Algebraic Cobordism
Lecture 1: Complex cobordism and algebraic cobordism

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Prelude: From homotopy theory to $\mathbb{A}^1$-homotopy theory

A basic object in homotopy theory is a *generalized cohomology theory* E^*

$$X \mapsto E^*(X)$$

A generalized cohomology theory E^* has a unique representation as an object E (a *spectrum*) in the *stable homotopy category* $\mathcal{SH}$.

$\mathcal{SH}$ can be thought of as a linearization of the category of pointed topological spaces Sp_*:

$$\Sigma^\infty : Sp_* \to \mathcal{SH}$$

which inverts the suspension operator Σ, and

$$E^n(X) = \mathrm{Hom}_{\mathcal{SH}}(\Sigma^\infty X_+, \Sigma^n E);\ \ n \in \mathbb{Z}.$$

Examples

$\mathcal{SH}$ is the homotopy category of *spectra*.

- Singular cohomology $H^*(-, A)$ is represented by the Eilenberg-Maclane spectrum HA
- Topological K-theory K^*_{top} is represented by the K-theory spectrum K_{top}
- Complex cobordism MU^* is represented by the Thom spectrum MU.

$\mathbb{A}^1$-homotopy theory

Morel and Voevodsky have defined a refinement of $\mathcal{SH}$ in the setting of algebraic geometry.

k: a field. $\mathbf{Sm}/k$: smooth varieties over k.

There is a sequence of functors:

$$\mathbf{Sm}/k \to Sp(k)_* \xrightarrow{\Sigma^\infty_{\mathbb{P}^1}} \mathcal{SH}(k).$$

$Sp(k)_* =$ pointed spaces over k,
$\mathcal{SH}(k) =$ the homotopy category of $\mathbb{P}^1$-spectra, localized by $\mathbb{A}^1$-homotopy.

Two circles
In Sp_*, the circle S^1 is fundamental: $\Sigma X := S^1 \wedge X$.

In $Sp(k)_* \supset Sp_*$, there are *two* S^1's:

The usual circle $S^{1,0} := S^1$
and
The Tate circle $S^{1,1} := (\mathbb{A}^1_k \setminus \{0\}, \{1\})$.

Set $S^{p,q} := (S^{1,1})^{\wedge q} \wedge (S^{1,0})^{\wedge p-q}$,
$\Sigma^{p,q}(X) := S^{p,q} \wedge X$.

Note. 1. $(\mathbb{P}^1, \infty) \cong S^{1,0} \wedge S^{1,1} = S^{2,1}$.

2. $Sp(k)_* \xrightarrow{\Sigma^\infty_{\mathbb{P}^1}} \mathcal{SH}(k)$ inverts all the operators $\Sigma^{p,q}$.

Cohomology for varieties over k

Because of the two circles, $\mathcal{SH}(k)$ represents *bi-graded* cohomology theories on $\mathbf{Sm}/k$: For $\mathcal{E} \in \mathcal{SH}(k)$, have

$$X \mapsto \mathcal{E}^{p,q}(X) := [\Sigma^\infty_{\mathbb{P}^1} X_+, \Sigma^{p,q}\mathcal{E}];\ \ p, q \in \mathbb{Z}.$$

- Motivic cohomology $H^{*,*}(-, A)$ is represented by the Eilenberg-Maclane spectrum $\mathcal{H}A$
- Algebraic K-theory $K^{*,*}_{alg}$ is represented by the K-theory spectrum $\mathcal{K}$
- Algebraic cobordism $MGL^{*,*}$ is represented by the Thom spectrum MGL.

Remarks

1. Bott periodicity yields $K_n^{alg}(X) = K_{alg}^{n+2m,m}(X)$ for all m.

2. $K_{alg}^{2*,*}(X) = K_0^{alg}(X)[\beta, \beta^{-1}]$, $\deg \beta = -1$

3. The Chow ring $\mathrm{CH}^*(X)$ of cycles modulo rational equivalence is the same as $H^{2*,*}(X, \mathbb{Z})$.

Main goal

To give an algebro-geometric description of the "classical part" $MGL^{2*,*}$ of algebraic cobordism.

Outline:

- Recall the main points of complex cobordism
- Describe the setting of "oriented cohomology over a field k"
- Describe the fundamental properties and main applications of algebraic cobordism
- Sketch the construction of algebraic cobordism

Complex cobordism

Quillen's viewpoint

Quillen (following Thom) gave a "geometric" description of $MU^*(X)$ (for X a C^∞ manifold):

$$MU^n(X) = \{(f : Y \to X, \theta)\}/\sim$$

1. $f : Y \to X$ is a proper C^∞ map

2. $n = \dim X - \dim Y :=$ codim$f$.

3. θ is a "$\mathbb{C}$-orientation of the virtual normal bundle of f" :

a factorization of f through a closed immersion $i : Y \to \mathbb{C}^N \times X$ plus a complex structure on the normal bundle N_i of Y in $\mathbb{C}^N \times X$ (or on $N_i \oplus \mathbb{R}$ if n is odd).

$\sim$ is the *cobordism relation*:

For $(F : Y \to X \times \mathbb{R}, \Theta)$, transverse to $X \times \{0, 1\}$, identify the fibers over 0 and 1:

$$(F_0 : Y_0 \to X, \Theta_0) \sim (F_1 : Y_1 \to X, \Theta_1).$$

$Y_0 := F^{-1}(X \times 0)$, $Y_1 := F^{-1}(X \times 1)$.

To identify $MU^n(X) \cong \{(f : Y \to X, \theta)\}/\sim$:

$$x \in MU^n(X) \leftrightarrow x : (X \times S^{2N-n}, X \times \infty) \to (\mathrm{Th}(U_N), *)$$
$$\to Y := x^{-1}(\text{0-section}) \to X$$

where we make Y a manifold by deforming x to make the intersection with the 0-section transverse.

To reverse (n even):

$$(Y \xrightarrow{i} 1_{\mathbb{C}}^N \to X) \to f : 1_{\mathbb{C}}^N \to \mathrm{Th}(U_{N+n/2}) \text{ classifying } Y \xrightarrow{0} N_i$$
$$\to \Sigma^{2N} X = \mathrm{Th}(1_{\mathbb{C}}^N) \to MU_{2N+n}$$

Properties of MU^*

- $X \mapsto MU^*(X)$ is a contravariant ring-valued functor:
  For $g : X' \to X$ and $(f : Y \to X, \theta) \in MU^n(X)$,
  $$g^*(f) = X' \times_X Y \to X'$$
  after moving f to make f and g transverse.

- For $(g : X \to X', \theta)$ a proper $\mathbb{C}$-oriented map, we have
  $$g_* : MU^*(X) \to MU^{*+n}(X'); \quad (f : Y \to X) \mapsto (gf : Y \to X')$$
  with $n = \text{codim} f$.

Definition Let $L \to X$ be a $\mathbb{C}$-line bundle with 0-section $s : X \to L$. The first Chern class of L is:
$$c_1(L) := s^* s_*(1_X) \in MU^2(X).$$

These satisfy:

- $(gg')_* = g_* g'_*$, $\mathrm{id}_* = \mathrm{id}$.

- Compatibility of g_* and f^* in transverse cartesian squares.

- Projective bundle formula: $E \to X$ a rank $r+1$ vector bundle, $\xi := c_1(\mathcal{O}(1)) \in MU^2(\mathbb{P}(E))$. Then

$$MU^*(\mathbb{P}(E)) = \oplus_{i=0}^r MU^{*-2i}(X) \cdot \xi^i.$$

- Homotopy invariance: $MU^*(X) = MU^*(X \times \mathbb{R})$.

Definition A cohomology theory $X \mapsto E^*(X)$ with push-forward maps g_* for $\mathbb{C}$-oriented g which satisfy the above properties is called $\mathbb{C}$-*oriented*.

Theorem 1 (Quillen) *MU^* is the universal $\mathbb{C}$-oriented cohomology theory*

Proof. Given a $\mathbb{C}$-oriented theory E^*, let $1_Y \in E^0(Y)$ be the unit. Map

$$(f : Y \to X, \theta) \in MU^n(X) \to f_*(1_Y) \in E^n(X).$$

□

The formal group law

E: a $\mathbb{C}$-oriented cohomology theory. The projective bundle formula yields:

$$E^*(\mathbb{CP}^\infty) := \varprojlim_n E^*(\mathbb{CP}^n) = E^*(pt)[[u]]$$

where the variable u maps to $c_1(\mathcal{O}(1))$ at each finite level. Similarly

$$E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) = E^*(pt)[[c_1(\mathcal{O}(1,0)), c_1(\mathcal{O}(0,1))]].$$

where

$$\mathcal{O}(1,0) = p_1^*\mathcal{O}(1); \ \mathcal{O}(0,1) = p_2^*\mathcal{O}(1).$$

Let $\mathcal{O}(1,1) = p_1^*\mathcal{O}(1) \otimes p_2^*\mathcal{O}(1) = \mathcal{O}(1,0) \otimes \mathcal{O}(0,1)$. There is a unique

$$F_E(u,v) \in E^*(pt)[[u,v]]$$

with

$$F_E(c_1(\mathcal{O}(1,0)), c_1(\mathcal{O}(0,1))) = c_1(\mathcal{O}(1,1)) \in E^2(\mathbb{CP}^\infty \times \mathbb{CP}^\infty).$$

Since $\mathcal{O}(1)$ is the universal $\mathbb{C}$-line bundle, we have

$$F_E(c_1(L), c_1(M)) = c_1(L \otimes M) \in E^2(X)$$

for *any* two line bundles $L, M \to X$.

Properties of $F_E(u, v)$

- $1 \otimes L \cong L \cong L \otimes 1 \Rightarrow F_E(0, u) = u = F_E(u, 0)$.

- $L \otimes M \cong M \otimes L \Rightarrow F_E(u, v) = F_E(v, u)$.

- $(L \otimes M) \otimes N \cong L \otimes (M \otimes N) \Rightarrow F_E(F_E(u, v), w) = F_E(u, F_E(v, w))$.

so $F_E(u, v)$ defines a *formal group* (commutative, rank 1) over $E^*(pt)$.

Note: c_1 is not necessarily additive!

The Lazard ring and Quillen's theorem

There is a universal formal group law $F_{\mathbb{L}}$, with coefficient ring the *Lazard ring* $\mathbb{L}$. Let

$$\phi_E : \mathbb{L} \to E^*(pt);\ \phi(F_{\mathbb{L}}) = F_E.$$

be the ring homomorphism classifying F_E.

Theorem 2 (Quillen) $\phi_{MU} : \mathbb{L} \to MU^*(pt)$ *is an isomorphism, i.e., F_{MU} is the universal group law.*

Note. Let $\phi : \mathbb{L} = MU^*(pt) \to R$ classify a group law F_R over R. If ϕ satisfies the "Landweber exactness" conditions, form the $\mathbb{C}$-oriented spectrum $MU \wedge_\phi R$, with

$$(MU \wedge_\phi R)(X) = MU^*(X) \otimes_{MU^*(pt)} R$$

and formal group law F_R.

Examples

1. $H^*(-,\mathbb{Z})$ has the additive formal group law $(u+v,\mathbb{Z})$.

2. K^*_{top} has the multiplicative formal group law $(u+v-\beta uv, \mathbb{Z}[\beta,\beta^{-1}])$, $\beta =$ Bott element in $K^{-2}_{top}(pt)$.

Theorem 3 (Conner-Floyd) $K^*_{top} = MU \wedge_{\times} \mathbb{Z}[\beta,\beta^{-1}]$; K^*_{top} *is the universal multiplicative oriented cohomology theory.*

Oriented cohomology over k

We now turn to the algebraic theory.

Definition k a field. An *oriented cohomology theory A over k* is a functor

$$A^* : \mathbf{Sm}/k^{\text{op}} \to \textbf{GrRing}$$

together with pushforward maps

$$g_* : A^*(Y) \to A^{*+n}(X)$$

for each projective morphism $g : Y \to X$; $n = \text{codim}g$, satisfying the algebraic versions of the properties of MU:

- functoriality of push-forward,
- compatibility of f^* and g_* in transverse cartesian squares,
- projective bundle formula,
- homotopy.

Remarks

1. For $L \to X$ a line bundle with 0-section $s : X \to L$,

$$c_1(L) := s^* s_*(1_X)).$$

2. The required homotopy property is

$$A^*(X) = A^*(V)$$

for $V \to X$ an $\mathbb{A}^n$-bundle.

3. There is no “Mayer-Vietoris” property required.

Examples

1. $X \mapsto \mathrm{CH}^*(X)$.

2. $X \mapsto K_0^{alg}(X)[\beta, \beta^{-1}]$, $\deg \beta = -1$.

3. For $\sigma : k \to \mathbb{C}$, E a (topological) oriented theory,

$$X \mapsto E^{2*}(X_\sigma(\mathbb{C})).$$

4. $X \mapsto MGL^{2*,*}(X)$. *Note.* Let $\mathcal{E}$ be a $\mathbb{P}^1$-spectrum. The cohomology theory $\mathcal{E}^{*,*}$ has good push-forward maps for projective g exactly when $\mathcal{E}$ is an MGL-module. In this case

$$X \mapsto \mathcal{E}^{2*,*}(X)$$

is an oriented cohomology theory over k.

The formal group law

Just as in the topological case, each oriented cohomology theory A over k has a formal group law $F_A(u,v) \in A^*(\mathrm{Spec}\, k)[[u,v]]$ with

$$F_A(c_1^A(L), c_1^A(M)) = c_1^A(L \otimes M)$$

for each pair $L, M \to X$ of algebraic line bundles on some $X \in \mathbf{Sm}/k$. Let

$$\phi_A : \mathbb{L} \to A^*(k)$$

be the classifying map.

Examples

1. $F_{\mathrm{CH}}(u,v) = u + v$.

2. $F_{K_0[\beta,\beta^{-1}]}(u,v) = u + v - \beta uv$.

Algebraic cobordism

The main theorem

Theorem 4 (L.-Morel) *Let k be a field of characteristic zero. There is a universal oriented cohomology theory Ω over k, called* algebraic cobordism*. Ω has the additional properties:*

1. Formal group law. *The classifying map $\phi_\Omega : \mathbb{L} \to \Omega^*(k)$ is an isomorphism, so F_Ω is the universal formal group law.*

2. Localization *Let $i : Z \to X$ be a closed codimension d embedding of smooth varieties with complement $j : U \to X$. The sequence*

$$\Omega^{*-d}(Z) \xrightarrow{i_*} \Omega^*(X) \xrightarrow{j^*} \Omega^*(U) \to 0$$

is exact.

For an arbitrary formal group law $\phi : \mathbb{L} = \Omega^*(k) \to R$, $F_R := \phi(F_{\mathbb{L}})$, we have the oriented theory

$$X \mapsto \Omega^*(X) \otimes_{\Omega^*(k)} R := \Omega^*(X)_\phi.$$

$\Omega^*(X)_\phi$ is universal for theories whose group law factors through ϕ.

The Conner-Floyd theorem extends to the algebraic setting:

Theorem 5 *The canonical map*

$$\Omega^*_\times \to K_0^{alg}[\beta, \beta^{-1}]$$

is an isomorphism, i.e., $K_0^{alg}[\beta, \beta^{-1}]$ *is the universal multiplicative theory over* k*. Here*

$$\Omega^*_\times := \Omega^* \otimes_{\mathbb{L}} \mathbb{Z}[\beta, \beta^{-1}].$$

Not only this but there is an additive version as well:

Theorem 6 *The canonical map*

$$\Omega_+^* \to \mathrm{CH}^*$$

is an isomorphism, i.e., CH^* *is the universal additive theory over* k. *Here*

$$\Omega_+^* := \Omega^* \otimes_{\mathbb{L}} \mathbb{Z}.$$

Remark
Define "connective algebraic K_0", $k_0^{alg} := \Omega^* \otimes_{\mathbb{L}} \mathbb{Z}[\beta]$.

$$k_0^{alg}/\beta = \mathrm{CH}^*$$
$$k_0^{alg}[\beta^{-1}] = K_0^{alg}[\beta, \beta^{-1}].$$

This realizes $K_0^{alg}[\beta, \beta^{-1}]$ as a deformation of CH^*.

Degree formulas

Definition Let X be an irreducible smooth variety over k with generic point η. Define

$$\deg : \Omega^*(X) \to \Omega^*(k)$$

as the composition

$$\begin{array}{ccc} \Omega^*(X) & & \Omega^*(k) \\ {\scriptstyle i_\eta^*}\downarrow & & \uparrow{\scriptstyle \phi_\Omega} \\ \Omega^*(k(\eta)) & \xleftarrow[\phi_{\Omega/k(\eta)}]{} & \mathbb{L} \end{array}$$

Note. Let $f : Y \to X$ be a projective morphism with $\dim X = \dim Y$. Then f has a degree, $\Omega^0(X) = \mathbb{Z}$ and

$$\deg(f_*(1_Y)) = \deg(f).$$

M. Rost first considered *degree formulas*, which express interesting congruences satisfied by characteristic numbers of smooth projective algebraic varieties. These all follow from

Theorem 7 (Generalized degree formula) *Given $\alpha \in \Omega^*(X)$, there are projective maps $f_i : Z_i \to X$ and elements $\alpha_i \in \Omega^*(k)$ such that*

1. *The Z_i are smooth over k and $\dim Z_i < \dim X$.*

2. *$f_i : Z_i \to f_i(Z_i)$ is birational*

3. *$\alpha = \deg(\alpha) \cdot 1_X + \sum_i \alpha_i \cdot f_{i*}(1_{Z_i})$.*

Proof

1. By definition, $j^*\alpha = \deg(\alpha) \cdot 1_U$ for some open $U \xrightarrow{j} X$.

2. Let $\tilde{W} \to W := X \setminus U$ be a resolution of singularities.
$f : \tilde{W} \to X$ the structure morphism.
Since $j^*(\alpha - \deg(\alpha) \cdot 1_X) = 0$,
use localization to find $\alpha_1 \in \Omega^{*-1}(\tilde{W})$ with

$$f_*(\alpha_1) = \alpha - \deg(\alpha) \cdot 1_X.$$

3. Use induction on $\dim X$ to conclude.

One applies the generalized degree formula by taking $\alpha := f_*(1_Y)$ for some morphism $f : Y \to X$ and evaluating "primitive" characteristic classes on both sides of the identity for α to yield actual degree formulas for characteristic numbers.

The construction of algebraic cobordism

The idea

We build $\Omega^*(X)$ following roughly Quillen's basic idea, defining generators: "cobordism cycles" and relations. However, there are some differences:

1. We construct a "bordism theory" Ω_* with projective push-forward and "1st Chern class operators" built in. At the end, we show Ω_* has good pull-back maps, yielding
$$\Omega^*(X) := \Omega_{\mathrm{dim}\, X - *}(X).$$
2. The formal group law doesn't come for free, but needs to be forced as an explicit relation.

Cobordism cycles

$\mathbf{Sch}_k :=$ finite type k-schemes.

Definition Take $X \in \mathbf{Sch}_k$.

1. A *cobordism cycle* is a tuple $(f : Y \to X; L_1, \ldots, L_r)$ with
(a) $Y \in \mathbf{Sm}/k$, irreducible.
(b) $f : Y \to X$ a projective morphism.
(c) $L_1, \ldots, L_r$ line bundles on Y ($r = 0$ is allowed).

Identify two cobordism cycles if they differ by a reordering of the L_j or by an isomorphism $\phi : Y' \to Y$ over X:

$$(f : Y \to X; L_1, \ldots, L_r) \sim (f\phi : Y' \to X; \phi^* L_{\sigma(1)}, \ldots, \phi^* L_{\sigma(r)})$$

2. The group $\mathcal{Z}_n(X)$ is the free abelian group on the cobordism cycles $(f : Y \to X; L_1, \ldots, L_r)$ with $n = \dim Y - r$.

Structures

- For $g : X \to X'$ projective, we have

$$g_* : \mathcal{Z}_*(X) \to \mathcal{Z}_*(X')$$
$$g_*(f : Y \to X; L_1, \ldots, L_r) := (g \circ f : Y \to X'; L_1, \ldots, L_r)$$

- For $g : X' \to X$ smooth of dimension d, we have

$$g^* : \mathcal{Z}_*(X) \to \mathcal{Z}_{*+d}(X')$$
$$g^*(f : Y \to X; L_1, \ldots, L_r) := (p_2 : Y \times_X X' \to X'; p_1^* L_1, \ldots, p_1^* L_r)$$

- For $L \to X$ a line bundle, we have the *1st Chern class operator*

$$\tilde{c}_1(L) : \mathcal{Z}_*(X) \to \mathcal{Z}_{*-1}(X)$$
$$\tilde{c}_1(L)(f : Y \to X; L_1, \ldots, L_r,) := (f : Y \to X; L_1, \ldots, L_r, f^* L)$$

Relations

We impose relations in three steps:

1. Kill all cobordism cycles of negative degree:

$$\dim Y - r < 0 \Rightarrow (f : Y \to X; L_1, \ldots, L_r) = 0.$$

2. Impose a "Gysin isomorphism": If $i : D \to Y$ is smooth divisor on a smooth Y, then

$$(i : D \to Y) = (Y, \mathcal{O}_Y(D)).$$

Denote the resulting quotient of $\mathcal{Z}_*$ by $\underline{\Omega}_*$.

Note. The identities (1) and (2) generate all the relations defining $\underline{\Omega}_*$ by closing up with respect to the operations g_*, g^* and $\tilde{c}_1(L)$.

Thus, these operations pass to $\underline{\Omega}_*$.

The formal group law

For $Y \in \mathbf{Sm}/k$, $1_Y := (\mathrm{id} : Y \to Y) \in \underline{\Omega}_{\dim Y}(Y)$.
The third type of relation is:

3. Let $F_{\mathbb{L}}(u, v) \in \mathbb{L}[[u, v]]$ be the universal formal group law. On $\mathbb{L} \otimes \underline{\Omega}_*$, impose the relations generated by the the identities

$$F_{\mathbb{L}}(\tilde{c}_1(L), \tilde{c}_1(M))(1_Y) = 1 \otimes \tilde{c}_1(L \otimes M)(1_Y)$$

in $\mathbb{L} \otimes \underline{\Omega}_*(Y)$, for each $Y \in \mathbf{Sm}/k$ and each pair of line bundles L, M on Y.

The quotient is denoted Ω_*.

Concluding remarks

1. The Gysin relation (2) implies a "naive cobordism relation":

Let $F : Y \to X \times \mathbb{P}^1$ be a projective morphism with Y smooth and with F transverse to $X \times \{0, 1\}$. Then in $\underline{\Omega}_(X)$, we have

$$(F_0 : Y_0 \to X \times 0 = X) = (F_1 : Y_1 \to X \times 1 = X).$$

2. The formal group law relation (3) seems artificial. But, in the definition of CH^* as cycles modulo rational equivalence, one needs to pass from a subscheme to a cycle, by taking the "associated cycle" of a subscheme. This turns out to be the same as imposing the *additive* formal group law.

3. The formal group law relation is *necessary*: each smooth projective curve C over k has a class $[C] \in \underline{\Omega}_1(k)$. However, even though $[C] = (1 - g(C))[\mathbb{P}^1]$ in the Lazard ring, this relation is not true in $\underline{\Omega}_1(k)$.

4. Even though it looks like we have enlarged $\underline{\Omega}$ greatly by taking $\mathbb{L} \otimes \underline{\Omega}$, $\underline{\Omega}_* \to \Omega_*$ is surjective. In fact, $\Omega_*(X)$ is generated by cobordism cycles $(f : Y \to X)$ *without* any line bundles.