Motivic homotopy theory of algebraic stacks.

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1 Introduction.

The six functor formalism was formulated by Grothendieck to give a framework for the basic operations and duality statements for cohomology theories. In brief, a six functor formalism is a theory of coefficient systems relative to any scheme with a collection of six functors $f^*, f_*, f^!, f_!, \otimes, \mathrm{Hom}$ which satisfy a set of relations. This formalism is usually formulated in the language of triangulated categories. In [MV99], Morel and Voevodsky define the general theory of $\mathbb{A}^1$-homotopy theory of schemes which incorporates homotopy theoryin the field of algebraic geometry. To a scheme S, they associate a triangulated category $\mathrm{SH}(S)$ which is defined by applying $\mathbb{A}^1$-localization and $\mathbb{P}^1$-stabilization to the category of simplicial Nisnevich sheaves. Voevodsky and Ayoub ([Ayo07a] and [Ayo07b]) constructed a six functor formalism of $\mathbb{A}^1$-homotopy theory. In this seminar, we study the extension of SH on a specific class of algebraic stacks and the six functor formalism using the language of ∞-categories due to Lurie ([Lur09]).

In order to motivate the need of language of ∞-categories, let us recall the six functor formalism of derived categories of ℓ-adic sheaves over an algebraic stack. To an algebraic stack $\mathcal{X}$, one can define the derived category of the algebraic stack $\mathcal{X}$ as derived category of ℓ-adic étale sheaves over $\mathcal{X}$. For example, if $\mathcal{X} = B\mathbb{G}_m$, then the derived category of $B\mathbb{G}_m$ is the derived category of $\mathbb{G}_m$-equivariant ℓ-adic étale sheaves over a point. As the connected group $\mathbb{G}_m$ cannot act non-trivially on locally constant sheaves, this is equivalent to the category of sheaves over a point. Thus this naive definition implies that $D(B\mathcal{G}_m)$ is equivalent to the derived category of a point. But we have

$$H^*(B\mathbb{G}_m) \cong \mathbf{Q}_l[c]$$

where c is in degree 1 ([Tot99]).

In [LO08b] and [LO08a], Laszlo and Olsson define derived categories of algebraic stacks and construct the six functor formalism using the lisse-étale topos. They use simplicial methods to to construct the derived category that gives the expected answer for the cohomology of $B\mathbb{G}_m$. The fact that the lisse-étale topos is not functorial makes the construction of derived pullback bit technical.The language of ∞-categories allows us to circumvent this problem.

In [LZ17], Liu and Zheng construct a six functor formalism of derived ∞-categories of ℓ-adic sheaves for any algebraic stack. To any scheme X, the derived ∞-category $\mathcal{D}_{et}(X, \mathbf{Q}_l)$

is the ∞-categorical enhancement of the usual derived category. The major advantange of the ∞-categorical language is that the derived ∞-category satisfies étale descent. For any algebraic stack $\mathcal{X}$, the ∞-category $\mathcal{D}_{\mathrm{et}}(\mathcal{X}, \mathbf{Q}_l)$ consturcted by Liu and Zheng is isomorphic to the limit of derived ∞-categories over Čech nerve of any atlas $x : X \to \mathcal{X}$. In other words, we have

$$\mathcal{D}_{\mathrm{et}}(\mathcal{X}, \mathbf{Q}_l) \cong \lim \Big(\mathcal{D}_{\mathrm{et}}(X, \mathbf{Q}_l) \overset{\longrightarrow}{\underset{\longrightarrow}{\longleftarrow}} \mathcal{D}_{\mathrm{et}}(X \times_{\mathcal{X}} X, \mathbf{Q}_l) \overset{\longrightarrow}{\underset{\longrightarrow}{\longleftarrow}} \mathcal{D}_{\mathrm{et}}(X \times_{\mathcal{X}} X \times_{\mathcal{X}} X, \mathbf{Q}_l) \qquad \cdots \Big) \tag{1}$$

where the maps in the limit are the derived pullback maps. Their construction uses abstract descent theory of the language of ∞-categories. This also allows to construct the pullback functor in a canonical way. Morever, they prove that their formalism agrees with the one introduced by Laszlo and Olsson once one passes to homotopy categories of the derived ∞-categories. Thus the language of ∞-categories seem advantageous to extend ∞-sheaves from schemes to algebraic stacks. We shall use a similar technique in our setting of motivic homotopy theory but in this case extra care is needed because motivic invariants usually do not satisfy étale descent.

To a Noetherian scheme of finite Krull dimension S, the motivic stable homotopy category $\mathcal{SH}^{\otimes}(S)$ is a presentable stable symmetric monoidal ∞-category (we refer to [Rob15] for the notations). The functorial assignment makes $\mathcal{SH}^{\otimes}$ into a functor

$$\mathcal{SH}^{\otimes} : \mathrm{N}(\mathrm{Sch}_{fd})^{op} \to \mathrm{CAlg}(\mathrm{Pr}^{L}_{stb}) \tag{2}$$

where the target is the ∞-category of stable presentable symmetric monoidal ∞-categories. As mentioned above, we cannot use equation Eq. (1) as a definition of $\mathcal{SH}^{\otimes}$ for an algebraic stack because $\mathcal{SH}^{\otimes}$ does not satisfy étale descent and thus Eq. (1) would depend on the choice of the atlas X. This problem problem by specifying a class of smooth atlases for which we can prove descent. The resulting class of $(2,1)$-category of algebraic stacks Nis-locSt consists of algebraic stacks which admit an atlas admitting Nisnevich-local sections. This includes all quasi-compact, quasi-separated algebraic spaces, quotient stacks $[X/G]$ where G is an affine algebraic group, local quotient stacks, the moduli stack of vector bundles Bun_n, the moduli stack of G-bundles Bun_G and moduli space of stable maps. Using the formulation of enhanced operation map ([LZ17]), we also manage to extend the six functor formalism from schemes to Nis-locSt. The main result is as follows:

Theorem 1.1. *[Cho, Theorem 5.5.1] The functor $\mathcal{SH}^{\otimes}(-)$ extends to a functor*

$$\mathcal{SH}^{\otimes}_{\mathrm{ext}} : \mathrm{N}^{D}_{\bullet}(\text{Nis-locSt})^{op} \to \mathrm{CAlg}(\mathrm{Pr}^{L}_{stb}).$$

Morever,

1. *For any $\mathcal{X} \in$ Nis-locSt, there exist functors $\otimes, \mathrm{Hom} : \mathcal{SH}_{\mathrm{ext}}(\mathcal{X}) \times \mathcal{SH}_{\mathrm{ext}}(\mathcal{X}) \to \mathcal{SH}_{\mathrm{ext}}(\mathcal{X})$.*
2. *For any morphism $f : \mathcal{X} \to \mathcal{Y}$ in Nis-locSt, there is a pair of adjoint functors*

$$f^* : \mathcal{SH}_{\mathrm{ext}}(\mathcal{Y}) \to \mathcal{SH}_{\mathrm{ext}}(\mathcal{X}) \ , \ f_* : \mathcal{SH}_{\mathrm{ext}}(\mathcal{X}) \to \mathcal{SH}_{\mathrm{ext}}(\mathcal{Y}).$$

3. *For a morphism* $\mathsf{f} : \mathcal{X} \to \mathcal{Y}$ *in* Nis-locSt *which is separated of finite type and representable by algebraic spaces, there is a pair of adjoint functors*

$$\mathsf{f}_! : \mathcal{SH}_{\text{ext}}(\mathcal{X}) \to \mathcal{SH}_{\text{ext}}(\mathcal{Y})\ ,\ \mathsf{f}^! : \mathcal{SH}_{\text{ext}}(\mathcal{Y}) \to \mathcal{SH}_{\text{ext}}(\mathcal{X}).$$

These functors restrict to the known functors on the category of schemes. Furthermore, the projection formula, base change, localization, homotopy invariance and purity extend to Nis-locSt.

Along with this theory, we shall also briefly study the equivariant motivic homotopy theory due to Hoyois ([Hoy17]), the motivic homotopy theory on scalloped algebraic stacks as well as limit-extended cohomology theories due to Khan and Ravi([KR21]).

The seminar is divided into three parts. The first part consists of prereuqisite material that we need to define $\mathcal{SH}^{\otimes}_{\text{ext}}(-)$ and the six functors. This includes a short introduction to presentable stable symmetric monoidal ∞-categories, reviewing the definition of motivic homotopy theory of schemes in ∞-categorical setting ([Rob14]) and algebraic stacks (especially local quotient stacks).
The second part deals with defining $\mathcal{SH}^{\otimes}_{\text{ext}}(\mathcal{X})$ and understanding the six functors associated to it ([Cho]). This in particular shall include the so called enhanced operation map due to Liu and Zheng ([LZ17]) which will be used for constructing the exceptional functors and proving projection formula, base change.
In the last part, we discuss the equivariant motivic homotopy theory ([Hoy17]) and the generalized cohomology theories on algebraic stacks ([KR21]).

2 Outline of the talks.

Part 1: Prerequisites.

2.1 Talk 1: Overview of the seminar (12/10/21).

2.2 Talk 2: Introduction to ∞-categories (19/10/21).

Overview: In this talk, we recall the basic notion of ∞-categories. We shall begin with recalling the definition and introduce basic terminologies like objects, morphisms, compositions, homotopy category, initial/final objects, limits and mapping spaces. The two main examples of ∞-categories that we are interested in arise from simplicial categories and 2-categories. We shall end the talk by recalling the notion of fibrations of simplicial sets, categorical equivalence and Cartesian fibrations.

References: [Lur09, Chapter 1-2], [Lur18a, Chapter 1, 2.2,2.3] and [Rez, Part 4].

Outline: Briefly motivate the notion of ∞-categories ([Lur18a, Pages 11-13]). Define simplicial sets and the simplicial sets $\Delta^n, \partial\Delta^n$ and Λ^n_I ([Lur18a, Section 1.1.1-1.1.2]). Define ∞-categories and state the examples of Singular complex and the nerve of a small category ([Lur18a, Section 1.3.0]). Define objects, morphisms ([Lur18a, Definition 1.3.1.1]), homotopies

of morphisms([Lur18a, 1.3.3]), composition of morphisms ([Lur18a, 1.3.4]), homotopy category ([Lur18a, 1.3.5]), mapping space, over and undercategories and limits ([Lur09, Chapter 1]). If time permits, briefly explain the homotopy category of $\mathrm{Sing}_\bullet(X)$.
Define simplicial categories and briefly describe the homotopy coherent nerve $\mathfrak{C}[-]$ ([Lur09, Chapter 1]). Define 2-categories and briefly describe the Duskin nerve of a 2-category ([Lur18a, Section 2.2-2.3]). State [Lur18a, Theorem 2.3.2.1]. Define the ∞-category of spaces ([Lur09, Section 1.8]) and the ∞-category of ∞-categories Cat_∞ ([Lur09, Chapter 3]).
Define various notions of fibrations of simplicial sets ([Lur09, Chapter 2]). Define the notion of Cartesian morphisms and Cartesian fibrations([Lur09, Section 2.3]) and try to motivate it as analog of cartesian morphisms in the setting of fibered categories.
Define categorical equivalence and state some examples ([Rez, Part 4]).

2.3 Talk 2: Presentable stable ∞-categories(26/10/21).

Overview: In this talk, we define the notion of presentability and stability in the setting of ∞-categories. The talk shall start by recalling the notion of presheaves, Yoneda lemma and adjoint functors in the setting of ∞-categories. We then move on defining presentable ∞-categories by filtered categories and ind-objects. We briefly recall the notion of ∞-sheaves. The talk ends by defining the notion of stable ∞-categories and the ∞-category of presentable stable ∞-categories $\mathrm{Pr}^{\mathsf{L}}_{\mathsf{stb}}$.

References: [Lur09], [Lur17] and [Lur18b].

Outline: Define the ∞-category of presheaves ([Lur09, Definition 5.1.0.1]) and briefly explain the Yoneda embedding j ([Lur09, 5.1.3]). State [Lur09, Proposition 5.1.3.1] and [Lur09, Corollay 5.1.58]. Define adjoint functors by [Lur09, Definition 5.2.2.7] and [Lur09, Proposition 5.2.2.8]. Define adjointable squares ([Lur17, Definition 4.7.4.13]) and ∞-categories $\mathrm{Fun}^{\mathsf{LAd}}(S, \mathrm{Cat}_\infty)$, $\mathrm{Fun}^{\mathsf{RAd}}(S, \mathrm{Cat}_\infty)$) ([Lur17, Definition 4.7.4.16]). State [Lur17, Corollary 4.7.4.18 (3)].
Define filtered ∞-categories ([Lur09, Definition 5.3.1.7]), explain [Lur09, Example 5.3.1.8] and state some properties ([Lur09, Lemma 5.3.1.12, 5.3.1.18]). Define the ∞-category of Ind-objects by [Lur09, Corollary 5.3.5.4] and state [Lur09, Proposition 5.3.5.10]. Define presentable ∞-categories ([Lur09, Theorem 5.5.1.1 (4)]) and state examples ((see [Cho, Theorem 5.5.1] for a complete statement)[Lur09, Example 5.5.1.8] and [Lur17, Corollary 4.7.4.18 (1)]). Explain the representable functors ([Lur09, 5.5.2]) and state [Lur09, Proposition 5.5.1.9]. State the adjoint functor theorem ([Lur09, Corollary 5.5.2.9]). Define the ∞-categories Pr^{L} and Pr^{R} ([Lur09, Definition 5.5.3.1]) and state [Lur09, Corollary 5.5.3.4 , Proposition 5.5.3.13, Theorem 5.5.3.18].
Define pointed ∞-categories, fibers and cofibers ([Lur17, Definition 1.1.1.1-Definition 1.1.1.6]). Try to motivate the definition of fibers and cofibers as kernels and cokernels when $\mathcal{C}$ is nerve of an ordinary category. Define stable ∞-categories ([Lur17, Definition 1.1.1.9]). State some examples ([Lur17, Example 1.1.1.11, 1.1.1.12]). Briefly explain suspension and loop functors ([Lur17, Page 24]). State [Lur17, Theorem 1.1.2.14] and explain why the ∞-category of presentable stable ∞-categories admits small limits.

2.4 Talk 3: Symmetric monoidal ∞-categories and module objects. (2/11/21).

Overview: In this talk, we define the algebra and module objects in the setting of ∞-categories. In particular, we are interested in defining symmetric monoidal ∞-categories. In order to make sense of these notions, we start by defining ∞-operads which are generalized notions of colored operads. Symmetric monoidal ∞-categories are special kinds of ∞-operads. The talk ends by defining module objects over ∞-operads and explaning a higher categorical generalization of the fact that a morphism between algebra objects $A \to B$ gives B an A-module structure..

Reference: [Lur17, Chapter 2,3] and [Rob14, Section 9.4.1.2].

Outline: Motivate the definition of $\mathcal{C}^{\otimes}$ associated to a symmetric monoidal category $\mathcal{C}$ ([Lur17, Construction 2.0.0.1]) and briefly describe the properties of the forgetful functor $p : \mathcal{C} \to \mathrm{Fin}_*$. ([Lur17, 166-168]). Define ∞-operads ([Lur17, Definition 2.1.1.10]), state [Lur17, Remark 2.1.1.12] and state examples ([Lur17, Example 2.1.1.18, 2.1.1.20]). Define maps of ∞-operads ([Lur17, Definition 2.1.2.7]). Define symmetric monoidal ∞-categories ([Lur17, Definition 2.0.07] or [Lur17, Example 2.1.2.18]) and explain [Lur17, Remark 2.1.2.20, Example 2.1.2.21]. Define the ∞-category of commutative algebra objects ([Lur17, Definition 2.1.3.1]) and state [Lur17, Example 2.1.3.2]. State the existence of the symmetric monoidal ∞-category $\mathcal{C}^{\amalg}$ associated to a category admitting finite coproducts ([Lur17, Construction 2.4.3.1]).
Define the ∞-operad $\mathrm{Pf}^{\otimes}$ ([Rob14, Definition 9.4.1.2]) and define module objects over a symmetric monoidal ∞-category by [Rob14, Eq 9.4.44]. Describe the map $\mathrm{Pf}^{\otimes} \to \Delta[1]^{\amalg}$ and explain how a morphisms of algebra objects induces a module structure ([Rob14, 9.4.1.2]).

2.5 Talk 4: Motivic homotopy theory of schemes(9/11/21).

Overview: In this lecture, we shall define the motivic homotopy theory of schemes in the language of ∞-categories. We shall briefly recall the definition of unstable, pointed and stable homotopy theory. Then we recall the functoriality and six operations. We shall state the existence of the exceptional pushforward functors without proving them. This shall be later explained while discussing the enhanced operation map in Talk 10. The talk ends by stating properties like localization, homotopy purity, homotopy invariance and explaining the construction of α_f and purity transformation ρ_f.

References: [Rob14], [CD19] and [Hoy17].

Outline: Define unstable motivic homotopy category ([Rob14, Section 5.1]) and state [Rob14, Theorem 5.1.2]. Explain how the cartesian structure in $\mathcal{H}(S)$ can be transferred into a symmetric monoidal structure in $\mathcal{H}(S)_*$ and defined the pointed unstable motivic category ([Rob14, Section 5.2]).
Before defining $\mathcal{SH}^{\otimes}(S)$, recall the inversion of objects in symmetric monoidal ∞-category ([Rob14, Section 4.2.2]). In particular, state and give a brief idea of [Rob14, Theorem 4.2.5]. Define symmetric objects ([Rob14, Definition 4.2.7] and state [Rob14, Theorem 4.2.10, Corol-

lary 4.2.13].
Define stable motivic category ([Rob14, Definition 5.3.1]) and state [Rob14, Corollay 5.3.2]. State [Rob14, Proposition 5.3.3] by briefly explaining the notion of compact generators ([Rob14, Section 2.1.23]).
State the six operations and relations between them ([Hoy17, Theorem 6.18, Propsition 6.24] taking G to be trivial). State smooth (proper) base change and projection formula ([Rob14, Example 9.4.6]). Explain the construction of α_f ([CD19, Proposition 2.2.10]) and purity transformation ρ_f ([Rob14, Section 9.4.2.4]).

2.6 Talk 5: Algebraic stacks (16/11/21).

Overview: In this talk, we discuss the notion of algebraic spaces and algebraic stacks. We also discuss some examples of algebraic stacks like local quotient stacks, moduli stack of vector bundles. We also define the deformation to the normal cone in the setting of algebraic stacks. The talk ends by discussing the notion of resolution property of algebraic stacks.

References: [Sta21], [Knu71], [Hei10], [LMB00] [Tot04] and [Gro17].

Outline: Define algebraic spaces and algebraic stacks ([Knu71], [Sta21, Part 7] or [Hei10]). Motivate the definition of algebraic stacks by group actions on a scheme and hence defining quotient stacks. Give other examples of algebraic stacks like moduli space of vector bundles ([Hei10, Example 1.14]). Define geometric properties on stacks and morphisms of stacks ([Hei10, Section 2.1]). Define local quotient stacks ([FHT11]). State the result of Totaro stating that stacks admitting resolution property are quotient stacks ([Tot04, Theorem 1.1]). Define normal bundle, blow-up and deformation to normal cone for a closed immersion of algebraic stacks by local constructions ([LMB00, Section 14]).

Part 2: The functor $\mathcal{SH}^{\otimes}_{\mathrm{ext}}(-)$.

2.7 Talk 6: Descent along sections (23/11/21).

Overview: In this talk, we recall the ∞-categorical setup generalizing the classical statement that descent along morphisms admitting sections is automatic. The talks by recalling the classical statement in ordinary category theory using the notion of split forks. Then we move in explaining the skeletal descriptions of simplicial category Δ, augmented simplicial category Δ_+ and the split-simplicial category $\Delta_{-\infty}$. The talks end by stating the main result which is due to [Lur09, Lemma 6.1.3.16].

References: [Lur09, Section 6.1] and [Cho, Chapter 2].

Outline: Define fork, equalizer and give examples ([Cho, Definition 2.1.1, Example 2.1.2]). Define split forks ([Cho, Definition 2.1.3]), prove [Cho, Lemma 2.1.4] and state [Cho, Example 2.1.5, Remark 2.1.6].
Define augmented simplicial (Δ_+) and split-simplicial category ($\Delta_{-\infty}$) ([Cho, Definition 2.1.1-2.2.2]). Describe the skeletal description of Δ and Δ_+ ([Cho, Remark 2.2.7]) by stating the simplicial identites and proving [Cho, Proposition 2.2.5, Corollary 2.2.6]. Define splitting

maps ([Cho, Notation 2.2.8]) and describe the skeletal description of $\Delta_{-\infty}$ ([Cho, Remark 2.2.11]) by [Cho, Proposition 2.2.9, Corollary 2.2.10].
Define simplicial, augmented and split-simplicial objects ([Cho, Definition 2.2.12]). Motivate the relevance of split-simplicial objects via the Dold-Kan correspondence ([Cho, Section 4.7.2]). Prove [Cho, Lemma 2.2.14].
State [Lur09, Lemma 6.1.3.16] (give an idea of the proof if possible) and prove [Lur09, Corollary 2.3.2].

2.8 Talk 7: Kan extensions, descent theory and localization of ∞-categories (30/11/21).

Overview: In this lecture, we recall some important notions in higher category theory that we need for proving [Cho, Theorem 3.4.1] which allows us to extend sheaves from schemes to a large class of algebraic stacks. We start the talk by introducing the notion of Kan extensions in the setting of ∞-categories. Kan extensions allow us to associate limits of diagrams in a functorial manner. The second part of the lecture introduces the notion of descent theory and states specific conditions when ∞-sheaves can be realised by descent along Čech nerves of coverings. The last part of the lecture reviews localization of ∞-categories and explains the existence of localization along any class of morphisms.

References: [Lur09, Section 4.2,5.5], [Lur18b, Appendix A 3.1-3.3],[Cho, Appendix A] and [Lan21, Section 2.4].

Outline: Define relative colimits ([Lur09, Definition 4.3.1.1]) and give examples ([Lur09, Example 4.3.1.3, Example 4.3.1.4]). Define Kan extensions along inclusions ([Lur09, Definition 4.3.2.2]) and give example ([Lur09, Example 4.3.2.4]). State [Lur09, Proposition 4.3.2.9] and explain the functorial association of limits of diagrams by stating [Lur09, Corollary 4.3.2.16]. Prove commutativity of limits ([Lur09, Lemma 5.5.2.3]).
Define finitary Grothendieck topologies ([Lur18b, Definition A.3.1.1]) and explain how one gets finitary Grothendieck topologies by some conditions on coverings ([Lur18b, Proposition A.3.2.1]). Explain that sheaf condition in finitary Grothendieck topologies can be studied by descent along Čech nerves of coverings ([Lur18b, Proposition A.3.3.1]). Define the notion of F-descent [Cho, Definition A.16.7] and prove [Cho, Lemma A.16.8].
Define localization of ∞-categories ([Lan21, Definition 2.4.2]). State an example of localization ([Lan21, Lemma 2.4.5]) and briefly explain the existence of localizations along any class of morphisms ([Lan21, Lemma 2.4.6]).

2.9 Talk 8: Enhancement of sheaves along coverings with local sections (7/12/21).

Overview: In this talk, we prove [Cho, Theorem 3.4.1] and thus extend the stable homotopy functor $\mathcal{SH}^{\otimes}(-)$ from schemes to the $(2,1)$-category Nis-locSt. The talk starts by introducing $\mathcal{T}$-local sections in a site $(\mathcal{C},\mathcal{T})$ and stating some properties. We then move to defining the $(2,1)$-category of stacks admitting $\mathcal{T}$-local sections of which the category of qcqs algebraic spaces and the $(2,1)$-category Nis-locSt are examples. We state [Cho, Theorem 3.4.1] and explain the proof using the theory of Kan extensions and localizations explained in the previous lecture. The lecture ends defining the functor $\mathcal{SH}^{\otimes}_{\text{ext}}(-)$ ([Cho, Corollary 3.5.3]) and constructing the four functors $f^*, f_*, - \otimes -$ and $\text{Hom}(-,-)$ on Nis-locSt.

References: [Cho, Chapter 3].

Outline: Define a morphism admitting $\mathcal{T}$-local sections ([Cho, Definition 3.1.1]). State [Cho, Example 3.1.2] and [Cho, Lemma 3.1.3]. State [Cho, Corollary 3.1.4] and prove [Cho, Proposition 3.1.5].
Define the category of stacks admitting $\mathcal{T}$-local sections ([Cho, Definition 3.2.1]) and explain or state the properties of this category ([Cho, Remark 3.2.2-Lemma 3.2.8]). Define Nis-locSt ([Cho, Notation 3.3.2]). State [Cho, Lemma 3.3.3-3.3.4] and give examples of algebraic stacks which belong to Nis-locSt ([Cho, Corollary 3.3.6-3.3.11]) (try to explain why quotient stacks belong to this category).
State the main theorem ([Cho, Theorem 3.4.1]) and explain the idea of the proof of the theorem by proving [Cho, Proposition 4.3.2]. Complete the proof of [Cho, Theorem 3.4.1].
Prove [Cho, Corollary 3.5.3] and explain the four functors on $\mathcal{SH}^{\otimes}_{\text{ext}}(\mathcal{X})$ ([Cho, Notation 3.5.6-Notation 3.5.9]).

2.10 Talk 9: Compactification in the setting of ∞-categories (14/12/21).

Overview: In this talk, we give a brief idea of Deligne's compactification in ∞-categorical setting due to Liu and Zheng ([LZ12]). The talks starts with a brief recall of Deligne's argument of constructing the exceptional pushforward $f_!$ in étale cohomology. The rest of talk is introducing the terminology of multi-marked and multi-tiled simplicial sets which is needed to state the theorem of ∞-categorical compactification ([LZ12, Theorem 0.1]) and constructing the enhanced operation map (which shall be done in the next talk). The talk ends with a brief sketch of the idea of [LZ12, Theorem 0.1].

References: [Del73, Section 3][LZ12] and [Cho, Section 4.2, Appendix D].

Outline: Briefly describe Deligne's argument of glueing pseudo-categories ([Del73, Section 3, Expose XVIII]).Motivate the need of multi-simplicial sets ([Cho, Section 4.1, Appendix D]) Define multi-simplicial sets and the functors $\delta^k_*, \delta^*_k, \epsilon^I_J$. Explain these functors in case $k = 2$ ([Cho, Example 4.2.5]). Define I-marked simplicial set ([LZ12, Definition 3.9]), the functors $\delta^*_{I+}, \delta^{I+}_*$ ([Cho, Notation 4.2.8]). Define the restricted I-simplicial nerve ([LZ12, Definition 3.10]) and explain it with [Cho, Example 4.2.10]. Define I-tiled simplicial set ([LZ12, Definition 3.12]), the functors $\delta^*_{I\square}$, $\delta^{I\square}_*$ ([Cho, Notation 4.2.12]) and the Cartesian I-simplicial nerve ([Cho, Definition 4.2.14]) with [Cho, Example 4.2.15].
State [LZ12, Theorem 0.1]. State [Corollary D.1.4] [Cho] and explain [Cho, Remark D.1.5]. Explain the basic idea of the proof of [LZ12, Theorem 0.1] ([Cho, Remark D.1.8]).
If time permits, describe the simplicial sets Cpt^n , $\square^n$, $\mathcal{K}\text{pt}^{\alpha}(\tau)$([LZ12, Section 4]) and the basic idea of why the map p_{comm} is a categorical equivalence ([Cho, Appendix D.1.4]).

2.11 Talk 10 : Enhanced operations for stable homotopy theory of algebraic stacks (21/12/21).

Overview: In this talk, we extend the exceptional pushforward and pullback functors $f_!$ and $f^!$ from schemes to Nis-locSt. This is extended by the so called enhanced operation map due to Liu and Zheng ([LZ17, Section 2.2]). The talk starts with recalling the setting of six operations with smooth (and proper) base change and projection formula on the level schemes and constructs the enhanced operation map. Then we explain how the enhanced operation map encodes the exceptional pushforward functors, base change and projection formula. The rest of talk deals with explaining extending the enhanced operation map from schemes to Nis-locSt ([Cho, Proposition 4.4.2]) with a brief sketch of the proof.

References: [Cho, Chapter 4, Appendix D], [LZ17, Section 1, Section 3] and [Rob14, Section 9.4].

Outline: State the general setup of six operations ([Cho, Notation 4.3.1]). State the statement of partial adjoints ([Cho, Theorem D.2.1]) and explain it via examples ([Cho, Remark D.2.2] and [LZ17, Remark 1.4.5]). Explain the construction of the enhanced operation map ([Cho, Appendix D.3] and [Rob14, Section 9.4.1.3]).
Explain how the enhanced operation map encodes the extraordinary pushforward map and encodes projection formula and base change ([Cho, Section 4.3.2]). State [Cho, Theorem 4.1.1]State and give a brief idea of the proof by stating [Cho, Proposition 4.4.2].

2.12 Talk 11: Six operations for $\mathcal{SH}_{\text{ext}}(\mathcal{X})$ (11/01/22) .

Overview: In this talk, we prove relations between the six operations that we enocuntered in the previous talks in particular; localization, homotopy invariance and homotopy purity. The talk starts with proving smooth and proper base change and then move on to prove localization and homotopy invariance. We construct the natural transformations α_f and ρ_f. The talks ends with proving the homotopy purity theorem via the deformation to the normal cone.

References: [Cho, Chapter 5].

Outline: Prove the conservativeness of pullback map ([Cho, Lemma 5.1.1]) and proof smooth/proper base change ([Cho, Proposition 5.1.2]). Prove localization and homotopy invariance ([Cho, Section 5.2]).
Define compactifiable morphisms ([Cho, Definition 5.3.1]) and construct the natural transformation α_f ([Cho, Proposition 5.3.3]).
Prove the purity theorem ([Cho, Proposition 5.4.1]) and describe the explicit description of Σ_f by the deformation to the normal cone ([Cho, Proposition 5.4.5-Corollary 5.4.7]).
If time permits, explain [Cho, Remark 5.4.8].

Part 3: Other constructions.

2.13 Talk 12: Equivariant motivic homotopy theory (18/01/22).

Overview: In this talk, we recall the equivariant motivic homotopy theory due to Hoyois ([Hoy17]). The talk starts with defining the unstable homotopy category $\mathrm{H}^{\mathsf{G}}(\mathsf{S})$ and stating the importance of tame condition of group scheme while constructing the purity isomorphism. We state functoriality,smooth and proper projection formula and define the pointed motivic homotopy category $\mathrm{H}^{\mathsf{G}}_{\bullet}(\mathsf{S})$ and stating the unstable ambidexterity map. The talk ends with defining the equivariant stable homotopy theory $\mathrm{SH}^{\mathsf{G}}(\mathsf{S})$ and stating the six operations and descent properties of $\mathrm{SH}^{\mathsf{G}}(-)$.

References: [Hoy17].
Outline: Will be written later.

2.14 Talk 13: Generalized cohomology theories of algebraic stacks (25/01/22).

Overview: In this talk, we recall the construction of generalized cohomology theories on scalloped algebraic stacks and also define limit-extended cohomology theories.

References: [KR21].

Outline: Will be written later.

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