Trace maps in motivic homotopy

November 9, 2020

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by choosing a closed immersion $i : X \to M$ with M smooth of dimension n and letting

$$\begin{aligned} cc(\mathcal{F}) :=& \mathbb{P}(CC(i_*\mathcal{F}) \oplus \mathbb{A}^1) \\ &\in CH_n(\mathbb{P}(X \times_M T^*M \oplus \mathbb{A}^1)) \simeq \bigoplus_{i=0}^{n} CH_i(X) \end{aligned}$$

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- In characteristic 0, cc is proper covariant, and gives a solution of the Deligne-Grothendieck conjecture
- In positive characteristic, cc fails to be proper covariant, except possibly the 0-dimensional part

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- Related to Behrend's construction on DT-type invariants
- This construction also works in SH, and is related to $\mathbb{A}^1$-enumerative geometry

Thom spaces in motivic homotopy

- We work in the stable motivic homotopy category SH, but the construction works for any *motivic ∞-categories*, as SH is the universal such ∞-category (Robalo, Drew-Gallauer)

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- For a vector bundle V over a scheme X, the *Thom space* $Th(V)$ is the pointed presheaf $V/V - \{0\}$
- This construction passes through the $\mathbb{P}^1$-stabilization, and induces a map

$$Th : K(X) \to Pic(\mathrm{SH}(X))$$

from the K-theory space to the Picard groupoid of $\mathrm{SH}(X)$, sending a virtual vector bundle v on X to a $\otimes$-invertible object $Th(v)$

(Twisted) bivariant groups in motivic homotopy

- For $f : X \to S$ a separated morphism of finite type and v a virtual vector bundle on X, define the mapping spectrum

$$H(X/S, v) := Maps_{\mathrm{SH}(X)}(Th(v), f^! \mathbb{1}_S)$$

whose homotopy groups $\pi_n H(X/S, v)$ define the *twisted bivariant groups* or twisted Borel-Moore theory groups

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- In the category of KGL-modules, $\pi_n H(X/S, v) = G_n(X)$

Functoriality of bivariant groups

- Base change:

$$\begin{array}{ccc} Y & \xrightarrow{q} & X \\ {\scriptstyle g}\downarrow & \Delta & \downarrow{\scriptstyle f} \\ T & \xrightarrow{p} & S \end{array}$$

$$\Delta^* : H(T/S, v) \to H(Y/X, g^* v)$$

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- Product: $X \xrightarrow{f} Y \xrightarrow{g} S$

$$H(X/Y, w) \otimes H(Y/S, v) \to H(X/S, w + f^* v)$$

Local acyclicity

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  is an isomorphism.
- We say that K is **universally strongly locally acyclic** (abbreviated as **USLA**) over S if for any morphism $T \to S$, the base change $K_{|X \times_S T}$ is strongly locally acyclic over T.

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- The proof uses generation of $\mathrm{SH}(X)$ by Chow motives (Ayoub, Bondarko-Déglise, Elmanto-Khan)

Künneth formula over a base

- For $f : X \to S$ a separated morphism of finite type, denote $\mathcal{K}_{X/S} = f^! \mathbb{1}_S$ and $\mathbb{D}_{X/S}(-) = \underline{Hom}(-, \mathcal{K}_{X/S})$

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- Let X, Y be two separated S-schemes of finite type, and let $p_X : X \times_S Y \to X$ and $p_Y : X \times_S Y \to Y$ be the projections, and denote $A \boxtimes_S B = p_X^* A \otimes p_Y^* B$

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Theorem (Künneth formula)

For any $L \in \mathrm{SH}_c(X)$ constructible and any $M \in \mathrm{SH}(Y)$ be USLA over S, there is a canonical isomorphism

$$\mathbb{D}_{X/S}(L) \boxtimes_S M \simeq \underline{Hom}(p_X^* L, p_Y^! M)$$

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- These results are extended to singular schemes by Lu-Zheng for étale sheaves, and the arguments also work for SH with minor changes

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- Example: any S-endomorphism $f : X \to X$ is viewed as a corrrespondence via the transpose of the graph $(f, id) : X \to X \times_S X$
- Denote by $c_1, c_2 : C \to X$ the compositions of c with p_1 and p_2. Given $K \in \mathrm{SH}_c(X)$ USLA over S, a **(cohomological) correspondence over** c is a map of the form $u : c_1^* K \to c_2^! K$

The trace map

- Consider the following Cartesian diagram

$$\begin{array}{ccc} Fix(c) & \xrightarrow{c'} & X \\ \downarrow & & \downarrow \delta_{X/S} \\ C & \xrightarrow{c} & X \times_S X. \end{array}$$

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$$u' : \mathbb{1}_C \xrightarrow{u} \underline{Hom}(c_1^* K, c_2^! K) \simeq c^! \underline{Hom}(p_1^* K, p_2^! K)$$

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which gives rise to the following map

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$$\to \delta^*_{X/S}(K \boxtimes_k \mathbb{D}_{X/S}(K)) = \mathbb{D}_{X/S}(K) \otimes K \simeq K \otimes \mathbb{D}_{X/S}(K) \to \mathcal{K}_{X/S}$$

- The **trace** of u is the map $Tr(u/S) : \mathbb{1}_{Fix(c)} \to \mathcal{K}_{Fix(c)/S}$ obtained by adjunction

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- More generally, Verdier pairing (SGA5): given two S-schemes X_1, X_2, $X_{12} := X_1 \times_S X_2$, $C \to X_{12}$, $D \to X_{12}$, $K_i \in \mathrm{SH}_c(X_i)$ USLA over S, $u : c_1^* K_1 \to c_2^! K_2$, $v : d_2^* K_2 \to d_1^! K_1$, $E := C \times_{X_{12}} D$ then we have a pairing $\langle u, v \rangle : \mathbb{1}_E \to \mathcal{K}_{E/S}$

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- The Verdier pairing can always be reduced to the trace map, via the identity $\langle u, v \rangle = \langle vu, 1 \rangle$

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- Étale contravariance: similar formulation

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- The general case reduces to S a field by conservativity of the restriction to points

Application to $\mathbb{A}^1$-enumerative geometry

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- The trace of the fundamental class of c_2

$$u : c_1^* \mathbb{1}_X = \mathbb{1}_C \overset{\eta_{c_2}}{\simeq} c_2^! \mathbb{1}_X \otimes Th(-\tau_{c_2})$$

agrees with $\Delta^* \eta_{c_1} \in H(C_s/S, \tau_{c_1|C_s})$

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Theorem (J.)

In the case where $c = (c_1, c_2)$ satisfies the condition of being contracting near β, then the local terms can be computed by some simpler invariants called the naive local terms

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- The proof in SH additionally uses the Fulton-style specialization map on bivariant groups (Déglise-J.-Khan)