Algebraic Cobordism

Motives and Periods
Vancouver-June 5-12, 2006

Marc Levine

Prelude: Cohomology of algebraic varieties

The category of Chow motives is supposed to capture "universal cohomology", but:

What is cohomology?

k: a field. $\mathbf{Sm}/k$: smooth quasi-projective varieties over k.
What should "cohomology of smooth varieties over k" be?

This should be at least the following

D1. An additive contravariant functor A^* from $\mathbf{Sm}/k$ to graded (commutative) rings:
$X \mapsto A^*(X)$;
$(f : Y \to X) \mapsto f^* : A^*(X) \to A^*(Y)$.

D2. For each projective morphisms $f : Y \to X$ in $\mathbf{Sm}/k$, a push-foward map

$$f_* : A^*(Y) \to A^{*+\epsilon d}(X)$$

$d = \text{codim} f$, $\epsilon = 1, 2$.

These should satisfy some compatibilities and additional axioms:

A1. $(fg)_* = f_*g_*$; $\quad \mathrm{id}_* = \mathrm{id}$

A2. For $f : Y \to X$ projective, f_* is $A^*(X)$-linear:
$f_*(f^*(x) \cdot y) = x \cdot f_*(y)$.

A3. Let

$$\begin{array}{ccc} W & \xrightarrow{f'} & Y \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ Z & \xrightarrow[f]{} & X \end{array}$$

be a cartesian transverse square in $\mathbf{Sm}/k$, with g projective. Then

$$f^*g_* = g'_*f'^*.$$

Examples

- *singular cohomology:* $(k \subset \mathbb{C})$, $X \mapsto H^*_{sing}(X(\mathbb{C}), \mathbb{Z})$.
- *topological K-theory:* $X \mapsto K^*_{top}(X(\mathbb{C}))$
- *complex cobordism:* $X \mapsto MU^*(X(\mathbb{C}))$.
- *étale cohomology:* $X \mapsto H^*_{\text{ét}}(X, \mathbb{Q}_\ell)$.
- *the Chow ring:* $X \mapsto \mathrm{CH}^*(X)$;
  or *motivic cohomology:* $X \mapsto H^*(X, \mathbb{Z}(*))$
- *algebraic K_0:* $X \mapsto K_0(X)[\beta, \beta^{-1}]$
  or *algebraic K-theory:* $X \mapsto K_*(X)[\beta, \beta^{-1}]$
- *algebraic cobordism:* $X \mapsto MGL^{*,*}(X)$

Chern classes

Once we have f^* and f_*, we have the 1st Chern class of a line bundle $L \to X$:

Let $s : X \to L$ be the zero-section. Define

$$c_1(L) := s^*(s_*(1_X)) \in A^\epsilon(X).$$

If we want to extend to a good theory of A^*-valued Chern classes of vector bundles, we need two additional axioms.

Axioms for oriented cohomology

PB:
Let $E \to X$ be a rank n vector bundle,
$\mathbb{P}(E) \to X$ the projective-space bundle,
$O_E(1) \to \mathbb{P}(E)$ the tautological quotient line bundle.
$\xi := c_1(O_E(1)) \in A^1(\mathbb{P}(E))$.

Then

$A^*(\mathbb{P}(E))$ is a free $A^*(X)$-module with basis $1, \xi, \ldots, \xi^{n-1}$.

EH:
Let $p : V \to X$ be an affine-space bundle. Then

$p^* : A^*(X) \to A^*(V)$ is an isomorphism.

In fact, use Grothendieck's method:

Let $E \to X$ be a vector bundle of rank n. By (PB), there are unique elements $c_i(E) \in A^i(X)$, $i = 0, \ldots, n$, with $c_0(E) = 1$ and

$$\sum_{i=0}^{n} (-1)^i c_i(E) \xi^{n-i} = 0 \in A^*(\mathbb{P}(E)),$$

$\xi := c_1(O_E(1))$.

This works because the *splitting principle* holds for A^*, so all computations reduce to the case of a direct sum of line bundles.

Example The Whitney product formula holds: $c(E) = c(E')c(E'')$ for

$$0 \to E' \to E \to E'' \to 0$$

exact, $c(E) := \sum_i c_i(E)$.

Outline:

- Recall the main points of complex cobordism
- Describe the setting of “oriented cohomology over a field k”
- Describe the fundamental properties of algebraic cobordism
- Sketch the construction of algebraic cobordism

Complex cobordism

The data D1, D2 and axioms A1-A3, PB and EV can be interpreted for the topological setting:
One replaces $\mathbf{Sm}/k$ with the category of differentiable manifolds
One has push-forward maps for "complex oriented proper maps".

Quillen showed that complex cobordism, MU^*, is the universal such theory.

Quillen's viewpoint

Quillen (following Thom) gave a "geometric" description of $MU^*(X)$ (for X a C^∞ manifold):

$$MU^n(X) = \{(f : Y \to X, \theta)\}/\sim$$

1. $f : Y \to X$ is a proper C^∞ map

2. $n = \dim X - \dim Y := \mathrm{codim} f$.

3. θ is a "$\mathbb{C}$-orientation of the virtual normal bundle of f":

A factorization of f through a closed immersion $i : Y \to \mathbb{C}^N \times X$ plus a complex structure on the normal bundle N_i of Y in $\mathbb{C}^N \times X$ (or on $N_i \oplus \mathbb{R}$ if n is odd).

$\sim$ is the *cobordism relation*:

For $(F : Y \to X \times \mathbb{R}, \Theta)$, transverse to $X \times \{0, 1\}$, identify the fibers over 0 and 1:

$$(F_0 : Y_0 \to X, \Theta_0) \sim (F_1 : Y_1 \to X, \Theta_1).$$

$Y_0 := F^{-1}(X \times 0)$, $Y_1 := F^{-1}(X \times 1)$.

Properties of MU^*

• $X \mapsto MU^*(X)$ is a contravariant ring-valued functor:
For $g : X' \to X$ and $(f : Y \to X, \theta) \in MU^n(X)$,

$$g^*(f) = X' \times_X Y \to X'$$

after moving f to make f and g transverse.

• For $(g : X \to X', \theta)$ a proper $\mathbb{C}$-oriented map, we have

$$g_* : MU^*(X) \to MU^{*+2d}(X');$$
$$(f : Y \to X) \mapsto (gf : Y \to X')$$

with $d = \text{codim}_{\mathbb{C}} f$.

Definition Let $L \to X$ be a $\mathbb{C}$-line bundle with 0-section $s : X \to L$. The first Chern class of L is:

$$c_1(L) := s^* s_*(1_X) \in MU^2(X).$$

These satisfy:

- $(gg')_* = g_* g'_*$, $\mathrm{id}_* = \mathrm{id}$.
- projection formula.
- Compatibility of g_* and f^* in transverse cartesian squares.
- Projective bundle formula: $E \to X$ a rank $r+1$ vector bundle, $\xi := c_1(\mathcal{O}(1)) \in MU^2(\mathbb{P}(E))$.

$$MU^*(\mathbb{P}(E)) = \oplus_{i=0}^{r} MU^{*-2i}(X) \cdot \xi^i.$$

- Homotopy invariance:

$$MU^*(X) = MU^*(X \times \mathbb{R}).$$

Definition A cohomology theory $X \mapsto E^*(X)$ with push-forward maps g_* for $\mathbb{C}$-oriented g which satisfy the above properties is called $\mathbb{C}$-*oriented*.

Theorem (Quillen) *MU^* is the universal $\mathbb{C}$-oriented cohomology theory*

Proof. Given a $\mathbb{C}$-oriented theory E^*, let $1_Y \in E^0(Y)$ be the unit. Map

$$(f : Y \to X, \theta) \in MU^n(X) \to f_*(1_Y) \in E^n(X).$$

The formal group law

E: a $\mathbb{C}$-oriented cohomology theory. The projective bundle formula yields:

$$E^*(\mathbb{CP}^\infty) := \varprojlim_n E^*(\mathbb{CP}^n) = E^*(pt)[[u]]$$

where the variable u maps to $c_1(\mathcal{O}(1))$ at each finite level. Similarly

$$E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) = E^*(pt)[[u,v]].$$

where

$$u = c_1(\mathcal{O}(1,0)),\ v = c_1(\mathcal{O}(0,1))$$

$$\mathcal{O}(1,0) = p_1^*\mathcal{O}(1);\ \mathcal{O}(0,1) = p_2^*\mathcal{O}(1).$$

Let $\mathcal{O}(1,1) = p_1^*\mathcal{O}(1) \otimes p_2^*\mathcal{O}(1) = \mathcal{O}(1,0) \otimes \mathcal{O}(0,1)$. There is a unique

$$F_E(u,v) \in E^*(pt)[[u,v]]$$

with

$$F_E(c_1(\mathcal{O}(1,0)), c_1(\mathcal{O}(0,1))) = c_1(\mathcal{O}(1,1))$$

in $E^2(\mathbb{CP}^\infty \times \mathbb{CP}^\infty)$.

Since $\mathcal{O}(1)$ is the universal $\mathbb{C}$-line bundle, we have

$$F_E(c_1(L), c_1(M)) = c_1(L \otimes M) \in E^2(X)$$

for *any* two line bundles $L, M \to X$.

Properties of $F_E(u,v)$

- $1 \otimes L \cong L \cong L \otimes 1$
  $\Rightarrow F_E(0,u) = u = F_E(u,0)$.

- $L \otimes M \cong M \otimes L \Rightarrow F_E(u,v) = F_E(v,u)$.

- $(L \otimes M) \otimes N \cong L \otimes (M \otimes N)$
  $\Rightarrow F_E(F_E(u,v),w) = F_E(u, F_E(v,w))$.

so $F_E(u,v)$ defines a *formal group* (commutative, rank 1) over $E^*(pt)$.

Note: c_1 is not necessarily additive!

The Lazard ring and Quillen's theorem

There is a universal formal group law $F_{\mathbb{L}}$, with coefficient ring the *Lazard ring* $\mathbb{L}$. Let

$$\phi_E : \mathbb{L} \to E^*(pt);\ \phi_E(F_{\mathbb{L}}) = F_E.$$

be the ring homomorphism classifying F_E.

Theorem (Quillen) $\phi_{MU} : \mathbb{L} \to MU^*(pt)$ *is an isomorphism, i.e., F_{MU} is the universal group law.*

Note. Let $\phi : \mathbb{L} = MU^*(pt) \to R$ classify a group law F_R over R. If ϕ satisfies the "Landweber exactness" conditions, form the $\mathbb{C}$-oriented spectrum $MU \wedge_\phi R$, with

$$(MU \wedge_\phi R)(X) = MU^*(X) \otimes_{MU^*(pt)} R$$

and formal group law F_R.

Examples

1. $H^*(-,\mathbb{Z})$ has the additive formal group law $(u+v,\mathbb{Z})$.

2. K^*_{top} has the multiplicative formal group law $(u+v-\beta uv,\mathbb{Z}[\beta,\beta^{-1}])$, $\beta =$ Bott element in $K^{-2}_{top}(pt)$.

Theorem (Conner-Floyd)
$K^*_{top} = MU \wedge_\times \mathbb{Z}[\beta,\beta^{-1}]$; *$K^*_{top}$ is the universal multiplicative oriented cohomology theory.*

The construction of the Lazard ring

Take the polynomial ring $\mathbb{Z}[A_{ij}]$ in variables A_{ij}, $1 \leq i, j$. Let $F = u + v + \sum_{i,j \geq 1} A_{ij} u^i v^j$. Then

$$\mathbb{L} = \mathbb{Z}[A_{ij}]/\sim$$

where $\sim$ is the ideal of relations on the coefficients of F forced by

1. $F(u, v) = F(v, u)$
2. $F(F(u, v), w) = F(u, F(v, w))$

The universal group law $F_{\mathbb{L}} \in \mathbb{L}[[u, v]]$ is the image of F. Grade $\mathbb{L}$ by

$$\deg A_{ij} := 1 - i - j.$$

Oriented cohomology over k

We now turn to the algebraic theory.

Definition k a field. An *oriented cohomology theory A over k* is a functor $A^* : \mathbf{Sm}/k^{\mathrm{op}} \to$ **GrRing** together with push-forward maps

$$g_* : A^*(Y) \to A^{*+d}(X)$$

for each projective morphism $g : Y \to X$, $d = \mathrm{codim} g$, satisfying the axioms A1-3, PB and EV:

- functoriality of push-forward,
- projection formula,
- compatibility of f^* and g_* in transverse cartesian squares,
- projective bundle formula,
- homotopy.

Examples

1. $X \mapsto \mathrm{CH}^*(X)$.

2. $X \mapsto K_0^{alg}(X)[\beta, \beta^{-1}]$, $\deg \beta = -1$.

3. For $k \subset \mathbb{C}$, E a (topological) oriented theory: $X \mapsto E^{2*}(X(\mathbb{C}))$

4. $X \mapsto MGL^{2*,*}(X)$.

Note. Let $\mathcal{E}$ be a $\mathbb{P}^1$-spectrum. The cohomology theory $\mathcal{E}^{*,*}$ has good push-forward maps for projective g exactly when $\mathcal{E}$ is an MGL-module. In this case

$$X \mapsto \mathcal{E}^{2*,*}(X)$$

is an oriented cohomology theory over k.

The formal group law

Just as in the topological case, each oriented cohomology theory A over k has a formal group law $F_A(u,v) \in A^*(\mathrm{Spec}\, k)[[u,v]]$ with

$$F_A(c_1^A(L), c_1^A(M)) = c_1^A(L \otimes M)$$

for each pair $L, M \to X$ of algebraic line bundles on some $X \in \mathbf{Sm}/k$. Let

$$\phi_A : \mathbb{L} \to A^*(k)$$

be the classifying map.

Examples

1. $F_{\mathrm{CH}}(u,v) = u + v$.

2. $F_{K_0[\beta,\beta^{-1}]}(u,v) = u + v - \beta uv$.

Algebraic cobordism

The main theorem

Theorem (L.-Morel) *Let k be a field of characteristic zero. There is a universal oriented cohomology theory Ω over k, called* algebraic cobordism*. Ω has the additional properties:*

1. Formal group law. *The classifying map $\phi_\Omega : \mathbb{L} \to \Omega^*(k)$ is an isomorphism, so F_Ω is the universal formal group law.*

2. Localization *Let $i : Z \to X$ be a closed codimension d embedding of smooth varieties with complement $j : U \to X$. The sequence*

$$\Omega^{*-d}(Z) \xrightarrow{i_*} \Omega^*(X) \xrightarrow{j^*} \Omega^*(U) \to 0$$

is exact.

For an arbitrary formal group law $\phi : \mathbb{L} = \Omega^*(k) \to R$, $F_R := \phi(F_{\mathbb{L}})$, we have the oriented theory

$$X \mapsto \Omega^*(X) \otimes_{\Omega^*(k)} R := \Omega^*(X)_\phi.$$

$\Omega^*(X)_\phi$ is universal for theories whose group law factors through ϕ.

The Conner-Floyd theorem extends to the algebraic setting:

Theorem *The canonical map*

$$\Omega^*_\times \to K_0^{alg}[\beta, \beta^{-1}]$$

is an isomorphism, i.e., $K_0^{alg}[\beta, \beta^{-1}]$ *is the universal multiplicative theory over* k*. Here*

$$\Omega^*_\times := \Omega^* \otimes_{\mathbb{L}} \mathbb{Z}[\beta, \beta^{-1}].$$

Not only this but there is an additive version as well:

Theorem *The canonical map*

$$\Omega_+^* \to \mathrm{CH}^*$$

is an isomorphism, i.e., CH^* *is the universal additive theory over* k. *Here*

$$\Omega_+^* := \Omega^* \otimes_{\mathbb{L}} \mathbb{Z}.$$

Remark

Define "connective algebraic K_0", $k_0^{alg} := \Omega^* \otimes_{\mathbb{L}} \mathbb{Z}[\beta]$.

$$k_0^{alg}/\beta = \mathrm{CH}^*$$
$$k_0^{alg}[\beta^{-1}] = K_0^{alg}[\beta, \beta^{-1}].$$

This realizes $K_0^{alg}[\beta, \beta^{-1}]$ as a deformation of CH^*.

Relation with motivic homotopy theory

$$\mathrm{CH}^n(X) \cong H^{2n}(X, \mathbb{Z}(n)) = H^{2n,n}(X)$$

$$K_0(X) \cong K^{2n,n}(X)$$

The universality of Ω^* gives a natural map

$$\nu_n(X) : \Omega^n(X) \to MGL^{2n,n}(X).$$

Conjecture $\Omega^n(X) \cong MGL^{2n,n}(X)$ *for all* n, *all* $X \in \mathbf{Sm}/k$.

Note. (1) $\nu_n(X)$ is surjective, and an isomorphism after $\otimes\mathbb{Q}$.

(2) $\nu_n(k)$ is an isomorphism.

The construction of algebraic cobordism

The idea

We build $\Omega^*(X)$ following roughly Quillen's basic idea, defining generators and relations. The original description of Levine-Morel was rather complicated, but necessary for proving all the main properties of Ω^*. Following a suggestion of Pandharipande, we now have a very simple presentation, with the same kind of generators as for complex cobordism. The relations are also similar, but need to allow for "double-point degenerations".

The simplified presention requires the base-field k to have characteristic zero.

Generators

$\mathbf{Sch}_k$:= finite type k-schemes.

Definition Take $X \in \mathbf{Sch}_k$.

1. $\mathcal{M}(X)$:= the set of isomorphism classes of projective morphisms $f : Y \to X$, with $Y \in \mathbf{Sm}/k$.

2. Grade $\mathcal{M}(X)$:

$$\mathcal{M}_n(X) := \{f : Y \to X \in \mathcal{M}(X) \mid n = \mathrm{dim}_k Y\}.$$

3. $\mathcal{M}_*(X)$ is a graded monoid under $\coprod$; let $\mathcal{M}_*^+(X)$ be the group completion.

Explicitly: $\mathcal{M}_n^+(X)$ is the free abelian group on $f : Y \to X$ in $\mathcal{M}(X)$ with Y irreducible and $\mathrm{dim}_k Y = n$.

Double point degenerations

Definition Let C be a smooth curve, $c \in C$ a k-point. A morphism $\pi : Y \to C$ in $\mathbf{Sm}/k$ is a *double-point degeneration at c* if

$$\pi^{-1}(c) = S \cup T$$

with

1. S and T smooth,
2. S and T intersecting transversely on Y.

Shortly speaking: $\pi^{-1}(c)$ is a reduced strict normal crossing divisor without triple points.

The codimension two smooth subscheme $D := S \cap T$ is called the *double-point locus* of the degeneration.

The degeneration bundle

Let $\pi : Y \to C$ be a double-point degeneration at $c \in C(k)$, with

$$\pi^{-1}(c) = S \cup T; \; D := S \cap T.$$

Set $N_{D/S}$:= the normal bundle of D in S.

Set: $$\mathbb{P}(\pi, c) := \mathbb{P}(\mathcal{O}_D \oplus N_{D/S}),$$

a $\mathbb{P}^1$-bundle over D, called the *degeneration bundle*.

$\mathbb{P}(\pi, c)$ is well-defined:

Let $N_{D/T} :=$ the normal bundle of D in T.

$$N_{D/S} = \mathcal{O}_Y(T) \otimes \mathcal{O}_D; \; N_{D/T} = \mathcal{O}_Y(S) \otimes \mathcal{O}_D.$$

Since $\mathcal{O}_Y(S+T) \otimes \mathcal{O}_D \cong \mathcal{O}_D$,

$$N_{D/S} \cong N_{D/T}^{-1}.$$

So the definition of $\mathbb{P}(\pi, c)$ does not depend on the choice of S or T:

$$\mathbb{P}(\pi, c) = \mathbb{P}_D(\mathcal{O}_D \oplus N_{D/S}) = \mathbb{P}_D(\mathcal{O}_D \oplus N_{D/T}).$$

Double-point cobordisms

Definition Let $f : Y \to X \times \mathbb{P}^1$ be a projective morphism with $Y \in \mathbf{Sm}/k$. Call f a *double-point cobordism* if

1. $p_1 \circ f : Y \to \mathbb{P}^1$ is a double-point degeneration at $0 \in \mathbb{P}^1$.

2. $(p_1 \circ f)^{-1}(1)$ is smooth.

Double-point relations

Let $f : Y \to X \times \mathbb{P}^1$ be a double-point cobordism. Suppose $Y \to \mathbb{P}^1$ has relative dimension n. Write

$(p_1 \circ f)^{-1}(0) = Y_0 = S \cup T$, $(p_1 \circ f)^{-1}(1) = Y_1$,

giving elements

$$[S \to X], [T \to X], [\mathbb{P}(p_1 \circ f, 0) \to X], [Y_1 \to X]$$

of $\mathcal{M}_n(X)$. The element

$$[Y_1 \to X] - [S \to X] - [T \to X] + [\mathbb{P}(p_1 \circ f, 0) \to X]$$

is the *double-point relation* associated to the double-point cobordism f.

The definition of algebraic cobordism

Definition For $X \in \mathbf{Sch}_k$, $\Omega_*(X)$ is the quotient of $\mathcal{M}_*^+(X)$ by the subgroup of all double-point relations associated to double-point cobordisms $f : Y \to X \times \mathbb{P}^1$:

$$\Omega_*(X) := \mathcal{M}_*^+(X)/[Y_1 \to X] \sim [S \to X] + [T \to X] - [\mathbb{P}(p_1 \circ f, 0) \to X]$$

for all double-point cobordisms $f : Y \to X \times \mathbb{P}^1$ with $Y_0 = S \cup T$.

Elementary structures

• For $g : X \to X'$ projective, we have

$$g_* : \mathcal{M}_*(X) \to \mathcal{M}_*(X')$$
$$g_*(f : Y \to X) := (g \circ f : Y \to X')$$

• For $g : X' \to X$ smooth of dimension d, we have

$$g^* : \mathcal{M}_*(X) \to \mathcal{M}_{*+d}(X')$$
$$g^*(f : Y \to X) := (p_2 : Y \times_X X' \to X')$$

• For $L \to X$ a globally generated line bundle, we have the *1st Chern class operator*

$$\tilde{c}_1(L) : \Omega_*(X) \to \Omega_{*-1}(X)$$
$$\tilde{c}_1(L)(f : Y \to X) := (f \circ i_D : D \to X)$$

$D :=$ the divisor of a general section of f^*L.

Concluding remarks

1. These structures extend to give the desired properties of $\Omega^*(X) := \Omega_{\dim X - *}(X)$.

2. Smooth degenerations yield a "naive cobordism relation":

Let $F : Y \to X \times \mathbb{P}^1$ be a projective morphism with Y smooth and with F transverse to $X \times \{0, 1\}$. Then in $\Omega_*(X)$, we have

$$[F_0 : Y_0 \to X \times 0 = X] = [F_1 : Y_1 \to X \times 1 = X].$$

These relations do NOT suffice to define Ω_*:
For C a smooth projective curve of genus g, $[C] = (1 - g)[\mathbb{P}^1] \in \Omega_1(k)$, but this relation is impossible to realize using only naive cobordisms.

An application: Donaldson-Thomas theory
(with R. Pandharipande)

X: a smooth projective threefold over $\mathbb{C}$
$Hilb(X, n) :=$ the Hilbert scheme of "n-points" in X
$I_0(X, n) \in \mathsf{CH}_0(Hilb(X, n))$ the "virtual fundamental class
(Maulik-Nekrasov-Okounkov-Pandharipande, Thomas).

$$Z(X, q) := 1 + \sum_{n \geq 1} \deg I_0(X, n) \cdot q^n$$

Conjecture (MNOP)

$$Z(X, q) = M(-q)^{\deg c_3(T_X \otimes K_X)}$$

where $M(q) := \prod_n (1 - q^n)^{-n}$ *is the MacMahon function, i.e., the generating function of 3-dimensional partitions.*

The conjecture is related to $\Omega^*(\mathbb{C})$ by the

Proposition (DT double-point relation) *Let* $\pi : Y \to C$ *be a projective double-point degeneration over* $0 \in C$*, and suppose that* $Y_c := \pi^{-1}(c)$ *is smooth for some point* $c \in C$*. Write*

$$\pi^{-1}(0) = S \cup T.$$

Then

$$Z(Y_c, q) = Z(S, q) Z(T, q) Z(\mathbb{P}(\pi, 0), q)^{-1}.$$

This is proven by MNOP.

To prove the conjecture:
We'll see later that $X \mapsto \deg c_3(T_X \otimes K_X)$ descends to a homomorphism $c_{DT} : \Omega^{-3}(\mathbb{C}) \to \mathbb{Z}$.

Thus, sending X to $M(-q)^{\deg c_3(T_X \otimes K_X)}$ descends to a homomorphism

$$M(-q)^{c_{DT}(-)} : \Omega^{-3}(\mathbb{C}) \to (1 + q\mathbb{Z}[[q]])^{\times}.$$

By the DT double-point relation, sending X to $Z(X, q)$ descends to a homomorphism

$$Z(-, q) : \Omega^{-3}(\mathbb{C}) \to (1 + q\mathbb{Z}[[q]])^{\times}.$$

But $\Omega^{-3}(\mathbb{C})_{\mathbb{Q}} = \mathbb{L}_{\mathbb{Q}}^{-3}$ has $\mathbb{Q}$-basis $[(\mathbb{P}^1)^3]$, $[\mathbb{P}^1 \times \mathbb{P}^2]$, $[\mathbb{P}^3]$, so it suffices to check the conjecture for these three varieties.

This was done in work of MNOP.

Advertisement

Lecture 2: We'll show how to use Ω^* to understand Riemann-Roch theorems, and how construct the Voevodsky/Brosnan Steenrod operations on CH^*/p. We'll describe the generalized degree formula, how to get lot's of interesting degree formulas from the generalized degree formula and give applications to quadratic forms and other varieties.

Lecture 3:
Part A is on the extension to singular varieties, with applications to Riemann-Roch for singular varieties. We'll also discuss the problem of fundamental classes, and how this relates to the problem of constructing a cobordism-valued Gromov-Witten theory

Part B is on the category of cobordism motives, its relation to Chow motives, and applications to the computation of the algebraic cobordism of Pfister quadrics, due to Vishik-Yagita.

Thank you!