A calculus of characteristic classes in Witt cohomology

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Enumerative geometry involves

1. Intersection theory on X via the Chow ring $\mathrm{CH}^*(X)$.
2. Degrees, via the pushforward

$$\deg_k = \pi_{X*} : CH^{\dim X}(X) \to \mathrm{CH}^0(\mathrm{Spec}\, k) = \mathbb{Z}$$

for $\pi_X : X \to \mathrm{Spec}\, k$ smooth and proper over k.

3. Characteristic classes of an algebraic vector bundles $V \to X$, for instance the Chern class $c_n(V) \in \mathrm{CH}^n(X)$.

We want to describe a "quadratic refinement" of this package.

We have the *Grothendieck-Witt ring* of a field F (of characteristic $\neq 2$),

$$\mathrm{GW}(F) = (\{\text{non-degenerate quadratic forms}\}/\text{isom}, \perp)^+,$$

the hyperbolic form $H(x, y) = xy \sim x^2 - y^2$, and the *Witt ring*

$$W(F) = \mathrm{GW}(F)/([H]) = \mathrm{GW}(F)/\mathbb{Z} \cdot [H].$$

$\mathrm{GW}(F)$ is additively generated by the one-dimensional forms $<u>$, $u \in F^\times$, $<u>(x) = ux^2$.

There is the rank homomorphism $\text{rnk} : \mathrm{GW}(F) \to \mathbb{Z}$ and for $F = \mathbb{R}$, the signature $\text{sig} : \mathrm{GW}(\mathbb{R}) \to \mathbb{Z}$.

For X smooth over k, the Milnor K-theory sheaf $\mathcal{K}^M_*$ has the quadratic refinement $\mathcal{K}^{MW}_*$, the *Milnor-Witt* sheaf, and a twisted version $\mathcal{K}^{MW}_*(L)$ for $L \to X$ a line bundle. There is an element $\eta \in K^{MW}_{-1}(k)$ with

$$\mathcal{K}^{MW}_*(L)/(\eta) = \mathcal{K}^M_*.$$

There are isomorphisms

$$\mathcal{K}^{MW}_0 \cong \mathcal{GW}, \quad \mathcal{K}^{MW}_{-n} \cong \mathcal{W}, \ n > 0,$$

with $\mathcal{W}$ the sheaf of Witt groups $\mathcal{GW}/(H)$. For $n \geq 0$, there is an exact sequence

$$0 \to \mathcal{I}^{n+1} \to \mathcal{K}^{MW}_n \to \mathcal{K}^M_n \to 0$$

$\mathcal{I} = \ker[\mathrm{rnk} : \mathcal{GW} \to \mathbb{Z}]$.

The global sections of the Gersten resolution for $\mathcal{K}^M_n$ computes $H^n(X, \mathcal{K}^M_n)$ as the cohomology in

$$\cdots \xrightarrow{\partial} \oplus_{x \in X^{(n-1)}} K^M_1(k(x)) = k(x)^\times \xrightarrow{\partial} \oplus_{x \in X^{(n)}} K^M_0(k(x)) = \mathbb{Z}$$

giving Kato's isomorphism

$$\mathrm{CH}^n(X) = H^n(X, \mathcal{K}^M_n).$$

This refines to define the *Chow-Witt groups* (Barge-Morel, Fasel)

$$\tilde{\mathrm{CH}}^n(X; L) := H^n(X, \mathcal{K}^{MW}_n(L)).$$

For $x \in X^{(n)}$, twisting by the local orientation line bunde $or_x := \Lambda^n \mathfrak{m}_x/\mathfrak{m}_x^2$ refines the Gersten resolution for $\mathcal{K}^M_n$ to the "Rost-Schmidt" resolution of $\mathcal{K}^{MW}_n(L)$.

Taking global sections in the Rost-Schmidt resolution describes $\tilde{\mathrm{CH}}^n(X;L) = H^n(X, \mathcal{K}_n^{MW}(L))$ as the cohomology in

$$\cdots \xrightarrow{\partial} \oplus_{x\in X^{(n-1)}} K_1^{MW}(L\otimes or_x)(k(x)) \xrightarrow{\partial} \oplus_{x\in X^{(n)}} K_0^{MW}(L\otimes or_x)(k(x)) \xrightarrow{\partial} \oplus_{x\in X^{(n+1)}} K_{-1}^{MW}(L\otimes or_x)(k(x)) \xrightarrow{\partial} \cdots$$

Via isomorphisms

$$K_0^{MW}(L\otimes or_x)(k(x)) \cong \mathrm{GW}(L\otimes or_x)(k(x)) \cong \mathrm{GW}(k(x)),$$

this represents $\mathcal{Z} \in \tilde{\mathrm{CH}}^n(X;L)$ as

$$\mathcal{Z} = \sum_i \alpha_i \cdot Z_i$$

with the $Z_i \subset X$ codimension n subvarieties and $\alpha_i \in \mathrm{GW}(k(Z_i))$.

For $f : Y \to X$ proper, X, Y smooth$/k$, there is a pushforward

$$f_* : H^m(Y, \mathcal{K}_n^{MW}(f^*L \otimes \omega_{Y/X})) \to H^{m-d}(X, \mathcal{K}_{n-d}^{MW}(L))$$

$d =$ relative dimension of f, $\omega_{Y/X}; = \omega_{Y/k} \otimes f^*\omega_{X/k}^{-1}$.

This gives the quadratic degree map

$$\tilde{\deg}_k = \pi_{X*} : \tilde{\mathrm{CH}}^d(X, \omega_{X/k}) \to \tilde{\mathrm{CH}}^0(\mathrm{Spec}\, k) = \mathrm{GW}(k)$$

for X smooth and proper of dimension d over k.

This refines the integral degree map via the canonical map

$$\mathcal{K}_*^{MW}(L) \to \mathcal{K}_*^{MW}(L)/(\eta) = \mathcal{K}_*^M.$$

For $V \to X$ a rank n vector bundle, we have the *Euler class*

$$e(V) := s^* s_*(1_X) \in \tilde{\mathrm{CH}}^n(X, \overset{-1}{\det}(V))$$

$s =$ zero section, $1_X \in \tilde{\mathrm{CH}}^0(X) = H^0(X, \mathcal{GW})$ the unit section.

The Euler class refines the top Chern class, and $e(\det V)$ refines $c_1(V)$ but there are no classes refining the other Chern classes.

For this we pass to classes in $\mathcal{W}$-cohomology.

The map $\times\eta : \mathcal{K}_n^{MW} \to \mathcal{K}_{n-1}^{MW}$ is the surjection $\mathcal{GW} \to \mathcal{W}$ for $n = 0$ and an isomorphism for $n < 0$.

Inverting $\eta \in K_{-1}^{MW}(k)$ gives

$$\mathcal{K}_*^{MW}(L)[\eta^{-1}] \cong \mathcal{W}(L) = \mathcal{GW}(L)/(h)$$

A rank n vector bundle $V \to X$ has *Pontryagin classes*

$$p_i(V) \in H^{4i}(X, \mathcal{W}); 2 \leq 2i \leq n.$$

For $n = 2m$, we have

$$p_m(V) = e(V)^2$$

The Pontryagin classes are defined via the Borel classes (Panin-Walter).

Let $V \to X$ be a rank $2n+2$ symplectic bundle over X with symplectic form ω. Let $H\mathbb{P}(V) \subset \mathrm{Gr}(2, V)$ the open subscheme of 2-planes $E \subset V$ with $\omega_{|E}$ non-degenerate. $\mathcal{E} \to H\mathbb{P}(V)$ the tautological symplectic 2-plane bundle.

Set $\zeta := e(\mathcal{E}) \in H^2(H\mathbb{P}(V), \mathcal{W})$. Then

$$H^*(H\mathbb{P}(V), \mathcal{W}) = \oplus_{i=0}^{n} H^*(X, \mathcal{W}) \cdot \zeta^i.$$

We get *Borel classes* $b_1(V), b_2(V), \ldots, b_i(V) \in H^{2i}(X, \mathcal{W})$, by the Grothendieck method.

For $V \to X$ a vector bundle, define $p_i(V) := b_{2i}(V \oplus V^\vee)$ in $H^{4i}(X, \mathcal{W})$.

Refining the classical enumerative package of the Chow ring, the degree map and Chern classes is the *quadratic enumerative package*:

1. (quadratic intersection theory) For X smooth over k, the twisted Chow-Witt groups $\tilde{\mathrm{CH}}^*(X; L)$

2. (quadratic degree) For $\pi_X : X \to \mathrm{Spec}\, k$ smooth and proper, the quadratic degree map

$$\tilde{\deg}_k = \pi_{X*} : \tilde{\mathrm{CH}}^{\dim X}(X; \omega_{X/k}) \to \tilde{\mathrm{CH}}^0(\mathrm{Spec}\, k) = \mathrm{GW}(k).$$

3. (quadratic characteristic classes) For $V \to X$ a rank n vector bundle, the Euler class

$$e(V) \in \tilde{\mathrm{CH}}^n(X, \det^{-1} V) = H^n(X, \mathcal{K}_n^{MW}(\det^{-1}(V))$$

and the Pontryagin classes $p_i(V) \in H^{4i}(X, \mathcal{W})$.

Ananyevskiy has computed $H^*(BSL_n, \mathcal{W})$.

Let $E_n \to BSL_n$ be the universal vector bundle. We have the Pontryagin classes $p_i(E_n) \in H^{4i}(BSL_n, \mathcal{W})$ and the Euler class $e(E_n) \in H^n(BSL_n, \mathcal{W})$ with $e^2 = p_m$ for $n = 2m$, $e = 0$ for n odd.

Theorem (Ananyevskiy)

1. *For $n = 2m$*

$$H^*(BSL_n, \mathcal{W}) \cong W(k)[p_1, \ldots, p_{m-1}, e].$$

2. *For $n = 2m + 1$*

$$H^*(BSL_n, \mathcal{W}) \cong W(k)[p_1, \ldots, p_m].$$

Let $i : (\mathrm{SL}_2)^m \hookrightarrow \mathrm{SL}_{2m}$ be the block-diagonal inclusion, and

$$e_j := \pi_j^* e(E_2) \in H^*(B(\mathrm{SL}_2)^m, \mathcal{W});\ j = 1, \ldots, m.$$

Theorem (SL$_2$-splitting principle-Ananyevskiy)

i induces an injection

$$i^* : H^*(B\mathrm{SL}_{2m}, \mathcal{W}) \hookrightarrow H^*(B(\mathrm{SL}_2)^m, \mathcal{W}) = W(k)[e_1, \ldots, e_m],$$

with

$$i^* e := \prod_{j=1}^m e_j,\ \ i^* p_j = \sigma_j(e_1^2, \ldots, e_m^2).$$

Question: What are the characteristic classes of $\mathrm{Sym}^\ell E_2$?

Theorem (Main)

Let $E_2 \to BSL_2$ be the tautological rank 2 bundle. Then

$$e(\mathrm{Sym}^\ell E_2) = \begin{cases} 0 & \text{for } \ell \geq 2 \text{ even.} \\ \ell \cdot (\ell - 2) \cdots 3 \cdot 1 \cdot e^m & \text{for } \ell = 2m - 1 \geq 1 \text{ odd.} \end{cases}$$

The total Pontryagin class $p(\mathrm{Sym}^\ell E_2) := 1 + \sum_{i \geq 1} p_i(\mathrm{Sym}^\ell E_2)$ is

$$p(\mathrm{Sym}^\ell E_2) = \prod_{j=0}^{[\ell/2]} (1 + (\ell - 2j)^2 e^2).$$

We consider the problem of counting the (finite number of) lines on a smooth hypersurface $X \subset \mathbb{P}^{d+1}$ of degree $2d-1$. The “answer” is given by

$$\tilde{\deg}_k(e(\mathrm{Sym}^{2d-1}(E_2))) \in \mathrm{GW}(k)$$

where $E_2 \to \mathrm{Gr}(2, d+2)$ is the tautological rank 2 quotient bundle of $\mathcal{O}^{d+2}$. The classical computation in the Chow ring gives the integer count N_d

$$N_d := \deg_k(c_{2d}(\mathrm{Sym}^{2d-1}E_2)) \in \mathbb{Z}$$

By our theorem, we have

$$e(\mathrm{Sym}^{2d-1}(E_2)) = (2d-1)!!e(E_2)^d \in H^d(\mathrm{Gr}(2, d+2), \mathcal{W}(O(-d))),$$

so

$$\pi_*^W(e(\mathrm{Sym}^{2d-1}(E_2))) = (2d-1)!! \in W(k).$$

Thus $\pi_*(e(\mathrm{Sym}^{2d-1}(E_2))) \in \mathrm{GW}(k)$ is given by

$$\pi_*(e(\mathrm{Sym}^{2d-1}(E_2))) = (2d-1)!! < 1 > + \frac{N_d - (2d-1)!!}{2} \cdot H$$

More explicitly,

$$N_d = \deg \left[\prod_{j=0}^{d-1} j \cdot (2d - j - 1)c_1^2 + (2(d-j) - 1)^2 c_2 \right]$$

and

$$\deg(c_1^{2(d-a)} c_2^a) = \frac{2(d-a)!}{(d-a+1)!(d-a)!}$$

The $W(k)$-part gives a lower bound $(2d-1)!!$ for the number of real lines (counted with a positive multiplicity) and a mod 2 congruence for "types" of lines over a finite field.

This recovers and extends work of (Kass-Wickelgren) who discuss the case of lines on a cubic surface, using a different method.

In the case of a topological real $\mathrm{SL}_2(\mathbb{R})$-bundle, one uses the homotopy equivalence

$$S^1 = SO(2) \hookrightarrow \mathrm{SL}_2(\mathbb{R})$$

and the $\mathbb{C}$-structure on a real S^1-bundle to reduce to complex line bundles:

$$BSL_2(\mathbb{R}) \sim BSO(2) = BS^1 \cong \mathbb{CP}^\infty$$
$$E_2 \leftrightarrow O(1); \quad e(E_2) \leftrightarrow c_1(O(1)) \in H^2(\mathbb{CP}^\infty, \mathbb{Z}).$$

As $SO(2)$-bundle $\mathrm{Sym}^\ell E_2$ splits

$$\mathrm{Sym}^\ell E_2 \cong \oplus_{j=0}^{[\ell/2]} O(\ell - 2j)$$

which explains the formula in the theorem.

We replace $S^1 \subset \mathrm{SL}_2(\mathbb{R})$ with the normalizer N_T of the standard torus $T \subset \mathrm{SL}_2$.

$$N_T = <\{\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}> \subset \mathrm{SL}_2.$$

N_T has (roughly) the same algebraic representation theory as the real representation theory of S^1: every irreducible representation is either 1- or 2-dimensional. We concentrate on the 2-dimenisonal representations $\rho_\ell : N_T \to \mathrm{GL}_2$, $\ell \geq 1$,

$$\rho_\ell(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}) = \begin{pmatrix} t^\ell & 0 \\ 0 & t^{-\ell} \end{pmatrix};\ \rho_\ell(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & 1 \\ (-1)^\ell & 0 \end{pmatrix}.$$

For $k = 0$, there are two 1-dimensional representations $\rho_0^\pm$.

$$\rho_0^\pm(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}) = \text{id}$$

$$\rho_0^\pm(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) = \pm\text{id}.$$

Let $\pi : BN_T \to B\mathrm{SL}_2$ be the canonical map, $E(\rho) \to BN_T$ the bundle associated to a representation $\rho : N_T \to \mathrm{GL}_n$. Then $\pi^* E_2 = E(\rho_1)$ and

$$\pi^*\mathrm{Sym}^\ell E_2 = \begin{cases} \oplus_{j=0}^{[\frac{\ell}{2}]} E(\rho_{\ell-2j}) & \ell \text{ odd} \\ \oplus_{j=0}^{\frac{\ell}{2}-1} E(\rho_{\ell-2j}) \oplus \rho_0^+ & \ell \equiv 0 \mod 4 \\ \oplus_{j=0}^{\frac{\ell}{2}-1} E(\rho_{\ell-2j}) \oplus \rho_0^- & \ell \equiv 2 \mod 4 \end{cases}$$

The main theorem reduces to

Theorem (N_T-splitting principle)

1. *$\pi^* : H^*(BSL_2, \mathcal{W}) \to H^*(BN_T, \mathcal{W})$ is a split injection.*
2. *$e(E(\rho_\ell)) = \ell \cdot e(E(\rho_1))$ for $\ell \geq 1$.*
3. *$e(E(\rho_0^\pm)) = 0$.*

(3) is just the vanishing of $e(V)$ in $H^*(-, \mathcal{W}(\det^{-1} V))$ for odd rank V.

1. BN_T is a (Zariski locally trivial) $N_T\backslash\mathrm{SL}_2$-bundle over $B\mathrm{SL}_2$ and

$$N_T\backslash\mathrm{SL}_2 \cong [\mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta]/\mathbb{Z}/2 \cong \mathbb{P}^2 \setminus C$$

where $C \subset \mathbb{P}^2$ is the conic $q := T_1^2 - 4T_0T_2 = 0$ $(\mathrm{char}\, k \neq 2)$.
The double cover $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2 = \mathrm{Sym}^2\mathbb{P}^1$ is $\mathrm{Spec}\,\mathcal{O}_{\mathbb{P}^2}(\sqrt{q})$.

2. q defines an SL_2-invariant section of $\mathcal{W}$ over $\mathbb{P}^2 \setminus C$. The Leray spectral sequence gives

$$H^n(BN_T, \mathcal{W}) = \begin{cases} H^n(B\mathrm{SL}_2, \mathcal{W}) & \text{for } n > 0 \\ W(k) \oplus [q]W(k) = & \\ \ H^0(B\mathrm{SL}_2, \mathcal{W}) \oplus [q]H^0(B\mathrm{SL}_2, \mathcal{W}) & \text{for } n = 0 \end{cases}$$

3. The map $(x, y) \mapsto (x^\ell, y^\ell)$ gives a fiberwise polynomial map of bundles $m_\ell : E(\rho_1) \to E(\rho_\ell)$, and reduces the theorem to showing

$$m_\ell^*(s_*^\ell(1_{BN_T})) = \ell \cdot s_*^1(1_{BN_T}))$$

in $H^2(E(\rho_1), \mathcal{W}) \cong H^2(BN_T, \mathcal{W})$.

4. We have a Thom isomorphism

$$W(k) \oplus [q]W(k) = H^0(BN_T, \mathcal{W}) \cong H^2_{0_{BN_T}}(E(\rho_1), \mathcal{W})$$

and a surjection

$$p : H^2_{0_{BN_T}}(E(\rho_1), \mathcal{W}) \to H^2(E(\rho_1), \mathcal{W})$$

5. One computes the kernel of this surjection as

$$\ker p = W(k)(1, [q]).$$

This is the same as the kernel of the evaluation map

$$ev_x : H^0(BN_T, \mathcal{W}) \to W(k)$$

where $x = (1:0:1) \in \mathbb{P}^2 \setminus C \subset BN_T(\mathrm{Spec}\, k)$, because

$$<q(x)> = <-4> = -1 \text{ in } W(k).$$

Note: x comes from $((1:i),(1:-i)) \in \mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta$.

6. This gives: $e(E(\rho_\ell)) = \lambda_\ell \cdot e(E(\rho_1))$ with

$\lambda_\ell =$ the "local degree" at 0 of $m_{\ell,x} : E(\rho_1)_x \to E(\rho_\ell)_x$,

$=$ the "degree" (in $\mathrm{End}_{\mathrm{SH}(k)}(\Sigma^\infty_T \mathbb{P}^1)[\eta^{-1}] = W(k)$) of

$$m_{\ell,x} : \mathbb{P}^1_k = \mathbf{Proj}(E(\rho_1)_x) \to \mathbf{Proj}(E(\rho_\ell)_x) = \mathbb{P}^1_k.$$

The map $m_{\ell,x}$ is

$$m_{\ell,x}(x:y) = (Re^{(\ell)}(x,y) : Im^{(\ell)}(x,y))$$

where

$$(x+iy)^\ell = Re^{(\ell)}(x,y) + i \cdot Im^{(\ell)}(x,y).$$

It suffices to make the computation for $\ell = p$ a prime.

7. By Morel's "motivic Brouwer degree formula",

$$\lambda_p = \mathrm{Tr}_{k(m_{p,x}^{-1}(y))/k(y)}(<\partial m_{p,x}/\partial t>)$$

for $y \in \mathbb{P}^1(k)$ a regular value of k, t a "normalized parameter" at y.

For $y = (0:1)$ the cover is

$$(0:1) \amalg \mathrm{Spec}\,\mathbb{Q}[\zeta_{4p} + \zeta_{4p}^{-1}] \to (0:1) = \mathrm{Spec}\,k$$

and an explicit calculation (via a theorem of Serre, with help from Eva Bayer) gives the trace form as

$$\lambda_p = \left[\sum_{i=1}^{p} x_i^2\right] = p \cdot <1> = p \in W(k).$$