# Welschinger invariants and quadratic degrees

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July 11, 2019

For $f : Y \to X$ a dominant morphism of $k$-varieties of the same dimension $d$:

$$\deg(f) := [k(Y) : k(X)]$$

We have fundamental classes $[Y] \in \mathrm{CH}_d(Y)$, $[X] \in \mathrm{CH}_d(X)$ and for $f$ proper

$$f_* : \mathrm{CH}_*(Y) \to \mathrm{CH}_d * (Y); \quad f_*([Y]) = \deg(f) \cdot [X]$$

Conservation of number:
For $f$ flat and finite, $x \in X$ a closed point, we have fundamental classes $[x] \in \mathrm{CH}^0(x) = \mathbb{Z} \cdot [x]$, $[f^{-1}(x)] \in \mathrm{CH}_0(f^{-1}(x)_{\mathrm{red}})$ and

$$f_{x*} : \mathrm{CH}_0(f^{-1}(x)_{\mathrm{red}}) \to \mathrm{CH}_0(x); \quad f_{x*}([f^{-1}(x)]) = \deg(f) \cdot [x].$$

This extends to define the functorial pushforward in $\mathrm{CH}_*$: for $f : Y \to X$ proper of relative dimension $d$, we have

$$f_* : \mathrm{CH}_n(Y) \to \mathrm{CH}_n(X)$$

$\mathrm{CH}^*$ is part of the *oriented cohomology theory* motivic cohomology $X \mapsto H^{*,*}(X)$. Other oriented cohomology theories include: algebraic $K$-theory $\mathrm{KGL}^{**}$, algebraic cobordism $\mathrm{MGL}^{**}$, ....

The term "oriented" has a technical meaning, but practically speaking, $E^{**}(-)$ is oriented if there are pushforward (Gysin) maps for proper morphisms.

In topology, the situation is more complicated. Even a surjective map of compact manifolds of the same dimenison $d$, $f : M \to N$ may not have a well-defined degree in $\mathbb{Z}$.
For $M$ to have a fundamental class $[M] \in H_d(M, \mathbb{Z})$, $M$ needs to be oriented; if $M$ and $N$ are oriented, we do have $\deg(f) \in \mathbb{Z}$ defined by

$$f_*([M]) = \deg(f) \cdot [N]$$

More generally, suppose we have an isomorphism $\theta : f^*(o_N) \to o_M$: a *relative orientation of* $f$. This gives $f_* : H_d(M, o_M) \to H_d(N, o_N)$ and a $\deg(f) \in \mathbb{Z}$ with

$$f_*([M]) = \deg(f) \cdot [N] \in H_d(N, o_N).$$

Both definitions of degree depend on a choice of relative orientation.

Real algebraic geometry exhibits features of both algebraic geometry and topology concerning degrees.

**Example** Let $f : \mathbb{A}^1_{\mathbb{R}} \to \mathbb{A}^1_{\mathbb{R}}$ be the map $f(x) = x^2$. $f$ has algebraic degree 2, $f^{-1}(1)$ has two real points, but $f^{-1}(-1)$ has none, a failure of conservation of number?

The two frameworks, algebraic geometry and topology, can be united by refining degrees to quadratic forms.

We have the *Grothendieck-Witt ring* of a field $F$ (of characteristic $\neq 2$),

$$\mathrm{GW}(F) = (\{\text{non-degenerate quadratic forms}\}/\text{isom}, \perp)^{+},$$

the hyperbolic form $H(x, y) = xy \sim x^2 - y^2$, and the *Witt ring*

$$W(F) = \mathrm{GW}(F)/([H]) = \mathrm{GW}(F)/\mathbb{Z} \cdot [H].$$

$\mathrm{GW}(F)$ is additively generated by the one-dimensional forms $<u>$, $u \in F^{\times}$, $<u>(x) = ux^2$.

There is the rank homomorphism $\mathrm{rnk} : \mathrm{GW}(F) \to \mathbb{Z}$ and for $F = \mathbb{R}$, the signature $\mathrm{sig} : \mathrm{GW}(\mathbb{R}) \to \mathbb{Z}$.

These both extend to (Nisnevich) sheaves on smooth schemes over a base-field $k$, $\mathcal{GW}$ and $\mathcal{W}$.

We have $\mathbb{G}_m \to \mathcal{GW}^\times$ sending a unit $u$ to the quadratic form $<u>$. For $L \to X$ a line bundle, we have the $L$-twisted sheaves on $X_{\mathrm{Nis}}$

$$\mathcal{GW}(L) := \mathcal{GW} \times^{\mathbb{G}_m} L^\times,\ \mathcal{W}(L) := \mathcal{W} \times^{\mathbb{G}_m} L^\times$$

Since $<u^2> = <1>$, we have canonical isomorphisms

$$\mathcal{GW}(L \otimes M^{\otimes 2}) \cong \mathcal{GW}(L),\ \mathcal{W}(L \otimes M^{\otimes 2}) \cong \mathcal{W}(L).$$

For $F \subset K$ a finite separable extension of field we have the trace map

$$\mathrm{Tr}_{K/F} : \mathrm{GW}(K) \to \mathrm{GW}(F)$$

sending a quadratic form $q : V \to K$ on a $K$-vector space $V$ to $\mathrm{Tr}_{K/F} \circ q : V_{/F} \to F$.

For $f : Y \to X$ a proper surjective map of smooth $k$-scheme of the same dimension, this extends to

$$f_* : H^0(Y, \mathcal{GW}(\omega_{Y/k} \otimes f^* L)) \to H^0(X, \mathcal{GW}(\omega_{X/k} \otimes L))$$

**Example** Our map $f : \mathbb{A}^1_k \to \mathbb{A}^1_k$, $f(x) = x^2$. On function fields, this is $k(t) \subset k(x)$ by $t = x^2$ and

$$\mathrm{Tr}_{k(X)/k(T)}(<1>) = q, \quad q(X_1, X_2) = 2X_1^2 + 2tX_2^2.$$

Note that $q$ does not extend to a non-degenerate form over $k[t]$: $disc(q) = 4t$.

But: $f^*dt = 2xdx$, so $<2x>dt = <2x><2x>dx = <1>dx$ and

$$f_*(<1>dx) = \mathrm{Tr}_{k(x)/k(t)}(<2x>)dt = q'(X_1, X_2)dt;$$

$$q'(X_1, X_2) = 4tX_1X_2 \Rightarrow q \sim H(X_1, X_2) \in \mathrm{GW}([k[t]).$$

If we work over $\mathbb{R}$, we have $\mathrm{sig}(f_*(<1>dx)) = 0$, which is the oriented degree of the map $\mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$: we have *oriented* conservation of number.

# A digression on motivic homotopy theory

## SL-oriented theories

Just as $\mathrm{CH}^*$ is part of the "complete" theory of motivic cohomology $H\mathbb{Z}^{**}$, the cohomology $H^*(-, \mathcal{GW})$ is part of a larger theory,

$$X \mapsto \mathrm{EM}(\mathcal{K}^{MW})^{a,b}(X) := H^{a-b}(-, \mathcal{K}^{MW}_b).$$

This is an example of an *SL-oriented theory*. $\mathcal{K}^{MW}_*$ is a quadratic refinement of the sheaf of Milnor $K$-groups $\mathcal{K}^M_*$.

Other examples: hermitian $K$-theory $\mathrm{KQ}^{**}$, Witt theory $\mathrm{KT}^{**}$, special linear cobordism $\mathrm{MSL}^{**}$.

Characteristic of SL-oriented theories: Twisting by line bundles and pushforwards for proper maps $f : Y \to X$ of relative dimension $d$:

$$f_* : E^{a,b}(Y, f^*L \otimes \omega_f) \to E^{a-2d,b-d}(X, L); \ \omega_f := \omega_{Y/k} \otimes f^*\omega^{-1}_{X/K}.$$

We have the algebraic Hopf map $\eta$ and the switch map $\tau$:

$$\eta : \mathbb{A}^2 \setminus \{0\} \to \mathbb{P}^1, \quad \tau : \mathbb{P}^1 \wedge \mathbb{P}^1 \to \mathbb{P}^1 \wedge \mathbb{P}^1.$$

$\mathrm{SH}(k)[1/2] = \mathrm{SH}(k)^+ \times \mathrm{SH}(k)^-$: $\tau$-$\pm$-eigenspace decomposition.

$$\mathrm{SH}(k)^- = \mathrm{SH}(k)[1/2, 1/\eta]; \quad \eta \cdot H\mathbb{Z} = 0$$

$\Rightarrow$ Motives do not see $\mathrm{SH}(k)^-$.

For $k = \mathbb{R}$, $Re_{\mathbb{R}}(\eta)$ is $\times 2 : S^1 = \mathbb{R}^2 \setminus \{0\} \to \mathbb{RP}^1 = S^1 \Rightarrow$ Motives only see 2-torsion phenomena under real realization.

SL-oriented theories allow one to view the minus part of motivic homotopy theory.

For example: $\mathcal{W} = \mathcal{K}_*^{MW}[1/\eta] \Rightarrow \mathcal{W}[1/2] \in \mathrm{SH}(k)^-$.

**Theorem (Tom Bachmann)**

*Real realization gives a equivalence* $\mathrm{SH}(\mathbb{R})^- \cong \mathrm{SH}[1/2]$. *This induces* $H^*(X_{\mathrm{Nis}}, \mathcal{W}[1/2]) \cong H^*_{sing}(X(\mathbb{R}), \mathbb{Z}[1/2])$ *for* $X$ *smooth over* $\mathbb{R}$.

Since $\mathcal{GW}$ maps to both $\mathbb{Z} = \underline{\mathrm{CH}}^0$ and to $\mathcal{W}$, the $\mathcal{GW}$-degree has a foot in both worlds.

Fix a smooth projective del Pezzo surface $S$ over $\mathbb{C}$ and an effective divisor $D$ on $S$ with $D^{(2)} \geq -1$. Let $n = -D \cdot K_S - 1$. For $y_1, \ldots, y_n$ general points on $S$, there are finitely many ($N_{S,D}$) rational curves in $|D|$ passing through all the $y_i$, and these are all integral with only ordinary double points (odp) as singularities.

**Question**: if $S$ is defined over $\mathbb{R}$ and the $n$ points consist of $r$ real points $p_1, \ldots, p_r$ and $s$ $\mathbb{C}$-conjugate pairs $q_1, \bar{q}_1, \ldots, q_s, \bar{q}_s$, how many of the $N_{S,D}$ rational curves in $|D|$ passing through the $p_i, q_j, \bar{q}_j$ are *real* ?

**Answer**: It depends! And not just on the real type of $(p_*, q_*, \bar{q}_*)$

**Reason**: The open subset of $S(\mathbb{C})^n$ parametrizing the "general" configurations $s_*$ is connected, but the corresponding open subset of $S(\mathbb{R})^n$ is not (even fixing the real type).

Welschinger corrected this by defining a "mass" $m(C) \in \mathbb{N}$ for each integral smooth curve $C$ on $S$ having only odp and showed

**Theorem (Welschinger ∼ 2005)**

*For $S$, $D$ and $(p_*, q_*, \bar{q}_*)$ as above, with $(p_*, q_*, \bar{q}_*)$ general,*

$$Wel_{S,D}(p_*, q_*, \bar{q}_*) := \sum_{C \supset \{p_*, q_*, \bar{q}_*\}, C \in |D|, C\, rational} (-1)^{m(C)}$$

*depends only on the real type of $(p_*, q_*, \bar{q}_*)$.*

Welschinger proved this in the setting of symplectic manifolds with real structure and almost complex structure.
Itenberg-Kharlamov-Shustin proved the version stated above.

**Goal**. For a del Pezzo $S/k$, and $p_* = \sum_i p_i$ a reduced effective 0-cycle of degree $n$, define a (natural) quadratic form $W_{S,D}(p_*) \in \mathrm{GW}(k(p_*))$ such that

1. $\mathrm{rank}(W_{S,D}(p_*)) = N_{S,D}$
2. for $k = k(p_*) = \mathbb{R}$, $\mathrm{sig}(W_{S,D}(p_*)) = Wel_{S,D}(p_*)$.

Moreover, the assignment $p_* \to W_{S,D}(p_*)$ should be "$\mathbb{A}^1$-invariant".

**Idea**. Define $W_{S,D}$ as a section of $\mathcal{GW}$ over the unordered configuration space $\mathrm{Sym}^n(S)^0$ by taking the pushforward of a fundamental class by an evaluation map.

**Technical points**. We may freely remove codimension 2 subsets of $\mathrm{Sym}^n(S)^0$ because $\mathcal{GW}$ is unramified. We assume $\mathrm{char} k \neq 2, 3$.

- $\bar{\mathcal{M}}_{0,n}(S,D)$ = Kontsevich moduli stack of stable maps to $S$ of $n$-pointed genus 0 curves, in the curve class $D$.
- $\mathrm{ev}:\bar{\mathcal{M}}_{0,n}(S,D)\to S^n$ the evaluation map: $\mathrm{ev}(f:C\to S,x_1,\ldots,x_n)=(f(x_1),\ldots,f(x_n))$.
- $\mathrm{ev}:\bar{\mathcal{M}}^{\Sigma}_{0,n}(S,D)\to \mathrm{Sym}^n(S)$: take the quotient by the $S_n$-action permuting the marked points.
- $\mathrm{Sym}^n(S)^0\subset \mathrm{Sym}^n(S)$ the unordered configuration space: $\mathrm{Sym}^n(S)^0=S_n\backslash(S^n\setminus\{\text{diagonals}\})$
- Cartesian diagram:

$$\begin{array}{ccccc}
\bar{\mathcal{M}}^{\Sigma}_{0,n}(S,D)^{\mathrm{gen}} & \subset & \bar{\mathcal{M}}^{\Sigma}_{0,n}(S,D)^0 & \subset & \bar{\mathcal{M}}^{\Sigma}_{0,n}(S,D)\\
\downarrow{\scriptstyle \mathrm{ev}^{\mathrm{gen}}} & & \downarrow{\scriptstyle \mathrm{ev}^0} & & \downarrow{\scriptstyle \mathrm{ev}}\\
\mathrm{Sym}^n(S)^{\mathrm{gen}} & \subset & \mathrm{Sym}^n(S)^0 & \subset & \mathrm{Sym}^n(S)
\end{array}$$

To describe $\mathrm{Sym}^n(S)^{\mathrm{gen}} \subset \mathrm{Sym}^n(S)^0$:

$p_* \in \mathrm{Sym}^n(S)^0$ is in $\mathrm{Sym}^n(S)^{\mathrm{gen}}$ if

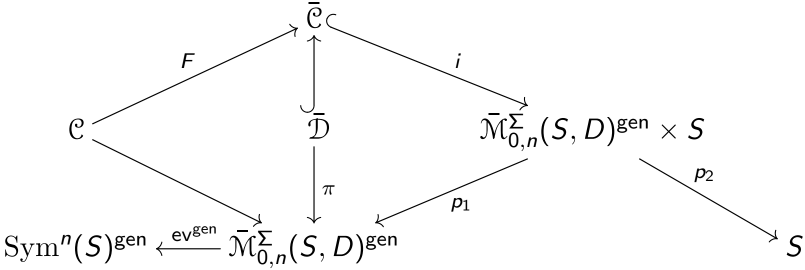
for all $(f : C \to S, x_*)$ with $\mathrm{ev}((f : C \to S, x_*)) = p_*$ we have

1. $C$ is smooth and irreducible
2. $f : C \to f(C)$ is birational
3. $f(C) \subset S$ has only odp's as singularities

# Quadratic Welschinger invariants

## The quadratic mass

A commutative diagram:




$\mathcal{C}$ = the universal curve, $i \circ F$ the universal map, $\bar{\mathcal{C}}$ the family of image curves, $\bar{\mathcal{D}}$ the subscheme of double points of $\bar{\mathcal{C}}$.
All objects except $\bar{\mathcal{C}}$ are smooth $k$-schemes.
$\pi$ and $\mathrm{ev}^{\mathrm{gen}}$ are finite and étale.

Taking the (relative) Hessian of a defining equation for $\bar{\mathcal{C}}$ along $\bar{\mathcal{D}}$ gives the map of sheaves on $\bar{\mathcal{D}}$

$$\mathrm{Hess} : \mathcal{I}_{\bar{\mathcal{C}}}/\mathcal{I}^2_{\bar{\mathcal{C}}} \otimes \mathcal{O}_{\bar{\mathcal{D}}} \to (p_2^*\Omega^1_{S/k})^{\otimes 2} \otimes \mathcal{O}_{\bar{\mathcal{D}}}$$

$$\mathrm{Hess} : \mathcal{O}_{\bar{\mathcal{D}}} \to \mathcal{H}om(p_2^*\Omega^{1\vee}_{S/k} \otimes \mathcal{O}(\bar{\mathcal{C}}), p_2^*\Omega^1_{S/k}) \otimes \mathcal{O}_{\bar{\mathcal{D}}}$$

Taking determinants gives the (nowhere vanishing) Hessian determinant

$$\det \mathrm{Hess} : \mathcal{O}_{\bar{\mathcal{D}}} \xrightarrow{\sim} [p_2^*\omega_{S/k}(\bar{\mathcal{C}})]^{\otimes 2} \otimes \mathcal{O}_{\bar{\mathcal{D}}}$$

Write $p_2^*\omega_{S/k}(\bar{\mathcal{C}}) \otimes \mathcal{O}_{\bar{\mathcal{D}}} = \mathcal{O}_{\bar{\mathcal{D}}}(A)$ for some Cartier divisor $A$ on $\bar{\mathcal{D}}$ and take the norm of $\det \mathrm{Hess}$ down to $\bar{\mathcal{M}}^{\Sigma}_{0,n}(S, D)^{\mathrm{gen}}$, giving

$$\mu \in H^0(\bar{\mathcal{M}}^{\Sigma}_{0,n}(S, D)^{\mathrm{gen}}, \mathcal{O}(2\pi_*(A))),$$

nowhere 0.

We mention three divisors on $\bar{\mathcal{M}}^{\Sigma}_{0,n}(S,D)^0$ having empty intersection with $\bar{\mathcal{M}}^{\Sigma}_{0,n}(S,D)^{\mathrm{gen}}$:

- $D_{cusp}$: The closure of the generic map $f : C \to S$ with $C$ smooth, $f : C \to f(C)$ birational and $f(C)$ having a single ordinary cusp (+ odp's).
- $D_{tac}$: The closure of the generic map $f : C \to S$ with $C$ smooth, $f : C \to f(C)$ birational and $f(C)$ having a single ordinary tacnode (+ odp's).
- $D_{trip}$: The closure of the generic map $f : C \to S$ with $C$ smooth, $f : C \to f(C)$ birational and $f(C)$ having a single ordinary triple point (+ odp's).

### Lemma

*After removing a closed* $F \subset \mathrm{Sym}^n(S)^0$, $\mathrm{codim} F \geq 2$*:*

*1. Let* $B$ *be the closure of* $\pi_*(A)$ *in* $\bar{\mathcal{M}}^{\Sigma}_{0,n}(S,D)^0$*. Then* $\mu \in H^0(\bar{\mathcal{M}}^{\Sigma}_{0,n}(S,D)^0, \mathcal{O}(2B))$ *has divisor* $D_{cusp} + 2D_{tac} + 6D_{trip}$*.*

*2. The map* $\mathrm{ev}^0 : \bar{\mathcal{M}}^{\Sigma}_{0,n}(S,D)^0 \to \mathrm{Sym}^n(S)^0$ *is étale away from* $D_{cusp}$ *and is ramified to order 2 along* $D_{cusp}$*.*

### Proposition

$\omega_{\mathrm{ev}^0} \cong \mathcal{O}(D_{cusp})$ *and* $\mu$ *defines an isomorphism*

$$\theta_\mu : L^{\otimes 2} \xrightarrow{\sim} \omega_{\mathrm{ev}^0}.$$

*with* $L = \mathcal{O}_{\bar{\mathcal{M}}^{\Sigma}_{0,n}(S,D)^0}(B + D_{tac} + 3D_{trip})$*.*

Via $\theta_\mu$, we have the isomorphism

$$\mathcal{GW} \cong \mathcal{GW}(L^{\otimes 2}) \xrightarrow{\theta_\mu} \mathcal{GW}(\omega_{\mathrm{ev}^0})$$

## Definition

$W_{S,D} \in H^0(\mathrm{Sym}^n(S)^0, \mathcal{GW})$ is defined as the image of $<1> \in H^0(\bar{\mathcal{M}}^{\Sigma}_{0,n}(S,D)^0, \mathcal{GW})$ via

$$H^0(\bar{\mathcal{M}}^{\Sigma}_{0,n}(S,D)^0, \mathcal{GW}) \xrightarrow{\theta_\mu} H^0(\bar{\mathcal{M}}^{\Sigma}_{0,n}, \mathcal{GW}(\omega_{\mathrm{ev}^0})) \xrightarrow{\mathrm{ev}^0_*} H^0(\mathrm{Sym}^n(S)^0, \mathcal{GW})$$

$W_{S,D}$ extends $\mathrm{Tr}_{k(\bar{\mathcal{M}}^{\Sigma}_{0,n})/k(\mathrm{Sym}^n(S))}(<\mu>) \in \mathrm{GW}(k(\mathrm{Sym}^n(S)))$.

**Theorem**

*Let $k$ be a perfect field of characteristic $\neq 2, 3$.*

*1. $\mathrm{rank}(W_{S,D}) = N_{S,D}$*

*2. Suppose $k = \mathbb{R}$. Then for $p_* \in \mathrm{Sym}^n(S)^0(\mathbb{R})$,*
$\mathrm{sig}(W_{S,D})(p_*) = (-1)^g Wel_{S,D}(p_*)$*, where $g$ is the genus of the generic (smooth) curve in $|D|$.*

Experimental error: one can correct by replacing $W_{S,D}$ with $<(-1)^g> W_{S,D}$.

**Proof.**

(1): $N_{S,D} =$ usual degree of $\mathrm{ev}^0 =$ rank of $\mathrm{ev}^0_*(<1>)$. □

## Proof.

(2): For $\bar{C} \subset S$, integral with only odp's, defined over $\mathbb{R}$,

$$m(\bar{C}) = \#\{\text{isolated points of } \bar{C}(\mathbb{R})\}$$

At $y \in \bar{C}_{sing}(\mathbb{R})$ we have

$$\det \mathrm{Hess}(y) \text{ is } \begin{cases} > 0 & \text{if } y \text{ is an isolated point} \\ < 0 & \text{if } y \text{ is a non-isolated point} \end{cases}$$

For $f : C \to \bar{C}$, we thus have $\mu(f) = (-1)^{\#\{\text{non-isolated singular points}\}}$ mod $\mathbb{R}^{\times 2}$. As $\bar{C}$ has $g$ singular points:

$$\mu(f) = (-1)^g \cdot (-1)^{m(\bar{C})} \mod \mathbb{R}^{\times 2}$$

and $\mathrm{sig}(W_{S,D}(p_*)) = \mathrm{sig}(\mathrm{Tr}_{k(\bar{\mathbb{M}}^{\Sigma}_{0,n})/k(\mathrm{Sym}^n(S))}(<\mu>)(p_*))$ just adds these up over all $f$ with $\mathrm{ev}(f) = p_* \Rightarrow$
$\mathrm{sig}(W_{S,D}(p_*)) = (-1)^g Wel_{S,D}(p_*)$. □

## Corollary

*$Wel_{S,D}(p_*)$ depends only on the real type of $p_*$*

## Proof.

The real connected components of $S(\mathbb{R})$ are real surfaces, so the connected components of $\mathrm{Sym}^n(S)^0(\mathbb{R})$ are exactly the real types.

$W_{S,D} \in H^0(\mathrm{Sym}^n(S)^0, \mathcal{GW}) \Rightarrow \mathrm{sig}(W_{S,D})$ is constant on each connected component of $\mathrm{Sym}^n(S)^0(\mathbb{R})$.

$Wel_{S,D}(p_*) = (-1)^g \mathrm{sig}(W_{S,D}(p_*)) \Rightarrow Wel_{S,D}(p_*)$ depends only on the real type of $p_*$. □

### Definition ($K$-type/$\mathbb{A}^1$-$K$-type)

Let $K$ be an extension field of $k$.
1. For $p_* = \sum_{i=1}^r p_i \in \mathrm{Sym}^n(S)^0(K)$, the $K$-type of $p_*$ is the equivalence class of the function on $\{1, \ldots, r\}$ sending $i$ to the isomorphism class of the field extension $K \subset K(p_i)$, where two such functions are equivalent if there is a $\sigma \in S_r$ with $K(p_{\sigma(i)}) \cong K(p_i)$ for all $i$.
2. For $p_* = \sum_{i=1}^r p_i \in \mathrm{Sym}^n(S)^0(K)$, the $\mathbb{A}^1$-$K$-type of $p_*$ is the equivalence class of the function on $\{1, \ldots, r\}$ sending $i$ to the class $[p_i] \in \pi_0^{\mathbb{A}^1}(S)(K(p_i))$, where two such functions are equivalent if there is a permutation $\sigma \in S_r$ and isomorphisms $\theta_i : K(p_i) \to K(p_{\sigma(i)})$ over $K$ with $\theta_{i*}([p_i]) = [p_{\sigma(i)}]$.

**Theorem (?-In progress)**

*Let $K$ be a field. For $p_*, q_* \in \mathrm{Sym}^n(S)^0(K)$ of the same $\mathbb{A}^1$-$K$-type, we have $W_{S,D}(p_*) = W_{S,D}(q_*)$.*

## Proof.

Let $\kappa = ((K_1, n_1), \ldots, (K_s, n_s))$ denote the $K$-type of $p_*$: exactly $n_j$ of the $p_i$ have $K(p_i) \cong K_j$. There is an associated restriction of scalars $S_\kappa := \prod_{j=1}^s \mathrm{Res}_{K_j/K} S$ and $(p_1, \ldots, p_r)$, $(q_1, \ldots, q_r)$ determine a $K$-points $\tilde{p}_*, \tilde{q}_*$ of $S_\kappa^0 := S_\kappa \setminus \{\text{diagonals}\}$.

We have $\pi_\kappa : S_\kappa^0 \to \mathrm{Sym}^n(S)^0$ and $W_{S,D}(p_*) = \pi_\kappa^*(W_{S,D})(\tilde{p}_*)$, etc.

Since $S_\kappa$ is smooth and $\{\text{diagonals}\}$ has codimension $\geq 2$, $\pi_\kappa^*(W_{S,D})$ extends to a section of $\mathcal{GW}$ over $S_\kappa$.

The condition that $p_*$ and $q_*$ have the same $\mathbb{A}^1$-$K$-type says that $[\tilde{p}_*] = [\tilde{q}_*]$ in $\pi_0^{\mathbb{A}^1}(S_\kappa)(K)$, so

$$W_{S,D}(p_*) = \pi_\kappa^*(W_{S,D})(\tilde{p}_*) = \pi_\kappa^*(W_{S,D})(\tilde{q}_*) = W_{S,D}(q_*)$$

□