Virtual localization formula of Graber – Pandharipande

Context.

Main goal of enumerative geometry: count curves on a scheme $X/\mathbb{C}$ (in different ways & setups)

A popular approach: consider $M_{g,n}(X,\beta)$ – stack of smooth proper genus g curves with n marked pts with a map to X, whose class in $H_2(X,\mathbb{Z})$ is β.

This stack is not proper, so people take its compactification $\overline{M}_{g,n}(X,\beta)$, by adding nodal curves with finite automorphisms.

When X is a projective variety, $\overline{M}_{g,n}(X,\beta)$ is a proper D-M stack, and it has a virtual fundamental class in $CH_*(\overline{M}_{g,n}(X,\beta))_{\mathbb{Q}}$ so a reasonable answer to curve counting problem on X is the „degree"

$$p_*[\overline{M}_{g,n}(X,\beta)]^{vir} \in CH_*(pt)\otimes\mathbb{Q} \cong \mathbb{Q}.$$

[We need to rationalize to have p_*, because $p: \overline{M}_{g,n}(X,\beta) \to pt$ is not representable!]

This number (degree of $\overline{M}_{g,n}(X,\beta)^{vir} \in \mathbb{Q}$) is an example of G-W invariant, and in simple cases (e.g. for CY 3-folds) that's the only one!

More generally, curve counting on X can be thought of as integration against $[\overline{M}_{g,n}(X,\beta)]^{vir}$:

you lift cohomology (cycle) classes from X^n and $M_{g,n}$, cup them on $\overline{M}_{g,n}(X,\beta)$ and then compute their degrees (rational numbers) by integrating against $[\overline{M}_{g,n}(X,\beta)]^{vir}$, this gives you G-W invariants.

Main question: how to compute $[\overline{M}_{g,n}(X,\beta)]^{vir}$ in practice?

Today we'll get a recipe for it when X has a $\mathbb{G}_m$-action (e.g. $X = \mathbb{P}^r$ with some choice of weights).

Virtual localization formula.

$\mathfrak{X}$ DM-stack with T-action and a $\overset{\mathbb{G}_m}{\overset{\shortparallel}{T}}$-equivariant perfect obstruction theory $E \to L_{\mathfrak{X}}$.

Denote $i: \mathfrak{X}^T \hookrightarrow \mathfrak{X}$ the fixed locus, $\mathfrak{X}^T = \sqcup \mathfrak{X}_j$ – conn. comps.

Suppose that $N^{vir} \simeq [N_0 \to N_1]$ – global resolution by v.b.

Then the following equality holds (in particular, RHS is well-defined):

$$[\mathfrak{X}]^{vir} = i_* \frac{[\mathfrak{X}^T]^{vir}}{\underset{e(N_0)/e(N_1)}{e(N^{vir})}} = i_* \sum_j \frac{[\mathfrak{X}_j]^{vir}}{e(N_j^{vir})} \in CH_*^T(\mathfrak{X}) \otimes_{\mathbb{Q}[t]} \mathbb{Q}[t^{\pm 1}],$$

where $\mathbb{Q}[t] = CH_*^T(pt) \otimes \mathbb{Q} = CH_*(BT) \otimes \mathbb{Q}$.

[We'll defined all the terms later on]

Rem. The original work by G-P has stronger assumptions:

- $\mathcal{X} \hookrightarrow \mathcal{Y}$ T-equiv. embedding in a smooth DM stack
- E has a global resolution $E = E_0 \to E_1$ by v.b.

I will follow the generalization by Chang-Kiem-Li „Torus localization and wall crossing for cosection localized virtual cycles", because I understand their proof better.

Rem: virtual localization formula is strong: it works in singular settings whereas B-B decomposition doesn't!

Application for $\overline{M}_{g,n}(\mathbb{P}^r, d)$: express $[\overline{M}_{g,n}(\mathbb{P}^r, d)]^{vir}$ in terms of $[\Pi \overline{M}_{g_r, n_r}]^{vir}$, where (g_r, n_r) are determined by the combinatorics of the fixed point locus.

Main idea: a point in the fixed locus has little freedom! If $f: C \to \mathbb{P}^r$ is a T-fixed map with finite autom, then the images of all marked pts, nodes, contracted comps, and ramification pts must be T-fixed pts in $\mathbb{P}^r$.

Moreover, each non-contracted component of C must map onto a T-invariant curve in $\mathbb{P}^r$, i.e. onto a line connecting two fixed points, and be ramified only over the two fixed pts $\Rightarrow$ this component must be rational ($\simeq \mathbb{P}^1$), and f restricted to this component is determined by its degree.

(Γ appears from decomposing curves into comps).

Standing assumptions: $k = \mathbb{C}$, $CH_* := CH_* \otimes \mathbb{Q}$.

Chow groups of stacks:

- for alg. spaces and DM stacks with $\mathbb{Q}$-coeffs, $CH_* = *$-dim integral closed substacks modulo $\sim_{rat}$
- for quotient stacks $\mathcal{Y}$

$$CH_*(\mathcal{Y}) = \operatorname*{colim}_{\mathcal{E} \to \mathcal{Y} \text{ v.b.}} CH_{*+rk(\mathcal{E})}(\mathcal{E})$$

graded group

in particular, $CH_*(BT) \overset{\simeq}{=} \mathbb{Q}[t]$, $t = [\mathcal{O}(1)] \in CH_{-2}(BT)$.

As a ring: $CH^*(BT) \simeq \mathbb{Q}[t]$, $CH^0 = \mathbb{Q}\cdot[BT]$, $CH^1 = \mathbb{Q}\cdot[\mathcal{O}(1)]$.

In topology: $H^*(B\mathbb{C}^*;\mathbb{Q}) = H^*(BS^1;\mathbb{Q}) \simeq H^*(\mathbb{CP}^\infty, \mathbb{Q}) \simeq \mathbb{Q}[t]$.

- for general Artin stacks: more complicated, generators are $f: U \xrightarrow{\text{projective}} \mathcal{E}$

- $\mathfrak{X}$ D-M stack with T-action.

$CH_*^T(\mathfrak{X}) := CH_*(\underbrace{[\mathfrak{X}/T]}_{\text{quotient stack}})$ – T-equivariant groups.

$i: \mathfrak{X}^T \subset \mathfrak{X}$ substack of the fixed locus, locally on affines given by $\operatorname{Spec} A/A^{mv} \hookrightarrow \operatorname{Spec} A$ where A^{mv} is the ideal generated by T-eigenfunctions with non-trivial characters.

Atiyah-Bott localization formula for stacks (Kresch)

(Originally Bott residue formula, Atiyah-Segal worked with KU_* instead of CH_* (1968), later Thomason worked out G_* for alg. spaces, and Edidin-Graham for CH_*; this result is also called Concentration theorem.)

Claim: the inclusion $i: \mathfrak{X}^T \hookrightarrow \mathfrak{X}$ induces

$$CH_*^T(\mathfrak{X}^T) \otimes_{\mathbb{Q}[t]} \mathbb{Q}[t^{\pm 1}] \xrightarrow{\sim} CH_*^T(\mathfrak{X}) \otimes_{\mathbb{Q}[t]} \mathbb{Q}[t^{\pm 1}].$$

$$\text{IS}$$

$$CH_*(\mathfrak{X}^T) \otimes_{\mathbb{Q}} \mathbb{Q}[t^{\pm 1}]$$

Proof: • We want to assume that the induced action of T on $\mathfrak{X}^T$ is trivial. Since these are stacks, it's not a given.

Example: $T \curvearrowright \mathbb{P}^1 \rightsquigarrow T \curvearrowright \overline{M}_0(\mathbb{P}^1, d) = \mathfrak{X}$

Take $p = [z \mapsto z^d] \in \overline{M}_0(\mathbb{P}^1, d)$,
it's a fixed point, its component in $\mathcal{X}^T$
is $B\mu_d$. Then $T \curvearrowright B\mu_d$ is non-trivial!
But: we can consider T'-action on $\mathcal{X}$,
where $T' \underset{x \mapsto x^d}{\xrightarrow{\sim}} T$ is a d-cover of T.

And T'-action on $\mathcal{X}^T$ is already trivial.

Fact: there is always an n-cover T' of T
that makes T'-action on $\mathcal{X}^T$ trivial,
and $\mathcal{X} - \mathcal{X}^T$ is a DM stack
(T'-action on $\mathcal{X} - \mathcal{X}^T$ has quasi-finite stabilizer)

Since passing to an n-cover doesn't
change CH_* with $\mathbb{Q}$-coeffs,
we can replace T by T' and assume these
properties.

- we want to use the localization sequence for stacks (Kresch):

$\mathcal{X}$ Artin stack, $\mathcal{Z} \hookrightarrow \mathcal{X} \hookleftarrow \mathcal{U}$ s.t.
$\mathcal{U}$ is a global quotient stack $\Rightarrow$
there's exact sequence

$$\underbrace{CH_*(\mathcal{U}, 1)}_{\text{is defined}} \to CH_*(\mathcal{Z}) \to CH_*(\mathcal{X}) \to CH_*(\mathcal{U}) \to 0.$$

With $\mathbb{Q}$-coeffs, it works when $\mathcal{U}$ is a DM stack too.

In particular, we get (after $[\frac{-}{T}]$):

$$CH_*^T(\mathcal{X}-\mathcal{X}^T,1) \to CH_*^T(\mathcal{X}^T) \xrightarrow{i_*} CH_*^T(\mathcal{X}) \to CH_*^T(\mathcal{X}-\mathcal{X}^T) \to 0.$$

The condition about $\mathcal{X}-\mathcal{X}^T$ DM stack with no fixed pts implies that

$CH_*^T(\mathcal{X}-\mathcal{X}^T)=0 \qquad * < 0$

$CH_*^T(\mathcal{X}-\mathcal{X}^T,1)=0 \qquad * < -1,$

so multiplying (invertible operation on $-\otimes\mathbb{Q}[t^{\pm1}]$) by powers of $t \in CH_{-2}^T(\operatorname{Spec}\mathbb{C}) = CH^{1,T}(\operatorname{Spec}\mathbb{C})$ will kill eventually Ker & Coker of i_*. ∎

⚠ $CH_*^T(\operatorname{Spec}\mathbb{C})=\mathbb{Q}[t]$ is unbounded in degrees, because BT isn't a D-M stack, since T isn't finite.

$\binom{\text{Non-}}{\text{Vir.}}$ Localization formula for smooth $\mathfrak{X}$ (and $E = \mathbb{L}_{\mathfrak{X}}$).

$\mathfrak{X}$ smooth $\Rightarrow [\mathfrak{X}]^{vir} = [\mathfrak{X}]$

$\mathfrak{X}$ smooth D-M stack $\overset{\text{fact}}{\Rightarrow}$ $\mathfrak{X}^T$ is smooth too, and so $[\mathfrak{X}^T]^{vir} = [\mathfrak{X}]$.

By A.-B. localization,

$[\mathfrak{X}] = i_* \alpha$ for some $\alpha \in CH_*(\mathfrak{X}^T) \otimes_{\mathbb{Q}} \mathbb{Q}[t^{\pm 1}]$.

We have:

$i^* i_* \alpha = \alpha \cdot \underbrace{e(N)}$, $N := N_{\mathfrak{X}} \mathfrak{X}^T$ — self-intersection formula

invertible because T-weights on N are non-zero (using rep. theory of T)

$i^*[\mathfrak{X}] = [\mathfrak{X}^T]$

$$\Rightarrow \alpha = \frac{[\mathfrak{X}^T]}{e(N)} \Rightarrow [\mathfrak{X}] = i_* \left(\frac{[\mathfrak{X}^T]}{e(N)} \right)$$

in $CH_*(\mathfrak{X}^T) \otimes_{\mathbb{Q}} \mathbb{Q}[t^{\pm 1}]$.

General case.

We'd like to imitate this proof, but we don't have i^* to start with, as well all $[-]^{vir}$ defined a priori...
But we'll get there :)

From last time:

$\mathcal{X} \rightsquigarrow \mathcal{C}_{\mathcal{X}}$ intrinsic normal cone, which étale-locally is given by $[C_{U/V}/T_V|_U]$ for

$$\begin{array}{c} U \hookrightarrow V_{\text{smooth}} \\ {\scriptstyle et} \downarrow \\ \mathcal{X} \end{array}$$

Perfect obstruction theory E gives a vector bundle stack $\mathcal{E}$ on $\mathcal{X}$ with
$\mathcal{C}_{\mathcal{X}} \hookrightarrow \mathcal{E}$ closed embedding

Virtual fundamental class is
$[\mathcal{X}]^{vir} := 0^!_{\mathcal{E}}[\mathcal{C}_{\mathcal{X}}]$, i.e. intersecting $\mathcal{C}_{\mathcal{X}}$ with the zero section $\mathcal{X} \hookrightarrow \mathcal{E}$ inside $\mathcal{E}$, and then applying htpy invariance
$CH_*(\mathcal{E}) \simeq CH_*(\mathcal{X})$.

Now T-equivariantly:

$\mathfrak{X}$ DM stack with T-action $\rightsquigarrow$

$\mathbb{L}_{\mathfrak{X}} \in D([\mathfrak{X}/_T])$ — T-equivariant complex of T-equivariant loc-free sheaves of $\mathcal{O}_{\mathfrak{X}}$-modules.

T-equivariant p.o.t.: $E \in D([\mathfrak{X}/_T])$ with $\varphi: E \to \mathbb{L}_{\mathfrak{X}} \in D([\mathfrak{X}/_T])$ that is a p.o.t. on $\mathfrak{X}$.

Now, we're given a T-equivariant p.o.t. on $\mathfrak{X}$ and we want to define an induced p.o.t. on $\mathfrak{X}^T$.

A T-equiv. sheaf of $\mathcal{O}_{\mathfrak{X}^T}$-modules $\rightsquigarrow$

A^{fix} – sheaf of T-fixed submodules

A^{mv} – subsheaf generated by T-eigensections with non-trivial characters

In particular, for $E \in D([\mathfrak{X}/_T])$

$$E|_{\mathfrak{X}^T} = E^{fix}_{\mathfrak{X}^T} \oplus E^{mv}_{\mathfrak{X}^T}$$

thanks to simple structure of T-reps!

Moreover, a T-equiv. p.o.t. $\varphi: E \to \mathbb{L}_{\mathfrak{X}}$ induces

$$\varphi^{fix}: E|^{fix}_{\mathfrak{X}^T} \to \mathbb{L}_{\mathfrak{X}}|^{fix}_{\mathfrak{X}^T} \qquad \varphi^{mv}: E|^{mv}_{\mathfrak{X}^T} \to \mathbb{L}_{\mathfrak{X}}|^{mv}_{\mathfrak{X}^T}$$

Lemma. The composition

$E|^{fix}_{\mathfrak{X}^T} \to \mathbb{L}_{\mathfrak{X}}|^{fix}_{\mathfrak{X}^T} \to \mathbb{L}_{\mathfrak{X}^T}$ is a p.o.t. on $\mathfrak{X}^T$.

Hence we get $[\mathcal{X}^T]^{vir}$ from this p.o.t.

Now, let $E_{\mathcal{X}^T} := E|_{\mathcal{X}^T}^{fix}$ and $N^{vir} := (E|_{\mathcal{X}^T}^{mv})^\vee$, which are both perfect complexes.
They fit in the diagram of cofiber sequences

$$\begin{tikzcd} E|_{\mathcal{X}^T} \arrow[r] \arrow[d] & E_{\mathcal{X}^T} \arrow[r, "\tau"] \arrow[d] & (N^{vir})^\vee[1] \arrow[d, "f"] & (*) \\ \mathbb{L}_{\mathcal{X}}|_{\mathcal{X}^T} \arrow[r] & \mathbb{L}_{\mathcal{X}^T} \arrow[r] & \mathbb{L}_{\mathcal{X}^T/\mathcal{X}} & \end{tikzcd}$$

By assumption, $N^{vir} \simeq [N_0 \to N_1]$, so the normal sheaf

$$N_{\mathcal{X}^T/\mathcal{X}} \hookrightarrow h^1/h^0(N^{vir}[-1]) = \ker(N_0 \to N_1) \subset N_0.$$

This allows to define the virtual pullback

$$i^! : CH_*(\mathcal{X}) \to CH_*(\mathcal{X}^T)$$

as $[B] \mapsto [\mathcal{C}_{B \times_{\mathcal{X}} \mathcal{X}^T / B}] \mapsto 0^!_{N_0}[\mathcal{C}_{B\times_{\mathcal{X}} \mathcal{X}^T/B}]$.

In general, virtual pull-back is a relative version of virtual fundamental class
i.e. $\forall\, h: \mathcal{X} \to \mathcal{Y}$ there's defined

$$h^! : CH_*(\mathcal{Y}) \to CH_*(\mathcal{X}) \text{ whenever}$$

given an embedding $\mathcal{C}_{\mathcal{X}/\mathcal{Y}} \hookrightarrow \mathcal{E}$, for $\mathcal{E}$ a v.b. stack on $\mathcal{X}$.
(for h qsmooth and $\mathcal{E} = \mathbb{L}$ it's Adeel's qsmooth Gysin map!)

These $h^!$ satisfy nice properties main of which is the following: when p.o.t. on $\mathcal{X}$ and $\mathcal{Y}$ are compatible with φ, we get

$$h^! [\mathcal{Y}]^{vir} = [\mathcal{X}]^{vir}.$$

Precisely: let h be a map of D-M stacks with p.o.t. $\varphi_{\mathcal{X}}: E_{\mathcal{X}} \to \mathbb{L}_{\mathcal{X}}$ and $\varphi_{\mathcal{Y}}: E_{\mathcal{Y}} \to \mathbb{L}_{\mathcal{Y}}$ that fit into cofiber seq.

$$\begin{tikzcd} h^* E_{\mathcal{Y}} \arrow[r, "\varphi"] \arrow[d, "h^*\varphi_{\mathcal{Y}}"] & E_{\mathcal{X}} \arrow[r] \arrow[d, "\varphi_{\mathcal{X}}"] & E_{\mathcal{X}/\mathcal{Y}} \arrow[d, "\varphi_{\mathcal{X}/\mathcal{Y}}"] \\ h^*\mathbb{L}_{\mathcal{Y}} \arrow[r] & \mathbb{L}_{\mathcal{X}} \arrow[r] & \mathbb{L}_{\mathcal{X}/\mathcal{Y}} \end{tikzcd}$$

We call h *virtually smooth* if $E_{\mathcal{X}/\mathcal{Y}}$ is of perfect amplitude contained in $[-1,0]$.
In that case, $\varphi_{\mathcal{X}/\mathcal{Y}}$ is a p.o.t. and we have an embedding of the intrinsic normal sheaf into the v.b. stack

$$h^1/h^0(\mathbb{L}_{\mathcal{X}/\mathcal{Y}}^\vee) \hookrightarrow h^1/h^0(E_{\mathcal{X}/\mathcal{Y}}^\vee) =: \mathcal{E}_{\mathcal{X}/\mathcal{Y}},$$

and $\mathcal{C}_{\mathcal{X}/\mathcal{Y}} \hookrightarrow \mathcal{E}_{\mathcal{X}/\mathcal{Y}}$.

Thm (technical core of virtual localization formula): $h: \mathcal{X} \to \mathcal{Y}$ virtually smooth map of D-M stacks $\Rightarrow h^! [\mathcal{Y}]^{vir} = [\mathcal{X}]^{vir}$.
(uses Vistoli's rational equivalences to compare cones...)

End of the proof of virtual localization formula

we have $i : \mathfrak{X}_T \hookrightarrow \mathfrak{X}$ giving $i^!$, which somewhat resembles i^* in the smooth case.

- Self-intersection holds by construction of $i^!$: for $\alpha \in CH_*(\mathfrak{X}^T) \otimes \mathbb{Q}[t^{\pm 1}]$,
$$i^! i_* \alpha = 0^!_{N_0} \alpha \overset{\mathbb{Q}}{=} \alpha \cdot e(N_0)$$
(true for any cycle in $\mathfrak{X}^T$).

- Now, if i was virtually smooth, we would say that
$$i^! [\mathfrak{X}]^{vir} = [\mathfrak{X}^T]^{vir}$$
and be done like in the smooth case (if there was no N_1).

Key idea: modify the p.o.t. of $\mathfrak{X}^T$ to make $\mathfrak{X}^T \hookrightarrow \mathfrak{X}$ virtually smooth!

We have:
$$N_0^\vee[1] \xrightarrow{g} (N^{vir})^\vee[1] \xrightarrow{h} N_1^\vee[2]$$
and define
$$\tilde{E}_{\mathfrak{X}_T} \xrightarrow{\alpha} E_{\mathfrak{X}^T} \xrightarrow{h \circ \tau} N_1^\vee[2]$$
where τ is from $(*)$.

p.o.t. on $\mathfrak{X}^T$ (exercise),

and now $i : \mathfrak{X}^T \hookrightarrow \mathfrak{X}$ is virtually smooth, so
$$i^! [\mathfrak{X}]^{vir}_E = [\mathfrak{X}^T]^{vir}_{\tilde{E}_{\mathfrak{X}^T}}.$$

Finally, we use

$$h^1/h^0(E^\vee_{\mathcal{X}^T}) \to h^1/h^0(\tilde{E}^\vee_{\mathcal{X}^T}) \to N_1$$

to conclude that

$$[\mathcal{X}^T]^{vir}_{\tilde{E}_{\mathcal{X}^T}} = [\mathcal{X}]^{vir}_{E_{\mathcal{X}^T}} \cdot e(N_1).$$

This allows to obtain the virtual localization formula just like in the smooth case. ∎

Hurrah!