Some Rough Remarks About Derived Schemes and Stacks.

§ Introduction

Recall: For $X \xrightarrow{f} Y$ a map of schemes we defined the notion of perfect obstruction theory:

$$\mathcal{E} \longrightarrow L_{X/Y} \in D_{qcoh}(X)$$

(1) $\mathcal{E}$ has perfect amplitude $[1,0]$.

(2) · $H_0(\mathcal{E})$ is an isomorphism

· $H_1(\mathcal{E})$ is a surjection.

Marc's Talk: Perfect obstruction theories ⇝ Virtual classes.

Problem: Perfect obstruction theories have bad functoriality properties.

(i.e. not obvious how to compose them and also gluing is a problem)

Solution: Work in the more structured setting of Derived Algebraic Geometry.

Philosophy: Perfect obstruction theories are "extra data" on X and not a property.

§ Derived Algebraic geometry

Here is an organizational picture I keep in my head.

$$\begin{tikzcd}
\mathrm{CAlg} \arrow[r, "\text{Schemes}"] \arrow[dr, "\substack{\text{Deligne-Mumford/}\\ \text{Artin stacks}}"'] \arrow[dddr, "\substack{\text{Higher}\\ \text{stacks}}"'] \arrow[dddd, leftrightarrow] & \mathrm{Sets} \arrow[d, hook] \\
& \mathrm{Grpds} \simeq S_{\leq 1} \arrow[d, phantom, "\vdots"] \\
& n\text{-}\mathrm{Grpds} \simeq S_{\leq n} \arrow[d, phantom, "\vdots"] \\
& \\
\mathrm{DerCAlg} \arrow[r, "\substack{\text{Derived stacks}\\ \cup\\ \text{Derived schemes.}}"'] & \mathrm{Anima}\ (= \text{Spaces, } \infty\text{-groupoids} \dots)
\end{tikzcd}$$

(terminology: Clausen, Scholze)

<u>Example</u>: Consider the group scheme $\mathbb{G}_m$.

we can form the Artin stack $B\mathbb{G}_m = [\bullet/\mathbb{G}_m]$ which classifies principal $\mathbb{G}_m$-bundles.

Notice that $B\mathbb{G}_m$ is a commutative group object in Stacks so we can consider

$$B^2\mathbb{G}_m = [\bullet/B\mathbb{G}_m] = \operatorname{colim}\left(\cdots \substack{\to\\ \to\\ \to} B\mathbb{G}_m^{\times 2} \substack{\to\\ \to\\ \to} B\mathbb{G}_m \rightrightarrows \bullet \right)$$

which is an example of a higher stack. (In fact this is an example of a 2-Artin stack.)

$$\mathrm{Der\,CAlg} = \{ \mathrm{cdga}_{\mathbb{Q}, \geq 0},\ \mathrm{CAlg}^{\mathrm{an}},\ \mathbb{E}_\infty\text{-rings} \}$$

<u>Remark</u>: Even for derived schemes we ~~and~~ really want our functors to take values in Anima since for a general animated ring

$$\pi_i \operatorname{Map}_{\mathrm{CAlg}^{\mathrm{an}}}(\mathbb{Z}[x], R) \simeq \pi_i R \neq 0.$$

We have an adjunction

$$\begin{tikzcd} CAlg^{an} \arrow[r, shift right, "\pi_0"'] & CAlg \arrow[l, shift right, "i"'] \end{tikzcd}$$

this induces adjunctions

$$\begin{tikzcd} HigherStacks \arrow[r, hook, shift right, "i"'] & DerStk \arrow[l, shift right, "\pi_0"'] \\ Schemes \arrow[u, phantom, "\cup"] \arrow[r, hook, shift right, "i"'] & DerSchemes \arrow[u, phantom, "\cup"] \arrow[l, shift right, "\pi_0"'] \end{tikzcd}$$

Warning I wrote the above adjunction the wrong way in the lecture!

The counit of the adjunction between Higher and derived Stacks gives a canonical map:

$$i\,\pi_0 X \longrightarrow X \quad \text{for } X \in DerStk$$

We will refer to $i\,\pi_0 X =: \pi_0 X$ as the underlying classical stack of X. Moreover when X is a derived Artin stack one can show that $\pi_0 X \to X$ is a closed immersion.

Remark: $DerScheme \subseteq DerArt \subseteq DerStk$.

§ Derived schemes

In more concrete terms, a derived scheme is given by

$$\underset{\text{topological space}}{(X}, \underset{\text{Sheaf of animated rings.}}{\mathcal{O}_X)}$$

(1) $(X, \pi_0(\mathcal{O}_X))$ is a scheme

(2) $\pi_i \mathcal{O}_X$ are quasi-coherent over $\pi_0 \mathcal{O}_X$

(3) $\mathcal{O}_X$ is hyper complete.

Remark: If X is Noetherian + finite Krull dimension

$\Rightarrow$ $\mathcal{O}_X$ is always hypercomplete.

Example: (Derived zero locus)

Let X be an affine scheme say, $X = \mathrm{Spec}(A)$ for A a k-algebra, $\mathrm{char}(k) = 0$ (so we can use the equivalence between $\mathrm{cdga}_{k,\geq 0}$ and CAlg_k^{an}). Moreover let $\mathbb{V}$ be a vector bundle over X, with section $s : X \longrightarrow \mathbb{V}$.

$$\begin{tikzcd} X \overset{R}{\underset{\mathbb{V}}{\times}} X \arrow[r] \arrow[d] & X \arrow[d, "s"] \\ X \arrow[r, "0"'] & \mathbb{V} \end{tikzcd} \quad \text{in DerSch.}$$

If $\mathbb{V} = \mathrm{Spec}(\mathrm{Sym}\, P^\vee)$ for some f.g. projective A-Module P.

$\Rightarrow$ Via the equivalence $(\mathrm{DerAff}_k)^{op} \simeq \mathrm{CAlg}_k^{an}$

$X \overset{R}{\underset{\mathbb{V}}{\times}} X$ corresponds to the ring given by...

$$A \overset{L}{\underset{Sym(P^\vee)}{\otimes}} A \simeq (\Lambda^\bullet P^\vee, d_s)$$

contraction with s

<u>upshot</u>: $\pi_0 \left(A \overset{L}{\underset{Sym(P^\vee)}{\otimes}} A \right) = Z(s)$ but now we have more interesting information about the intersection stored in the structure sheaf.

<u>Cotangent complex</u>:

Recall if $\varphi : A \to B$ is a map of commutative rings and M is a B-module

$$\Rightarrow \mathrm{Der}_A(B, M) \simeq \mathrm{Hom}_B(\Omega_{B/A}, M)$$

<u>i.e.</u> The Kähler differentials corepresent A-linear derivations.

<u>Observation</u>: $\mathrm{Der}_A(B, M) \cong \mathrm{Hom}_{CAlg_A/_B}(B, B \oplus M)$

Suppose now that $\varphi : A \to B$ is a map of animated rings i.e. in $CAlg^{an}$. Then for any connective $M \in \mathrm{Mod}_B$ we have:

$$\mathrm{Maps}_{CAlg^{an}_{A/B}}(B, B \oplus M) \simeq \mathrm{Maps}_{\mathrm{Mod}_B}(L_{B/A}, M)$$

and for derived schemes this sheafifies to: For $f : X \to Y$ in DerSch and $M \in \mathrm{Qcoh}(X)^{cn}$ we have

$$\mathrm{Maps}_{X/\mathrm{DerSch}_Y}(X[M], X) \simeq \mathrm{Maps}_{\underset{\underset{\mathrm{Spec}(A) \xrightarrow{f} X}{\lim} \mathrm{Mod}_A}{\overset{\mathrm{Qcoh}(X)}{\shortparallel}}}(L_{X/Y}, M).$$

§ Relation to introduction

Let X be a derived scheme

$$\pi_0 X \xhookrightarrow{j} X$$

<u>Lemma</u>: $\mathrm{Cofib}(j^* L_X \longrightarrow L_{\pi_0 X})$ is 2-connective.

(In particular H_1 is a surjection and H_0 is an isomorphism.)

Suppose now that L_X is perfect in $[1,0]$.

Lemma $\Rightarrow$ $j^* L_X \longrightarrow L_{\pi_0 X}$ is a perf. obs. thy for $\pi_0 X$.

<u>Definition</u>: A Derived Scheme is <u>quasi-smooth</u> if L_X is perfect in $[1,0]$. More generally a map of derived schemes $f: X \to Y$ is called <u>quasi-smooth</u> if $L_{X/Y}$ is perfect in $[1,0]$.

<u>UPSHOT</u>: If $f: X \to Y$ is quasi-smooth then $\pi_0 X$ gets a canonical perf. obs. theory.

<u>UPSHOT</u>: Being quasi-smooth is a property.

Example: (Derived Stack)

$$\underline{\mathrm{Perf}} : \mathrm{CAlg}^{an} \longrightarrow \mathrm{Anima}$$
$$R \longmapsto \mathrm{Perf}(R)^{\simeq}$$

This is a stack in the étale topology on CAlg^{an}.

Suppose that X is a smooth, proper scheme over a field k.

$$\underline{\mathrm{Map}}(X, \underline{\mathrm{Perf}}) : \mathrm{CAlg}_k^{an} \longrightarrow \mathrm{Anima}$$
$$R \longmapsto \mathrm{Perf}(X_R)^{\simeq}$$

Theorem: (Toën-Vaquié)

$\underline{\mathrm{Map}}(X, \underline{\mathrm{Perf}})$ is a derived stack which is "locally geometric."

Theorem: (PTVV) Suppose further that X is CY 3-fold (i.e. $\omega_X \simeq \mathcal{O}_X$)

$\Rightarrow$ $\underline{\mathrm{Map}}(X, \underline{\mathrm{Perf}})$ is (-1)-shifted symplectic.