# The Chern-Schwartz-MacPherson class

November 9, 2020

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  such that for $f : X \to Y$ a proper morphism between regular schemes,

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- This is a Riemann-Roch type formula, where the Todd class is replaced by the total relative Chern class
- For schemes of finite type over the field of complex numbers, the Deligne-Grothendieck conjecture is solved and extended to singular schemes by MacPherson

# Constructible functions

- For $X$ a scheme, let $Cons(X)$ be the ring of ($\mathbb{Z}$-valued) **constructible functions** on $X$, i.e. functions $f : X \to \mathbb{Z}$ such that there exists a finite stratification $X = \sqcup X_i$ into constructible subsets such that $f_{|X_i}$ is constant

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- As abelian group, $Cons(X) \simeq \bigoplus_Z \mathbb{Z} \cdot 1_Z$, where $Z$ runs through irreducible closed subsets of $X$
- We have a canonical map

$$\chi : K_0(D^b_{ctf}(X_{et}, \Lambda)) \to Cons(X) \otimes_{\mathbb{Z}} K_0(\Lambda)$$
$$\mathcal{F} \mapsto (x \mapsto [\mathcal{F}_{\bar{x}}])$$

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- Theorem 2 (MacPherson): there exists a unique natural transformation of additive functors $c^{SM} : Cons(-) \to CH_*(-)$ such that
  - $f$ proper, $f_* c^{SM} = c^{SM} f_*$
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- Theorem 2 implies the Deligne-Grothendieck conjecture by multiplying by $c(T_X)$ and by composing with $\chi$

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- Setting: $S_\alpha$, $S_\beta$ two strata, $x_i \in S_\alpha$, $y_i \in S_\beta$, $x \in S_\alpha \cap \overline{S_\beta}$ such that $x_i \to x$, $y_i \to x$, and the lines $\overline{x_i y_i} \to \ell$ in some projective space, and $T_{y_i} S_\beta \to V$ in some Grassmannian

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- Conditions: (A) $T_x S_\alpha \subset V$ (B) $\ell \subset V$
- We have (B) $\Rightarrow$ (A)

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2. *If $Z \subset X$ is a locally finite union of algebraic or analytic subvarieties, then there is a Whitney stratification such that $Z$ is a union of strata*
3. *If $f : X \to Y$ is an algebraic or analytic map, then there are Whitney stratifications such that for any stratum $S$ of $X$, the map $S \to f(S)$ induced by $f$ is a submersion to a stratum*

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  Then for any $y \in T_j$, $\chi_c(f^{-1}(y) \cap W) = \sum_k \chi_c(f^{-1}(y) \cap S_k)$ is constant $\Rightarrow f_* 1_W$ is constructible

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Uniqueness in Theorem 2: let $\alpha \in Cons(X)$. By resolution of singularities and induction, there exists $(n_i) \in \mathbb{Z}$ and proper morphisms $f_i : W_i \to X$ with $W_i$ smooth such that $\alpha = \sum n_i f_{i*} 1_{W_i}$ $\Rightarrow c^{SM}(\alpha) = \sum n_i f_{i*} c(T_{W_i})$.

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\begin{array}{ccc}
Z_*(X) & \xrightarrow[\sim]{Eu} & Cons(X) \\
 & \searrow^{c^M} & \downarrow{\scriptstyle c^{SM}} \\
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- The **Nash blow-up** $\widetilde{X}$ is the closure of the image of $s$, with a canonical morphism $\nu : \widetilde{X} \to X$ by restriction of $p$. There is a canonical vector bundle $T\widetilde{X} \to \widetilde{X}$ by restriction of the universal bundle on $Gr_n(i^* T_M)$

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- These data only depend on $X$ and are independent of $i$

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- Extends by linearity

$$\begin{aligned} c^M : Z_*(X) &\to CH_*(X) \\ \sum n_i Z_i &\mapsto \sum n_i \iota_{i*} c^M(Z_i) \end{aligned}$$

where $\iota_i : Z_i \to X$ is the closed immersion

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- This can be proved using Whitney condition (A) and the Bruhat-Whitney lemma

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- $Eu(Z)(P) = \langle Eu(T^*\widetilde{Z}, r), \mathcal{O} \rangle \in \mathbb{Z}$ **local Euler obstruction**
- Using stratification one show that $P \mapsto Eu(Z)(P)$ is constructible

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- $Eu(Z \times Z')((P, P')) = Eu(Z)(P) \times Eu(Z')(P')$
- If $Z$ is reducible at $P$ with $Z_i$ the irreducible components, then $Eu(Z)(P) = \sum_i Eu(Z_i)(P)$

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- $Eu(Z)(P) = 1$ if $P$ is a regular point of $Z$
- If $Z$ is a curve, then $Eu(Z)(P)$ is the multiplicity of $Z$ at $P$. If $Z$ is the cone on a smooth plane curve of degree $d$ and $P$ is the vertex, then $Eu(Z)(P) = 2d - d^2$
- $Eu(Z \times Z')((P, P')) = Eu(Z)(P) \times Eu(Z')(P')$
- If $Z$ is reducible at $P$ with $Z_i$ the irreducible components, then $Eu(Z)(P) = \sum_i Eu(Z_i)(P)$

By linearity we define the following map $Eu$

$$Eu : Z_*(X) \xrightarrow{\sim} Cons(X)$$
$$\sum_i a_i Z_i \mapsto (P \mapsto \sum_i a_i Eu(Z_i)(P))$$

which we can show to be an isomorphism by induction

# Gonzalez-Sprinberg's algebraic formula

**Theorem (Gonzalez-Sprinberg)**

$$Eu(Z)(P) = \deg(c(T\widetilde{Z}) \cap s(\nu^{-1}(P), \widetilde{Z}))$$

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Alternatively: $Z' = Bl_{\nu^{-1}(P)}\widetilde{Z}$, $D =$ exceptional divisor, $\xi = N_D Z'$

$$Eu(Z)(P) = \deg(c_{d-1}(T\widetilde{Z} - \xi) \cap [D])$$

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- For $\epsilon$ small, $V$ is a neighborhood of $\nu^{-1}(0)$ and $V - \nu^{-1}(0)$ retracts to $\partial V$, so $\sigma_s^*(\omega) = Eu(T\widetilde{Z}, \sigma_s) \in H^{2d}(V, \partial V)$, and we have $Eu(Z)(0) = \deg(\sigma_s^*(\omega) \cap [\widetilde{Z}])$

# Proof of Gonzalez-Sprinberg's formula (II)

- Find $W \subset Gr(n-d, E)$ open and an (algebraic) section $\sigma \in \Gamma(W, T\widetilde{Z}_{|W})$ such that the restriction $s_{|V} : V \to W$ satisfies $\sigma_s = \pi \circ \sigma \circ s_{|V}$ and $s_{|V}^*$ is an inverse of $p^*$ on cohomology, where $\pi : T\widetilde{Z}_{|W} \to T\widetilde{Z}$ and $p : W \to \widetilde{Z}$ are the canonical maps

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- Use Fulton's intersection theory to give an algebraic formula for $\deg[W \cdot_\sigma W]$

# The CSM class reinterpreted

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  Variants in the real case: Stiefel-Whitney classes (Fu-McCrory)

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$$\begin{aligned} \chi : K_0(D^b_c(X)) &\to Cons(X) \\ \mathcal{F} &\mapsto (x \mapsto \chi(\mathcal{F}_x)) \end{aligned}$$

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  More generally this holds for any section of $T^*X$ instead of the zero section

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- The characteristic cycle construction agrees with the microlocal approach, and can be interpreted in terms of vanishing cycles

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- For $L$ conic $\mathbb{C}^*$-invariant cycle on $T^*M$, $\mathbb{P}(L \oplus \mathbb{A}^1) :=$ projective completion of $L$. The *Segre class* of $L$

$$s_*(L) = \pi_*(c(\mathcal{O}(-1))^{-1} \cap [\mathbb{P}(L \oplus \mathbb{A}^1)]) \in CH_*(X)$$

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- For $L$ conic $\mathbb{C}^*$-invariant cycle on $T^*M$, $\mathbb{P}(L \oplus \mathbb{A}^1) :=$ projective completion of $L$. The *Segre class* of $L$

$$s_*(L) = \pi_*(c(\mathcal{O}(-1))^{-1} \cap [\mathbb{P}(L \oplus \mathbb{A}^1)]) \in CH_*(X)$$

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- Example (Sabbah): $c_*([\overline{T^*_{X^{sm}}M}]) = c^M(X)$
- We have $cc = c_* \circ CC$, and the proper covariance of $cc$ reduces to that of $c_*$ and $CC$

# Towards a theory in positive characteristic

- In characteristic 0, by the theory of characteristic cycles, the Euler characteristic $\chi(X, \mathcal{F})$ of a constructible sheaf $\mathcal{F}$ over a smooth proper scheme $X$ only depends on the local Euler-Poincaré index $\chi(\mathcal{F}) \in \mathit{Cons}(X)$

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- In positive characteristic, the singular support need not be Lagrangian (Deligne)
- Nevertheless, it is expected that there is an algebraic cycle $CC(\mathcal{F})$ associated to a constructible étale sheaf $\mathcal{F}$, which satisfies a generalized Milnor formula (SGA7, Deligne) and the index formula

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- $f : X \to Y \in Sm/k \Rightarrow df : T^*Y \times_Y X \to T^*X$

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The proof uses Brylinski's Radon transform

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$$-\operatorname{dimtot} \Phi_u(h^*\mathcal{F}, f) = (CC(\mathcal{F}), df)_{T^*U,u}$$

*where the left-hand side is the total dimension of vanishing cycles*

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- Recently Saito proved this conjecture assuming that the dimension of the image of the singular support is bounded by

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The proof uses the theory of $\epsilon$-*factors*