines a perfect symmetric obstruction theory on M (see R.P.Thomas, *A holomorphic Casson invariant for CalabiYau 3-folds, and bundles on K3 fibrations*, J. Differential Geom. 54:2 (2000), 367-438 and Behrend-Fantechi *Symmetric obstruction theories and Hilbert schemes of points on threefolds*, Algebra & Number Theory 2, no. 3 (2008))

Program

Lecture 1. (Levine) Motivation and background from Gromov-Witten theory

Lectures 2/3. (Jin/Aranha) Chern-MacPherson-Schwartz classes/Overview of stacks and derived schemes

Lecture 4. (Ravi) The Behrend-Fantechi virtual fundamental class

Lecture 5 (Yakerson) Localization of virtual classes

Lecture 6/7. Behrend's work on symmetric obstruction theories

Lecture 8. (Tabakov) Deglise-Jin-Khan Fundamental classes

Lecture 9. (Aranha) Virtual classes for Artin stacks

Lecture 10. Khan's virtual classes for quasi-smooth morphisms

Lecture 11. (D'Angelo) A comparison of the classes of Khan and those of Behrend-Fantechi

Lecture 12. Virtual fundamental classes in motivic homotopy theory