Fundamental classes in motivic homotopy theory

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Setting

Let S be a qcqs scheme. $\mathcal{SH}(S)$ is the stable ∞-category of motivic spectra. $\mathcal{SH}(S)$ is symmetric monoidal, denote the monoidal product and the monoidal unit by $\otimes$ and $\mathbb{S}_S$.
For any morphism of schemes $f\colon T \to S$ we get adjoint functors:

$$f^*\colon \mathcal{SH}(S) \to \mathcal{SH}(T),\ f_*\colon \mathcal{SH}(T) \to \mathcal{SH}(S),$$

If f is an s-morphism (separated morphism of finite type), we also get an another pair of adjoint functors:

$$f_!\colon \mathcal{SH}(T) \to \mathcal{SH}(S),\ f^!\colon \mathcal{SH}(S) \to \mathcal{SH}(T).$$

Six functors 1

$\mathcal{SH}(S)$ is equipped with the six functors formalism. Denote these functors as $(\otimes, \underline{\mathrm{Hom}}, f^*, f_*, f^!, f_!)$. They satisfy a variety of compatibilities. The most useful ones:

1. For every morphism f, the functor f^* is symmetric monoidal.
2. There is a natural transformation $f_! \to f_*$ which is invertible when f is proper.
3. There is an invertible natural transformation $f^* \to f^!$ when f is an open immersion.
4. There is a canonical isomorphism

$$\mathbb{E} \otimes f_!(\mathbb{F}) \to f_!(f^*(\mathbb{E}) \otimes \mathbb{F})$$

for any s-morphism $f\colon T \to S$ and any $\mathbb{E} \in \mathcal{SH}(S)$, $\mathbb{F} \in \mathcal{SH}(T)$.

Six functors 2

5. For any cartesian square

$$\begin{array}{ccc} T' & \xrightarrow{g} & S' \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ T & \xrightarrow{f} & S \end{array}$$

where f and g are s-morphisms, there are canonical isomorphisms

$$p^* f_! \to g_! q^*, \qquad q_* g^! \to f^! p_*.$$

Proposition

Let $f\colon X \to S$ be an s-morphism, $\mathbb{E}, \mathbb{F} \in \mathcal{SH}(S)$. Then there is a canonical morphism

$$Ex^{!*}_{\otimes}\colon f^!(\mathbb{E}) \otimes f^*(\mathbb{F}) \to f^!(\mathbb{E} \otimes \mathbb{F}).$$

Suspension functor

Definition

If $\mathscr{E}$ is a vector bundle of finite rank over S, define the equivalence $\Sigma^{\mathscr{E}} : \mathcal{SH}(S) \to \mathcal{SH}(S)$ as a smash product with the Thom space of $\mathscr{E}$. Denote the inverse as $\Sigma^{-\mathscr{E}}$.

Remark

The suspension functor is compatible with the six functors in the following way:

$$f^*\Sigma^{\mathscr{E}} \cong \Sigma^{f^*(\mathscr{E})} f^*, \quad f_*\Sigma^{f^*(\mathscr{E})} \cong \Sigma^{\mathscr{E}} f_*,$$

$$f^!\Sigma^{\mathscr{E}} \cong \Sigma^{f^*(\mathscr{E})} f^!, \quad f_!\Sigma^{f^*(\mathscr{E})} \cong \Sigma^{\mathscr{E}} f_!.$$

Purity

- If f is a smooth s-morphism and T_f is its relative tangent bundle, by Ayoub's purity theorem there is a canonical isomorphism of functors $\mathfrak{p}_f \colon \Sigma^{T_f} f^* \to f^!$.
- For any cartesian square

$$\begin{array}{ccc} T' & \xrightarrow{g} & S' \\ \downarrow q & & \downarrow p \\ T & \xrightarrow{f} & S \end{array}$$

there is a natural transformation $Ex^{*!} \colon q^* f^! \to g^! p^*$. If p or f is smooth, $Ex^{*!}$ is invertible.

Bivariant theories 1

- Bivariant theory. For any s-morphism $p\colon X \to S$ and any K-theory class $v \in \mathrm{K}(X)$, define the v-twisted bivariant spectrum of X over S as the mapping spectrum

$$\mathbb{E}(X/S, v) = \mathrm{Maps}_{\mathcal{SH}(S)}(\mathbb{S}_S, p_*(p^!(\mathbb{E}) \otimes \mathrm{Th}_X(-v))).$$

We also write $\mathbb{E}_n(X/S, v) = [\mathbb{S}_S[n], p_*(p^!(\mathbb{E}) \otimes \mathrm{Th}_X(-v))]$ for each $n \in \mathbb{Z}$.

Bivariant theories 2

- Cohomology theory. For any morphism $p\colon X \to S$, define the v-twisted cohomology spectrum of X over S as the mapping spectrum

$$\mathbb{E}(X,v) = \mathrm{Maps}_{\mathcal{SH}(S)}(\mathbb{S}_S, p_*(p^*(\mathbb{E}) \otimes \mathrm{Th}_X(v))).$$

- Bivariant theory with proper support. For any s-morphism $p\colon X \to S$, define the v-twisted bivariant theory with proper support of X over S as the mapping spectrum

$$\mathbb{E}^c(X/S,v) = \mathrm{Maps}_{\mathcal{SH}(S)}(\mathbb{S}_S, p_!(p^!(\mathbb{E}) \otimes \mathrm{Th}_X(-v))).$$

- Cohomology with proper support. For any s-morphism $p\colon X \to S$, define the v-twisted cohomology with proper support of X over S as the mapping spectrum

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Properties 1

1. Base change. For any cartesian square

$$\begin{array}{ccc} X_T & \xrightarrow{g} & X \\ \downarrow q & & \downarrow p \\ T & \xrightarrow{f} & S \end{array}$$

   there is a canonical base change map

$$\Delta\colon \mathbb{E}(X/S, v) \to \mathbb{E}(X_T/T, g^*v).$$

2. Proper covariance. For any proper morphism $f\colon X \to Y$ of s-schemes over S, there is a direct image map

$$f_*\colon \mathbb{E}(X/S, f^*v) \to \mathbb{E}(Y/S, v).$$

Properties 2

3. Étale contravariance. For any étale s-morphism $f\colon X \to Y$ of s-schemes over S, there is an inverse image map

$$f^{!}\colon \mathbb{E}(Y/S, v) \to \mathbb{E}(X/S, f^{*}v).$$

4. Product. If $\mathbb{E}$ is equipped with a multiplication map $\mu_{\mathbb{E}}\colon \mathbb{E} \otimes \mathbb{E} \to \mathbb{E}$, then for s-morphisms $p\colon X \to S$ and $q\colon Y \to X$ and $v \in \mathrm{K}(X)$, $w \in \mathrm{K}(Y)$, there is a map

$$\mathbb{E}(Y/X, w) \otimes \mathbb{E}(X/S, v) \to \mathbb{E}(Y/S, w + q^{*}v).$$

Localization triangle

Let $i\colon Z \to S$ be a closed immersion. Ther direct image functor $i_*\colon \mathcal{SH}(Z) \to \mathcal{SH}(S)$ is fully faithful. If the complementary closed immersion $j\colon U \to S$ is quasi-compact, by the Morel-Voevodsky localization theorem there is an exact trianle

$$i_*i^! \to \mathrm{Id} \to j_*j^*.$$

Proposition

Let $i\colon Z \to X$ be a closed immersion of s-schemes over S, with a quasi-compact complementary open immersion $j\colon U \to X$. Then for any $e \in \mathrm{K}(X)$ there exists a canonical exact triangle of spectra

$$\mathbb{E}(Z/S, e) \xrightarrow{i_*} \mathbb{E}(X/S, e) \xrightarrow{j^*} \mathbb{E}(U/S, e),$$

which is called the localization triangle.

Orientation

For any s-morphism $f : X \to S$ and any $v \in \mathrm{K}(X)$ set $H(X/S, v) = \mathbb{S}_S(X/S, v)$.

Definition

An orientation of f is a pair (η_f, e_f), where $e_f \in \mathrm{K}(X)$ and $\eta_f \in H(X/S, e_f)$. We also write η_f instead of (η_f, e_f).

Definition

If for any $v \in \mathrm{K}(X)$, cap-product with η_f induces an isomorphism

$$\gamma_{\eta_f} : H(X, v) \to H(X/S, e_f - v), \; x \mapsto x.\eta_f,$$

then we call γ_{η_f} the duality isomorphism.

Fundamental class

Let f be a smooth s-morphism with tangent bundle T_f.

Definition

The purity isomorphism $\mathfrak{p}_f \colon \Sigma^{T_f} f^* \to f^!$ induces a canonical isomorphism

$$\eta_f \colon \mathrm{Th}_X(T_f) \xrightarrow{\sim} f^!(\mathbb{S}_S).$$

The fundamental class of f is the orientation $\eta_f \in H(X/S, \langle T_f \rangle)$.

System of fundamental classes 1

Let S be a scheme and let $\mathcal{C}$ be a class of morphisms between s-schemes over S. A system of fundamental classes for $\mathcal{C}$ consists of the following data:

1. Fundamental classes. For each morphism $f: X \to Y$ in $\mathcal{C}$, there is an orientation $(\eta_f^{\mathcal{C}}, e_f)$.
2. Normalization. For each scheme S, with the identity morphism $f = id_S$, there are identifications $e_f \cong 0$ in $\mathrm{K}(S)$ and $\eta_f^{\mathcal{C}} \cong 1$ in $H(S/S, 0)$.
3. Associativity formula. Let $f: X \to Y$ and $g: Y \to Z$ be morphisms in $\mathcal{C}$ such that the composite $g \circ f$ is also in $\mathcal{C}$. Then there are identifications $e_{g\circ f} \cong e_f + f^*(e_g)$ in $\mathrm{K}(X)$ and $\eta_g^{\mathcal{C}} \cdot \eta_f^{\mathcal{C}} \cong \eta_{g\circ f}^{\mathcal{C}}$ in $H(X/Z, e_{g\circ f})$.

System of fundamental classes 2

We say that a system of fundamental classes $(\eta_f^{\mathcal{C}})_f$ is stable under transverse base change if it is equipped with the following data:

4. Transverse base change formula. For any tor-independent cartesian square

$$\begin{array}{ccc} Y & \xrightarrow{g} & T \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & S \end{array}$$

such that f and g are in $\mathcal{C}$, there are identifications $e_g \cong q^*(e_f)$ in $\mathrm{K}(Y)$ and $\Delta^*(\eta_f^{\mathcal{C}}) \cong \eta_g^{\mathcal{C}}$.

Main theorem for smooth morphisms

Example

Let S be a scheme. Then there exists a system of fundamental classes $(\eta_f, \langle T_f \rangle)$ on the class of smooth morphisms between s-schemes over S, satisfying the following properties:

- For every smooth morphism $f : X \to Y$, the orientation $\eta_f \in H(X/Y, \langle T_f \rangle)$ is the fundamental class defined earlier.
- The system is stable under transverse base change.

Virtual tangent bundle

If f is a smoothable lci, then f factors as $f = p \circ i$, where i is a regular closed immersion and p is smooth. The virtual tangent bundle $\langle L_f \rangle$ of f is $i^*\langle T_p \rangle - \langle N_i \rangle$, where T_p is the class of relative tangent bundle of p and N_i is the class of the normal bundle of i. $\langle L_f \rangle$ does not depend from the choice of p and i.
If we have two smoothable lci morphisms $f \colon X \to Y$ and $g \colon Y \to Z$, then $\langle L_{g \circ f} \rangle$ can be identified with $\langle L_f \rangle + f^*\langle L_g \rangle$.

Gysin maps

Let $f: X \to Y$ be a morphism of s-schemes over S, $e \in \mathrm{K}(Y)$. Any orientation $\eta_f \in H(X/Y, e_f)$ gives rise to a Gysin map

$$\eta_f^!: H(Y/S, e) \to H(X/S, f^*(e) + e_f), \quad x \mapsto \eta_f x$$

using the product in bivariant $\mathbb{A}^1$-theory. We also write $f^!$ instead of $\eta_f^!$.

Purity transformation 1

Definition

Let $f\colon X \to S$ be an s-morphism and (η_f, e_f) be its orientation. The class $\eta_f \in H(X/S, e_f)$ can be seen as a morphism in $\mathcal{SH}(X)$:

$$\eta_f\colon \mathrm{Th}(e_f) \to f^!(\mathbb{S}_S).$$

It gives rise to a natural transformation

$$\mathfrak{p}(\eta_f)\colon \Sigma^{e_f} f^* \to f^!$$

defined as

$$f^*(-) \otimes \mathrm{Th}_X(e_f) \xrightarrow{Id \otimes \eta_f} f^*(-) \otimes f^!(\mathbb{S}_S) \xrightarrow{Ex^{!*}_{\otimes}} f^!(- \otimes \mathbb{S}_S) \cong f^!.$$

Purity transformation 2

Remark

When f is smooth, it is the purity isomorphism.

Remark

(η_f, e_f) and $\mathfrak{p}(\eta_f)$ are essentially interchangeable. Evaluating $\mathfrak{p}(\eta_f)$ on $\mathbb{S}_S$ gives η_f.

Thom isomorphism

Let X be an s-scheme over S and $p\colon E \to X$ be a vector bundle.

Lemma

The Gysin map $p^!\colon H(X/S, e) \to H(E/S, p^*e + p^*\langle E\rangle)$ is invertible.

Definition

Define the Thom isomorphism

$$\phi_{E/X}\colon H(E/S, e) \to H(X/S, e - \langle E\rangle)$$

to be the inverse of $p^!\colon H(X/S, e - \langle E\rangle) \to H(E/S, e)$.

Euler class

If $\nu\colon F \to E$ is a monomorphism of vector bundles over X, one gets a canonical morphism of pointed sheaves $\nu_*\colon \mathrm{Th}_X(F) \to \mathrm{Th}_X(E)$.

Definition

Let E be a vector bundle over X and s be its zero section. Define the Euler class $e(E)$ of E/X as the induced map in $\mathcal{H}_\bullet(X)$:

$$s_*\colon \mathrm{Th}_X(X) \to \mathrm{Th}_X(E).$$

We will often view the Euler class as a class $e(E) \in H(Y, \langle E \rangle) \cong H(Y/Y, -\langle E \rangle)$, via the canonical map

$$\mathrm{Maps}_{\mathcal{H}_\bullet(X)}(X_+, \mathrm{Th}_X(E)) \to \mathrm{Maps}_{\mathcal{SH}(X)}(\mathbb{S}_X, \mathrm{Th}_X(E)).$$

For any scheme X and any $e \in \mathrm{K}(X)$ one has the following diagrams:

$$\begin{array}{ccc} \mathbb{G}_m X & \xrightarrow{\quad j \quad} & \mathbb{A}^1 X \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \overline{\pi}} & \\ & X & \end{array}$$

$$\begin{array}{ccccc} H(X, 1-e) & = & H(X/X, e-1) & \xrightarrow{\overline{\pi}^!} & H(\mathbb{A}^1 X/X, e) \\ \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^!} & & \downarrow{\scriptstyle j^!} \\ H(\mathbb{G}_m X, 1-e) & \xrightarrow{\gamma_{\eta_\pi}} & H(\mathbb{G}_m X/X, e) & = & H(\mathbb{G}_m X/X, e) \end{array}$$

where $\langle T_\pi \rangle \cong 1$ in $\mathrm{K}(\mathbb{G}_m X)$, $\langle T_{\overline{\pi}} \rangle \cong 1$ in $\mathrm{K}(\mathbb{A}^1 X)$.

Consider the localization triangle associated to the zero section $s_0 \colon X \to \mathbb{A}^1_X$:

$$H(\mathbb{A}^1 X/X, e)[-1] \xrightarrow{j^!} H(\mathbb{G}_m X/X, e)[-1] \xrightarrow{\partial_{s_0}} H(X/X, e).$$

It splits canonically and we get a section of ∂_{s_0}:

$$\gamma_t \colon H(X/X, e) \to H(\mathbb{G}_m X/X, e)[-1].$$

Let X be an S-scheme, $i\colon Z \to X$ a regular closed immersion, $e \in \mathrm{K}(X)$. Define $D_Z X$ to be the deformation space $B_{Z\times 0}(X \times \mathbb{A}^1)\backslash B_{Z\times 0}(X \times 0)$, where $B_Z X$ is a blow-up of X in Z. One also has

$$N_Z X \xrightarrow{k} D_Z X \xleftarrow{h} \mathbb{G}_m X,$$

where h and k are an open and a closed immersion. Consider the associated localization triangle:

$$H(N_Z X/S, e) \xrightarrow{k_*} H(D_Z X/S, e) \xrightarrow{h^!} H(\mathbb{G}_m X/S, e) \xrightarrow{\partial_{N_Z X/D_Z X}}$$

$$\xrightarrow{\partial_{N_Z X/D_Z X}} H(N_Z X/S, e)[-1]$$

Fundamental class for regular closed immersion

The fundamental class $\eta_i \in H(Z/X, -\langle N_Z X\rangle)$ associated to the regular closed immersion $i\colon Z \to X$ is the image of $1 \in H(X/X, 0)$ by the composite

$$H(X/X,0) \xrightarrow{\gamma_t} H(\mathbb{G}_m X/X,0)[-1] \xrightarrow{\partial_{N_Z X/D_Z X}}$$

$$\xrightarrow{\partial_{N_Z X/D_Z X}} H(N_Z X/X,0) \xrightarrow{\phi_{N_Z X/Z}} H(Z/X,-\langle N_Z X\rangle)$$

where $\phi_{N_Z X/Z}$ is the Thom isomorphism of $p\colon N_Z X \to Z$.

Main theorem for regular closed immersions

Theorem

Let S be a scheme. Then there exists a system of fundamental classes $(\eta_i, -\langle N_i \rangle)$ on the class of regular closed immersions between s-schemes over S, satisfying the following properties:

- For every regular closed immersion $i\colon Z \to X$, the orientation $\eta_i \in H(Z/X, -\langle N_i \rangle)$ is the fundamental class defined earlier.
- The system is stable under transverse base change.

Main theorem

Theorem

Let S be a scheme. Then there exists a system of fundamental classes $(\eta_f, \langle L_f \rangle)$ on the class of smoothable lci morphisms between s-schemes over S, satisfying the following properties:

- The restriction of the system $(\eta_f, \langle L_f \rangle)$ to the class of smooth s-morphisms coincides with the previous one.
- The restriction of the system $(\eta_f, \langle L_f \rangle)$ to the class of regular closed immersions coincides with the previous one.
- The system is stable under transverse base change.

Let S be a scheme and $\mathbb{E} \in \mathcal{SH}(S)$. Either using the unit map $\eta\colon \mathbb{S}_S \to \mathbb{E}$ or doing all constructions for $\mathbb{E}(X/S, v)$ instead of $H(X/S, v) = \mathbb{S}_S(X/S, v)$, one can define the fundamental classes of f with coefficients in $\mathbb{E}$:

$$\eta_f^{\mathbb{E}} \in \mathbb{E}(X/S, \langle L_f \rangle).$$

One can also define the Euler class with coefficients in $\mathbb{E}$:

$$e(E, \mathbb{E}) \in \mathbb{E}(X, \langle E \rangle) \cong \mathbb{E}(X/X, -\langle E \rangle).$$

Example

When $\mathbb{E}$ is oriented, the Euler class coincides with the top Chern class. When $\mathbb{E}$ is the Milnor-Witt spectrum $\widetilde{\mathbf{H}\mathbb{Z}}$, the Euler class is the classical Euler class in the Chow-Witt group.

Let $\mathscr{T}$ be a motivic ∞-category of coefficients, defined over the site $\mathscr{S}$ of qcqs schemes. All the constructions make sense in the setting of $\mathscr{T}$, as they use only six functors:

- One can define the four theories. For example, the bivariant theory is

$$\mathbb{E}(X/S, v, \mathscr{T}) := \mathrm{Maps}_{\mathscr{T}(S)}(\mathbb{1}_S, p_*(p^!(\mathbb{E}) \otimes \mathrm{Th}_X(-v, \mathscr{T}))),$$

  where $p\colon X \to S$ is an s-morphism and $v \in \mathrm{K}(X)$.
- We have a system of fundamental classes and purity transformations with coefficients in $\mathbb{E} \in \mathscr{T}(S)$, satisfying stability under transverse base change:

$$\eta_f{}^{\mathscr{T}} \in \mathbb{E}(X/Y, -\langle L_f \rangle, \mathscr{T}), \qquad \mathfrak{p}_f^{\mathscr{T}} : \Sigma^{L_f} f^* \to f^!$$

  for any smoothable lci s-morphism $f\colon X \to Y$ of s-schemes over S.
- There are functorial Gysin maps with coefficients in any $\mathbb{E} \in \mathscr{T}(S)$.

Examples

- If we consider the motivic cohomology sprectrum $\mathbf{H}\mathbb{Z} \in \mathcal{SH}(S)$, we obtain the bivariant theory with higher Chow groups.
- If we consider the Milnor-Witt motivic cohomology sprectrum $\widetilde{\mathbf{H}\mathbb{Z}} \in \mathcal{SH}(S)$, we obtain the bivariant theory with higher Chow-Witt groups.
- If 2 is invertible, we can consider the spectrum of homotopy invariant Hermitian K-theory $\mathbf{BO} \in \mathcal{SH}(S)$. We obtain a bivariant theory with it.

Chern–Gauss–Bonnet theorem

Let M be a compact orientable $2n$-dimensional Riemannian manifold without boundary and e be the Euler class. Then

$$\chi(M) = \int_M e(TM).$$

Categorical Euler characteristic

Definition

Let C be a symmetric monoidal category, $A \in C$ be a strogly dualizable object and $f \colon A \to A$ an endomorphism. Then the trace of f is an endomorphism of the unit $\mathbb{1}_C$ given by the composition

$$\mathbb{1}_C \xrightarrow{coev} A \otimes A^\vee \xrightarrow{f \otimes \mathrm{Id}} A \otimes A^\vee \cong A^\vee \otimes A \xrightarrow{ev} \mathbb{1}_C.$$

Definition

Let $p \colon X \to S$ be a smooth proper morphism. Define the categorical Euler characteristik $\chi^{cat}(X/S) \in \mathrm{Maps}_{\mathcal{SH}(S)}(\mathbb{S}_S, \mathbb{S}_S)$ as trace of $f = \mathrm{Id}$ and $A = p_! p^! (\mathbb{1}_S)$. We view $\chi^{cat}(X/S)$ as a class in $H(S/S, 0)$.

The motivic Gauss-Bonnet formula

Theorem

Let $p\colon X \to S$ be a smooth proper morphism. There is an identification $\chi^{cat}(X/S) \cong p_*(e(T_p))$ in the group $H(S,0)$.

Consider the diagonal regular closed immersion $\delta\colon X \to X \times_S X$ and the cartesian diagram

$$\begin{array}{ccc} X \times_S X & \xrightarrow{\pi_2} & X \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{p} & S \end{array}$$

Lemma

Define $\theta\colon \delta^! \to \delta^*$ to be the exchange transformation $Ex^{!*}\colon \mathrm{Id}^*\delta^! \to \mathrm{Id}^!\delta^*$. The endomorphism $\chi^{cat}(X/S)\colon \mathbb{S}_S \to \mathbb{S}_S$ is obtained by evaluating the following natural transformation of at the monoidal unit $\mathbb{S}_S$:

$$\begin{array}{ccccccccc} Id & \dashrightarrow & & & & & & & Id \\ \downarrow{\scriptstyle unit} & & & & & & & & \uparrow{\scriptstyle counit} \\ p_*p^* & = & p_*\delta^!(\pi_2)^!p^* & \xleftarrow[Ex^{*!}]{\simeq} & p_*\delta^!(\pi_1)^*p^! & \xrightarrow{\theta} & p_*\delta^*(\pi_1)^*p^! & = & p_*p^! \end{array}$$

$$\begin{array}{ccccc} p^* = \delta^!(\pi_2)^! p^* & \xleftarrow{Ex^{!*}} & \delta^!(\pi_1)^* p^! & \xrightarrow{\theta} & \delta^*(\pi_1)^* p^! = p^! \\ & \searrow^{\Sigma^{-T_p}\mathfrak{p}_p} & \uparrow \mathfrak{p}_\delta & \nearrow_{\mathfrak{e}_p p^!} & \\ & & \Sigma^{-T_p} p^! & & \end{array}$$

Here $\mathfrak{e}_p$ is the natural transformation $\Sigma^{-T_p} \to \mathrm{Id}$ induced by the Euler class $e(T_p)\colon \mathbb{S}_X \to \mathrm{Th}_X(T_p)$.