# LECTURES ON QUADRATIC ENUMERATIVE GEOMETRY

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ABSTRACT. We give a survey of some recent developments in quadratic enumerative geometry. This includes discussions of Euler characteristics, Euler classes, Riemann-Hurwitz formulas, Becker-Gottlieb transfers, and computations of Pontryagin and Euler classes in Witt cohomology.

## CONTENTS



## INTRODUCTION

These notes are based on four lectures I gave at the conference "Motivic homotopy theory and enumerative geometry" in May 2018. They give an introduction to an emerging area of study, in which one uses methods and results from motivic homotopy theory to enlarge the scope of enumerative geometry to allow for a more

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general type of invariant than the classical integer ones. We have in mind the invariants in the Grothendieck-Witt ring that arise via Morel's theorem identifying the endomorphism ring of the motivic sphere spectrum with the Grothendieck-Witt ring, or other "motivic" approaches, such as replacing the intersection theory given by the Chow ring with that formed by the Chow-Witt ring, or replacing algebraic $K$-theory with hermitian $K$-theory. These lectures explore some of the techniques that have been developed recently to give such a quadratic enumerative geometry a suitable toolkit to enable an effective exploration of classical enumerative problems, such as curve counting, in the quadratic setting. We have not discussed the many concrete examples that have been worked out already using these and other techniques, such as the quadratic count of lines on a cubic surface by Kass-Wickelgren [9] or our quadratic version of Welschinger invariants [12], leaving a survey of these results to another day.

This is mainly a survey of results already available in the literature or posted on the arXiv, but we hope that in its current form these notes will provide the reader with an accessible and enjoyable introduction to this area.

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## Lecture 1. Euler characteristics

In this first lecture, we discuss one of the most basic numerical invariants, the Euler characteristic, from a number of different points of view. We first look at the Euler characteristic of a dualizable object in a symmetric monoidal category, then specialize to the case of a perfect field $k$ and the suspension spectrum of a smooth (projective in the case of positive characteristic) $k$-scheme, viewed as an object in the motivic stable homotopy category $\mathrm{SH}(k)$. Via Morel's theorem computing the endomorphism ring of the motivic sphere spectrum as the Grothendieck-Witt ring, this gives us the Euler characteristic $\chi(X/k) \in \mathrm{GW}(k)$. We formulate a version of a "motivic Gauß-Bonnet formula" which relates this rather abstractly constructed invariant to more concrete versions, and use this to give "realizations" of the motivic Euler characteristics with respect to several different cohomology theories. Two particular cases, the cohomology of the Milnor-Witt sheaves, and hermitian $K$-theory, give two complementary methods for computing $\chi(X/k)$. The second realizes a suggestion of Serre, that $\chi(X/k)$ should be the quadratic form on $\oplus_{p,q} H^q(X, \Omega^p_{X/k})[p-q]$ (in the derived category) given by cup product and the trace map $H^{\dim X}(X, \Omega^{\dim X}_{X/k}) \to k$.

A general version of a motivic Gauß-Bonnet formula, which we do not use here, appears in the paper of Déglise-Jin-Khan [5]; many of the details of the ideas sketched here are worked out in a paper of ours with Arpon Raksit [11]. For a general discussion of the $\mathrm{SH}(k)$ Euler characteristic and its basic properties, we refer the reader to [10].

1.1. **Categorical Euler characteristics.** There is a general notion of a *trace* in a symmetric monoidal category, and the Euler characteristic of a dualizable object $x$ is defined as the trace of the identity map on $x$.

**Definition 1.1.1.** Let $(\mathcal{C}, \otimes, \tau, 1)$ be a symmetric monoidal category.

1. A *dual* of an object $x \in \mathcal{C}$ is a triple $(x^\vee, \delta_x, ev_x)$ with

$$\delta_x : \mathbb{1} \to x \otimes x^\vee,\ \ ev_x : x^\vee \otimes x \to \mathbb{1}$$

morphisms in $\mathcal{C}$ such that the compositions

$$x = \mathbb{1} \otimes x \xrightarrow{\delta_x \otimes \mathrm{Id}_x} x \otimes x^\vee \otimes x \xrightarrow{\mathrm{Id}_x \otimes ev_x} x \otimes \mathbb{1} = x$$

and

$$x^\vee = x^\vee \otimes \mathbb{1} \xrightarrow{\mathrm{Id}_{x^\vee} \otimes \delta_x} x^\vee \otimes x \otimes x^\vee \xrightarrow{ev_x \otimes \mathrm{Id}_{x^\vee}} \mathbb{1} \otimes x^\vee = x^\vee$$

are identity morphisms. The triple $(x^\vee, \delta_x, ev_x)$ is unique up to unique isomorphism; we call $x$ *dualizable* if $x$ has a dual.

2. Let $f : x \to y$ be a morphism of dualizable objects in $\mathcal{C}$. We have the dual morphism $f^\vee : y^\vee \to x^\vee$ defined as the composition

$$y^\vee = y^\vee \otimes \mathbb{1} \xrightarrow{\mathrm{Id}_{y^\vee} \otimes \delta_x} y^\vee \otimes x \otimes x^\vee \xrightarrow{\mathrm{Id}_{y^\vee} \otimes f \otimes \mathrm{Id}_x} y^\vee \otimes y \otimes x^\vee \xrightarrow{ev_y \otimes \mathrm{Id}_x} \mathbb{1} \otimes x^\vee = x^\vee$$

Then $(x, f) \mapsto (x^\vee, f^\vee)$ defines an involution on the full subcategory of dualizable objects $\mathcal{C}^\times$ in $\mathcal{C}$.

3. For $f : x \to x$ a morphism in $\mathcal{C}^\times$, we have $\mathrm{Tr}(f) \in \mathrm{End}_{\mathcal{C}}(\mathbb{1})$ defined as the composition

$$\mathbb{1} \xrightarrow{\delta_x} x \otimes x^\vee \xrightarrow{f \otimes \mathrm{Id}_{x^\vee}} x \otimes x^\vee \xrightarrow{\tau_{x,x^\vee}} x^\vee \otimes x \xrightarrow{ev_x} \mathbb{1}$$

The *Euler characteristic* $\chi_{\mathcal{C}}(x) \in \mathrm{End}_{\mathcal{C}}(\mathbb{1})$ is defined to be $\mathrm{Tr}(\mathrm{Id}_x)$.

Clearly $\mathrm{Tr}_{\mathcal{C}}(f)$ and $\chi_{\mathcal{C}}(x)$ are functorial: if $F : \mathcal{C} \to \mathcal{D}$ is a symmetric monoidal functor and $x \in \mathcal{C}^\times$, the $F(x) \in \mathcal{D}^\times$ and $F(\mathrm{Tr}_{\mathcal{C}}(f)) = \mathrm{Tr}_{\mathcal{D}}(F(f))$.

*Example* 1.1.2. For $X$ a finite simplicial set or finite CW complex, we have the chain complex $C_*(X;\mathbb{Z}) \in D(\mathbf{Ab})$, a dualizable object, and $\chi_{D(\mathbf{Ab})}(C_*(X;\mathbb{Z}))$ is the usual topological Euler characteristic $\chi^{top}(X)$. We have $\mathrm{End}_{D(\mathbf{Ab})}(\mathbb{Z}) = \mathbb{Z}$; as $\mathrm{End}_{D(\mathbb{Q})}(\mathbb{Q}) = \mathbb{Q}$, we can pass to $D(\mathbb{Q})$ and then note that $C_*(X;\mathbb{Q}) \cong \oplus_{i=0}^{\dim X} H_i(X,\mathbb{Q})[i]$; the dual is $C_*(X;\mathbb{Q})^\vee = \oplus_{i=0}^{\dim X} H_i(X,\mathbb{Q})[-i]$ and the trace of the identity gives the trace of the identity map on $H_i(X,\mathbb{Q})$, times the factor $(-1)^i$ coming from the symmetry $\tau(a \otimes b) = (-1)^{\deg a \deg b} b \otimes a$.

2. The same computation says that

$$\chi_{D(\mathrm{Sh}_{\text{ét}}(k))}(R\Gamma(X_{\text{ét}}, \mathbb{Q}_\ell)) = \chi^{\text{ét}}(X) = \sum_{i=0}^{2\dim X} (-1)^i \dim_{\mathbb{Q}_\ell} H^i_{\text{ét}}(X, \mathbb{Q}_\ell)$$

for $k = \bar{k}$ and $X$ a finite type $k$-scheme.

3. For $\mathcal{C} = \mathrm{DM}(k)$, we have $\mathrm{End}(\mathbb{Z}(0)) = \mathbb{Z}$, and we can see by either an étale realization or the Betti realization for $k \subset \mathbb{C}$, that for $X$ smooth over $k$

$$\chi_{\mathrm{DM}(k)}(M(X)) = \chi^{\text{ét}}(X_{\bar{k}}) = \chi^{top}(X(\mathbb{C})).$$

Similarly, for $\mathcal{C} = \mathrm{SH}$, we have $\mathrm{End}_{\mathrm{SH}}(\mathbb{S}) = \mathbb{Z}$, so we can compute in $D(\mathbf{Ab})$ and we get

$$\chi_{\mathrm{SH}}(\Sigma^\infty X_+) = \chi_{D(\mathbf{Ab})}(C_*(X;\mathbb{Z})) = \chi^{top}(X).$$

for $X$ a finite simplicial set/CW complex.

Our main object of study is the "motivic" Euler characteristic $\chi_{\mathrm{SH}(k)}(\Sigma^\infty_T X_+)$. For $k$ a field, we have the category of *pointed spaces over* $k$, $\mathbf{Spc}_\bullet(k)$, this being the category of presheaves on $\mathbf{Sm}/k$ with values in pointed simplicial sets. There is a good notion of "$\mathbb{A}^1$-weak equivalence" replacing the classical notion of weak homotopy equivalence, giving us the motivic unstable pointed homotopy category over $k$, $\mathcal{H}_\bullet(k) := \mathbf{Spc}_\bullet(k)[WE_{\mathbb{A}^1}^{-1}]$ [17]. The motivic analog of classical stabilization through $S^1$-suspension spectra is the so-called $T$-stabilization, $T := \mathbb{A}^1/(\mathbb{A}^1 - \{0\})$, giving rise to the symmetric monoidal triangulated category $\mathrm{SH}(k)$, the *motivic stable homotopy over* $k$ [8, 22]. Each $X \in \mathbf{Sm}/k$ gives an object $X_+ \in \mathbf{Spc}_\bullet(k)$ defined as the pointed representable sheaf of sets $Y \mapsto \mathrm{Hom}_{\mathbf{Sm}/k}(Y, X) \cup \{*\}$. There is an infinite suspension functor

$$\Sigma^\infty_T : \mathcal{H}_\bullet(k) \to \mathrm{SH}(k)$$

and for $X \in \mathbf{Sch}/k$, smooth and proper over $k$, the infinite suspension $\Sigma^\infty_T X_+$ is dualizable (see e.g. [21, §2], [8, §5]). We write $\chi(X/k)$ for $\chi_{\mathrm{SH}(k)}(\Sigma^\infty_T X_+)$.

For $k$ of characteristic zero, $\Sigma^\infty_T X_+$ is dualizable for all $X \in \mathbf{Sm}/k$; for simplicity, we will either assume $k$ has characteristic zero, or that "all relevant $X$" have $\Sigma^\infty_T X_+$ dualizable. We will assume as needed that $\mathrm{char} k \neq 2$.

The first point is the theorem of Morel:

**Theorem 1.1.3** (Morel [14, §6] [15, §6])**.** *For $k$ a perfect field there is a canonical isomorphism*

$$\pi_0(\mathbb{S}_k)_n \cong \mathcal{K}^{MW}_n.$$

To explain: $\mathcal{K}^{MW}_* = \oplus_{n\in\mathbb{Z}} \mathcal{K}^{MW}_n$ is a Nisnevich sheaf of graded rings on $\mathbf{Sm}/k$ with value $K^{MW}_*(F)$ on fields, defined by Hopkins-Morel [14, §6], [15, §6]. Here $K^{MW}_*(F)$ is defined by generators and relations.

**Generators**: for $u \in F^\times$, the generator $[u] \in K^{MW}_1(F)$ and an additional generator $\eta \in K^{MW}_{-1}(F)$.

**Relations**:
(0) $\eta[u] = [u]\eta$
(1) $[u][1-u] = 0$ for $u \in F - \{0, 1\}$
(2) $[uv] = [u] + [v] + \eta[u][v]$
(3) Let $h = 2 + \eta[-1]$. Then $\eta \cdot h = 0$.

There are "tame symbol maps" $\partial_t : K^{MW}_n(F) \to K^{MW}_{n-1}(f)$ for each DVR $\mathcal{O}$ with quotient field $F$, residue field $f$ and generator $t$ for the maximal ideal, and $\mathcal{K}^{MW}_n$ is the associated unramified sheaf on $\mathbf{Sm}/k$: For $X \in \mathbf{Sm}/k$ irreducible we have

$$\mathcal{K}^{MW}_n(X) = \{\alpha \in K^{MW}_n(k(X)) \mid \partial_{t_x}(\alpha) = 0\ \forall x \in X^{(1)}, (t_x) = \mathfrak{m}_x \subset \mathcal{O}_{X,x}\}$$

where $X^{(n)} \subset X$ is the set of codimension $n$ points of $X$. The fact that this does indeed form a (Nisnevich) sheaf on $\mathbf{Sm}/k$ is not obvious, but is proven in [13].

For $\mathcal{E} \in \mathrm{SH}(k)$, we have the bi-graded sheaf of homotopy groups, with $\pi_a(\mathcal{E})_b$ associated to the presheaf

$$U \mapsto [\Sigma^a_{S^1}\Sigma^\infty_T U_+, \Sigma^b_{\mathbb{G}_m}\mathcal{E}]_{\mathrm{SH}(k)}.$$

Morel's connectedness theorem [16] tells us that $\pi_a(\mathbb{S})_b = 0$ for $a < 0$ and all $b$, so the $\pi_0(\mathbb{S}_k)_*$ are the "first" non-vanishing sheaves; these should be thought of as the analog of $\pi_0(\mathbb{S}) = \mathbb{Z}$ in SH.

In addition, Morel computes $K^{MW}_n(F)$ as follows (for simplicity, we work away from characteristic 2):

**Theorem 1.1.4** (Morel [14, §6], [15, §6])**.** *Let* $\mathrm{GW}(F)$ *be the Grothendieck-Witt ring of quadratic forms over* $F$*,* $W(F)$ *the Witt ring* $\mathrm{GW}(F)/(H)$*,* $H$ *the hyperbolic form* $x^2 - y^2$ *and* $I(F) \subset \mathrm{GW}(F)$ *the ideal of forms of virtual rank zero. Sending the quadratic form* $q_u(x) = ux^2$*,* $u \in F^\times$*, to* $\langle u\rangle := 1 + \eta[u]$ *defines an isomorphism*

$$\mathrm{GW}(F) \to K_0^{MW}(F)$$

*For* $n > 0$*, sending* $q_u$ *to* $\eta^n(1 + \eta[u])$ *defines an isomorphism*

$$W(F) \to K_{-n}^{MW}(F)$$

*Sending* $[u_1] \cdot \ldots \cdot [u_n]$ *to the symbol* $\{u_1, \ldots, u_n\}$ *and sending* $\eta$ *to zero defines a surjection* $\pi : K_*^{MW}(F) \to K^M(F)$*. For* $n > 0$*, there is an exact sequence*

$$0 \to I(F)^{n+1} \to K_n^{MW}(F) \xrightarrow{\pi} K_n^M(F) \to 0$$

As consequence, we can identify $\mathrm{End}_{\mathrm{SH}(k)}(\mathbb{S}) = \pi_0(\mathbb{S})_0(k)$ with $\mathrm{GW}(k)$, and thus we have the Euler characteristic $\chi(X/k) \in \mathrm{GW}(k)$, or more generally, $\chi(\mathcal{X}/k) := \chi_{\mathrm{SH}(k)}(\Sigma^\infty\mathcal{X}) \in \mathrm{GW}(k)$ for a dualizable pointed space $\mathcal{X} \in \mathbf{Spc}_\bullet(k)$.

**Elementary properties of the Euler characteristic** (see [10, §§1.1, 1.2])

1. Let $S_k^{p,q} = S^{p-q} \wedge \mathbb{G}_m^{\wedge b}$. Then $\chi(S_k^{p,q}/k) = (-1)^p \cdot \langle -1\rangle^q$.

2. Let $F$, $X$ and $Y$ be in $\mathbf{Sm}/k$ and let $p : Y \to X$ be a Nisnevich (or Zariski) locally trivial fiber bundle with fiber $F$. Then

$$\chi(Y/k) = \chi(X/k) \cdot \chi(F/k).$$

3. Let $X$ be in $\mathbf{Sm}/k$ and let $p : V \to X$ be a rank $r$ vector bundle. Then $\mathrm{Th}(V)$ is dualizable and

$$\chi(\mathrm{Th}(V)/k) = \langle -1\rangle^r \cdot \chi(X/k).$$

4. Let $X$ be in $\mathbf{Sm}/k$ and let $p : V \to X$ be a rank $r$ vector bundle. Let $q : \mathbb{P}(V) \to X$ be the associated projective space bundle $\mathrm{Proj}_X(\mathrm{Sym}^* V^\vee)$. Then $\mathbb{P}(V)$ is dualizable and

$$\chi(\mathbb{P}(V)/k) = (\sum_{i=0}^{r-1} \langle -1\rangle^i) \cdot \chi(X/k).$$

5. Let $i : Z \to X$ be a codimension $c$ closed immersion in $\mathbf{Sm}/k$. Let $\tilde{X}$ be the blow up of $X$ along $Z$ and let $p : N_Z \to Z$ be the normal bundle of $i$. . Then

$$\chi(\tilde{X}/k) = \chi(X/k) + (\sum_{i=1}^{c-1} \langle -1\rangle^i) \cdot \chi(Z/k).$$

6. Let $\sigma : k \to F$ be an extension of fields, inducing the homomorphism $\sigma_* : \mathrm{GW}(k) \to \mathrm{GW}(F)$. Then

$$\chi(X_F/F) = \sigma_*(\chi(X/k)).$$

7. For $\sigma : k \to \mathbb{C}$, and for $X \in \mathbf{Sm}/k$, $\mathrm{rank}\chi(X/k) = \chi^{top}(X^\sigma(\mathbb{C}))$.

8. For $\sigma : k \to \mathbb{R}$, $\mathrm{sig}(\sigma_*\chi(X/k)) = \chi^{top}(X^\sigma(\mathbb{R}))$. In consequence $\chi^{top}(X^\sigma(\mathbb{R})) \equiv \chi^{top}(X^\sigma_{\mathbb{C}}(\mathbb{C})) \mod 2$.

1.2. **Motivic Gauss-Bonnet.** One can define Euler classes of vector bundles in a variety of situations. Déglise-Jin-Khan [5] have given a definition with values in the motivic cohomotopy of the Thom space of $-V$; we discuss here a version with values in an SL-oriented theory, following [11]. For notions concerning SL-orientations and the basic properties of SL-oriented theories, we refer the reader to [2].

**Definition 1.2.1.** An SL-*orientation* on a motivic commutative ring spectrum $\mathcal{E}\in \mathrm{SH}(k)$ is the functorial assignment of a *Thom class* $\theta_{V,\phi}\in\mathcal{E}^{2r,r}(\mathrm{Th}(V))$ for each rank $r$ vector bundle $V\to X$, $X\in\mathbf{Sm}/k$ endowed with a trivialization of the determinant $\phi:\det V\xrightarrow{\sim}\mathcal{O}_X$, satisfying

1. For $V=\oplus_{i=1}^r\mathcal{O}_X\cdot e_i$, $\phi$ the isomorphism $e_1\wedge\ldots\wedge e_r\mapsto 1$, $\theta_{V,\phi}\in\mathcal{E}^{2r,r}(\mathrm{Th}(V))$ is the image of the unit map $\mathbb{S}\to\mathcal{E}$ via the canonical isomorphisms

$$[\mathbb{S},\mathcal{E}]_{\mathrm{SH}(k)}\xrightarrow{\sim}[\Sigma^r_T\mathbb{S},\Sigma^r_T\mathcal{E}]_{\mathrm{SH}(k)}\xrightarrow{\sim}[\mathrm{Th}(\mathcal{O}_X^r),\Sigma^r_T\mathcal{E}]_{\mathrm{SH}(k)}=\mathcal{E}^{2r,r}(\mathrm{Th}(V)).$$

2. $\theta_{V_1\oplus V_2,\phi_1\wedge\phi_2}=p_1^*\theta_{V_1,\phi_1}\cup p_2^*\theta_{V_2,\phi_2}$.

For $\mathcal{E}$ an SL-oriented theory, and $V\to X$ a rank $r$ vector bundle with trivialized determinant, the map

$$\vartheta_{V,\phi}:=(-\cup\vartheta_{V,\phi})\circ\pi^*:\mathcal{E}^{a,b}(X)\to\mathcal{E}^{a+2r,b+r}(\mathrm{Th}(V))$$

is an isomorphism. For $L\to X$ a line bundle, one defines

$$\mathcal{E}^{a,b}(X;L):=\mathcal{E}^{a+2,b+1}(\mathrm{Th}(L)).$$

For $p:V\to X$ a vector bundle let $\pi:L\to X$ be the line bundle $\det V\to X$. Then we have canonical trivializations $\phi_1,\phi_2$ of the determinant bundles of $p_1:\pi^*(V\oplus L^{-1})\to L$ and $p_2:p^*(L^{-1}\oplus L)\to V$. The diagram

$$\mathcal{E}^{a+2,b+1}(\mathrm{Th}(L))\xrightarrow{\vartheta_{\pi^*(V\oplus L^{-1}),\phi_1}}\mathcal{E}^{a+2r+4,b+r+2}(\mathrm{Th}(V\oplus L\oplus L^{-1}))\xleftarrow{\vartheta_{p^*(L^{-1}\oplus L),\phi_2}}\mathcal{E}^{a+2r,b+r}(\mathrm{Th}(V))$$

gives an isomorphism

$$\vartheta_V:\mathcal{E}^{a,b}(X;\det V)\to\mathcal{E}^{a+2r,b+r}(\mathrm{Th}(V))$$

If we have $\phi:\det V\xrightarrow{\sim}\mathcal{O}_X$, the suspension isomorphism $\mathcal{E}^{a,b}(X)\cong\mathcal{E}^{a+2,b+1}(\mathrm{Th}(\mathcal{O}_X))$ gives the isomorphism $\phi_*:\mathcal{E}^{a,b}(X)\to\mathcal{E}^{a,b}(X;\det^{-1}V)$ and one checks that the composition

$$\mathcal{E}^{a,b}(X)\xrightarrow{\phi_*}\mathcal{E}^{a,b}(X;\det^{-1}V)\xrightarrow{\vartheta_V}\mathcal{E}^{2r,r}(\mathrm{Th}(V))$$

is $\vartheta_{V,\phi}$.

Let $\theta_V\in\mathcal{E}^{2r,r}(\mathrm{Th}(V);\det^{-1}V)$ be the element corresponding to $\theta_{V\oplus\det^{-1}V,can}\in\mathcal{E}^{2r+2,r+1}(\mathrm{Th}(V\oplus\det^{-1}V))$; we call $\theta_V$ the *canonical Thom class* for $V$. Again, if $\phi:\det V\to\mathcal{O}_X$ is a trivialization of $\det V$, the via the isomorphism

$$\mathcal{E}^{2r,r}(\mathrm{Th}(V);\det^{-1}V)\xrightarrow{\phi_*^{-1}}\mathcal{E}^{2r,r}(\mathrm{Th}(V))$$

as above, $\theta_V$ maps to $\theta_{V,\phi}$, so using the "orientation bundle" $\det^{-1}V$ we have canonical Thom classes without choice of a trivialization. See [2, §4] for details on these constructions.

**Definition 1.2.2.** Let $\mathcal{E}$ be an SL-oriented theory and let $V \to X$ be a rank $r$ vector bundle on $X \in \mathbf{Sm}/k$ with 0-section $s : X \to V$. Define the *Euler class* $e(V) \in \mathcal{E}^{2r,r}(X; \det^{-1} V)$ by

$$e(V) := s^*\theta_V.$$

For $\mathcal{E}$ an SL-oriented theory, one has functorial push-forward maps

$$p_* : \mathcal{E}^{a,b}(Y, \omega_{Y/k} \otimes f^*L) \to \mathcal{E}^{a-2d,b-d}(X, \omega_{X/k} \otimes L)$$

for each proper map $p : Y \to X$ of relative dimension $d$ in $\mathbf{Sm}/k$. To see this, we introduce a part of the Grothendieck six-functor formalism, available for the motivic stable homotopy category through work of Ayoub [1] and Hoyois [8]. This involves two sets of adjoint functors for each morphism of finite type $B$-schemes $f : Y \to X$ for some fixed base-scheme $B$:

$$f^* : \mathrm{SH}(X) \rightleftarrows \mathrm{SH}(Y) : f_*$$

$$f_! : \mathrm{SH}(Y) \rightleftarrows \mathrm{SH}(X) : f^!.$$

The left adjoints on the left, the ones with lower decorations are covariant in $f$, those with upper decorations are contravariant. One has the respective units and co-units of adjunction, and a natural transformation $\eta_!^f : f_! \to f_*$, which is an isomorphism if $f$ is proper.

Back to our base-scheme $B = \operatorname{Spec} k$ and a proper map $p : Y \to X$ in $\mathbf{Sm}/k$. Let $\pi_X : X \to \operatorname{Spec} k$, $\pi_Y : Y \to \operatorname{Spec} k$ be the structure maps. We have the natural transformation (*proper pullback*)

$$p^* : \pi_{X!} \to \pi_{Y!} \circ p^* : \mathrm{SH}(X) \to \mathrm{SH}(k) \tag{1.2.1}$$

defined as the composition

$$\pi_{X!} \xrightarrow{u_p} \pi_{X!}p_*p^* \xrightarrow{(\eta^p_{!*})^{-1}} \pi_{X!}p_!p^* \cong \pi_{Y!} \circ p^*.$$

Applying $p^*$ to $1_X$ and noting that $\pi_{X!}(1_X) = \mathrm{Th}(-T_{X/k})$, $\pi_{Y!}(1_Y) = \mathrm{Th}(-T_{Y/k})$ gives the morphism

$$p^* : \mathrm{Th}(-T_{X/k}) \to \mathrm{Th}(-T_{Y/k})$$

in $\mathrm{SH}(k)$. One checks easily that $(pq)^* = q^*p^*$ for composable proper morphisms $p$ and $q$.

Since $\det(-T_{X/k}) = \det^{-1}(T_{X/k}) = \omega_{X/k}$, and similarly for $Y$, we have

$$\mathcal{E}^{a,b}(Y, \omega_{Y/k} \otimes f^*L) = \mathcal{E}^{a+2\mathrm{dim}Y, b+\mathrm{dim}Y}(\mathrm{Th}(-T_{Y/k}); f^*L)$$

and

$$\mathcal{E}^{a',b'}(X, \omega_{X/k} \otimes L) = \mathcal{E}^{a'+2\mathrm{dim}X, b'+\mathrm{dim}X}(\mathrm{Th}(-T_{X/k}); L)$$

so the map $p^*$ induces the map

$$p_* : \mathcal{E}^{a,b}(Y, \omega_{Y/k} \otimes f^*L) \to \mathcal{E}^{a-2d,b-d}(X, \omega_{X/k} \otimes L)$$

on $\mathcal{E}$-cohomology.

Via these constructions, we have another description of $\theta_V \in \mathcal{E}^{2r,r}(X; \det^{-1} V)$ as

$$\theta_V = s_*(1_X^{\mathcal{E}})$$

where $1_X^{\mathcal{E}} \in \mathcal{E}^{0,0}(X)$ is the unit and $s : X \to V$ is the 0-section.

The diagonal $\Delta_X : X \to X \times_k X$ induces the map

$$\Delta_* : X \to X \times_k X/(X \times_k X - \Delta_X(X)) \cong \mathrm{Th}(N_{\Delta_X})$$

The composition

$$T_{X/k} \xrightarrow{p_2^*} \Delta_X^* p_2^* T_{X/k} \xrightarrow{i_2} \Delta_X^*(p_1^* T_{X/k} \oplus p_2^* T_{X/k}) \cong \Delta_X^* T_{X\times_k X/k} \xrightarrow{\pi} N_{\Delta_X}$$

is an isomorphism.

Letting $j : X \times_k X - \Delta_X(X) \to X \times_k X$ be the inclusion, the map $\Delta_*$ induces the map $\beta_X : \mathrm{Th}(-T_{X/k}) \to \Sigma^\infty_T X_+$ as the composition

$$\begin{aligned}\mathrm{Th}(-T_{X/k}) \xrightarrow{\mathrm{Th}(\Delta_X)} \mathrm{Th}(-p_1^*T_{X/k}) &\xrightarrow{q} \mathrm{Th}(-p_1^*T_{X/k})/\mathrm{Th}(-j^*p_1^*T_{X/k})\\ &\cong \mathrm{Th}(-T_{X/k} \oplus N_{\Delta_X}) \cong \mathrm{Th}(-T_{X/k} \oplus T_{X/k}) \cong \Sigma^\infty_T X_+\end{aligned}$$

**Lemma 1.2.3.** *For $X$ smooth and proper over $k$, $\chi(X/k)$ is equal to the composition*

$$\mathbb{S}_k \xrightarrow{\pi_X^*} \mathrm{Th}(-T_{X/k}) \xrightarrow{\beta_X} \Sigma^\infty_T X_+ \xrightarrow{\pi_X} \mathbb{S}_k$$

*Proof.* Let $p_i : X \times_k X \to X$, $i = 1, 2$, be the projections. The map

$$ev_X : (\Sigma^\infty_T X_+)^\vee \wedge \Sigma^\infty_T X_+ \to \mathbb{S}_k$$

is the composition

$$\begin{aligned}(\Sigma^\infty_T X_+)^\vee \wedge \Sigma^\infty_T X_+ &= \mathrm{Th}(-T_{X/k}) \wedge \Sigma^\infty_T X_+\\ &= \mathrm{Th}(-p_1^*T_{X/k}) \xrightarrow{q} \mathrm{Th}(-p_1^*T_{X/k})/\mathrm{Th}(-j^*p_1^*T_{X/k}) \cong \mathrm{Th}(-T_{X/k} \oplus N_{\Delta_X})\\ &\cong \mathrm{Th}(-T_{X/k} \oplus T_{X/k}) \cong \Sigma^\infty_T X_+ \xrightarrow{\pi_X} \mathbb{S}_k\end{aligned}$$

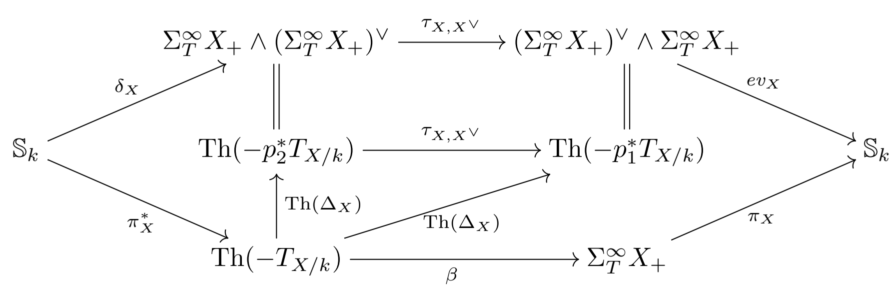
Thus $\pi_X \circ \beta_X = ev_X \circ \mathrm{Th}(\Delta_X)$. Also, $\pi_X^* = \pi_X^\vee$ (see [8, Corollary 6.13]) and $\pi_X^\vee$ is given by

$$\mathbb{S}_k \xrightarrow{\delta_X} \Sigma^\infty_T X_+ \wedge (\Sigma^\infty_T X_+)^\vee = \mathrm{Th}(-p_2^*T_{X/k}) \xrightarrow{p_2} \mathrm{Th}(-T_{X/k})$$

It follows from the construction of the map $\delta_X$ (see for example [8, §5.3, Corollary 6.13]) that

$$\delta_X = \mathrm{Th}(\Delta_X) \circ \pi_X^*.$$

This gives us the commutative diagram




and the result follows from the definition of $\chi(X/k)$ as $ev_X \circ \tau_{X,X^\vee} \circ \delta_X$. $\square$

**Theorem 1.2.4** (Motivic Gauss-Bonnet [11, Theorem 5.2])**.** *Let $\mathcal{E}$ be an* SL*-oriented theory, $\pi_X \to \mathrm{Spec}\, k$ a smooth and projective $k$-scheme, let $u_\mathcal{E} : \mathbb{S} \to \mathcal{E}$ be the unit map. Then*

$$\pi_{X*}(e^\mathcal{E}(T_{X/k})) = u_{\mathcal{E}*}(\chi(X/k)) \in \mathcal{E}^{0,0}(k).$$

*Proof.* We use a Jouanolou cover $p : \tilde{X} \to X$, that is, $p$ is an affine space bundle and $\tilde{X}$ is affine. There is then a vector bundle $p_\nu : \nu_{\tilde{X}} \to \tilde{X}$ and an isomorphism

$$p^*T_{X/k} \oplus \nu_{\tilde{X}} \cong \mathcal{O}_{\tilde{X}}^N,$$

which gives the isomorphism of virtual bundles (i.e. in $K_0(\tilde{X})$) $\nu_{\tilde{X}} \cong -p^*T_{X/k} \oplus \mathcal{O}_{\tilde{X}}^N$, giving in turn an isomorphism in $\mathrm{SH}(k)$

$$\Sigma_T^N \mathrm{Th}(-T_{X/k}) \cong \Sigma_T^\infty \mathrm{Th}(\nu_{\tilde{X}}).$$

Let $i_1 : \nu_{\tilde{X}} \to \nu_{\tilde{X}} \oplus p^*T_{X/k}$ be the inclusion. Via the projection $\pi_1 : \nu_{\tilde{X}} \oplus p^*T_{X/k} \to \nu_{\tilde{X}}$, we can view $\nu_{\tilde{X}} \oplus p^*T_{X/k}$ as the vector bundle $p_\nu^* p^* T_{X/k}$ on $\nu_{\tilde{X}}$ with 0-section $i_1$.

We have the maps $\Delta_X^1 : \tilde{X} \to \tilde{X} \times_k X$, $\Delta_X^2 : \tilde{X} \to X \times_k \tilde{X}$ induced by $p$. The map $\Sigma_T^N \beta_X$ is the stabilization of the map

$$\begin{aligned}\mathrm{Th}(\nu_{\tilde{X}}) &\xrightarrow{\mathrm{Th}(\Delta_X^1)} \mathrm{Th}(p_1^*\nu_{\tilde{X}}) \xrightarrow{q} \mathrm{Th}(p_1^*\nu_{\tilde{X}})/\mathrm{Th}(j^*p_1^*\nu_{\tilde{X}})\\ &\cong \mathrm{Th}(\nu_{\tilde{X}} \oplus N_{\Delta_X^1}) \cong \mathrm{Th}(\nu_{\tilde{X}} \oplus p^*T_{X/k}) \cong \mathrm{Th}(\mathcal{O}_{\tilde{X}}^N) = \Sigma_T^N \tilde{X}_+ \cong \Sigma_T^N X_+\end{aligned}$$

Thus the map on $\mathcal{E}^{a,b}$ induced by $\beta_X$ is the stabilization of $\mathrm{Th}(i_1)^* \circ susp$, with

$$\mathrm{Th}(i_1)^* : \mathcal{E}^{a+2N,b+N}(\mathrm{Th}(\nu_{\tilde{X}} \oplus p^*T_{X/k})) \to \mathcal{E}^{a+2N,b+N}(\mathrm{Th}(\nu_{\tilde{X}})) = \mathcal{E}^{a,b}(\mathrm{Th}(-T_{X/k}))$$

and *susp* the suspension isomorphism

$$\mathcal{E}^{a,b}(\tilde{X}) \cong \mathcal{E}^{a+2N,b+N}(\mathrm{Th}(\mathcal{O}_{\tilde{X}}^N)) \cong \mathcal{E}^{a+2N,b+N}(\mathrm{Th}(\nu_{\tilde{X}} \oplus p^*T_{X/k})).$$

Identifying $\mathcal{E}^{a+2N,b+N}(\mathrm{Th}(\nu_{\tilde{X}} \oplus p^*T_{X/k}))$ with $\mathcal{E}_0^{a+2N,b+N}(\nu_{\tilde{X}} \oplus p^*T_{X/k})$ and $\mathcal{E}^{a+2N,b+N}(\nu_{\tilde{X}})$ with $\mathcal{E}_0^{a+2N,b+N}(\nu_{\tilde{X}})$ ($0$ = respective 0-section) identifies $\mathrm{Th}(i_1)^*$ with the map

$$i_1^* : \mathcal{E}_0^{a+2N,b+N}(\nu_{\tilde{X}} \oplus p^*T_{X/k}) \to \mathcal{E}_0^{a+2N,b+N}(\nu_{\tilde{X}})$$

Via the isomorphism $\mathcal{E}_0^{a+2N,b+N}(\nu_{\tilde{X}}) \cong \mathcal{E}^{a,b}(\mathrm{Th}(-T_{X/k}))$, this gives the description of $\mathcal{E}^{a,b}(\beta_X)^* : \mathcal{E}^{a,b}(X) \to \mathcal{E}^{a,b}(\mathrm{Th}(-T_{X/k}))$ as $i_1^* \circ susp$.

The Thom classes $\theta_{p^*T_{X/k}} = p^*\theta_{T_{X/k}}$ and $\theta_{\nu_{\tilde{X}}}$ give the Thom isomorphisms

$$\vartheta_{p^*T_{X/k}} : \mathcal{E}^{0,0}(\tilde{X}) \to \mathcal{E}_0^{2d_X,d_X}(p^*T_{X/k})$$

$$\vartheta_{p_\nu^*p^*T_{X/k}} : \mathcal{E}^{2(N-d_X),N-d_X}(\nu_{\tilde{X}}) \to \mathcal{E}_0^{2N,N}(\nu_{\tilde{X}} \oplus p^*T_{X/k})$$

and

$$\vartheta_{\nu_{\tilde{X}}} : \mathcal{E}^{0,0}(\tilde{X}) \to \mathcal{E}_0^{2(N-d_X)+,N-d_X}(\nu_{\tilde{X}})$$

Moreover, we have

$$\pi_1^*\theta_{\nu_{\tilde{X}}} \cup \pi_2^*\theta_{p^*T_{X/k}} = \theta_{\nu_{\tilde{X}} \oplus p^*T_{X/k}} = \theta_{\mathcal{O}_{\tilde{X}}^N},$$

giving us the commutative diagram

$$\begin{array}{ccc}
\mathcal{E}^{0,0}(\tilde{X}) & \xrightarrow{\vartheta_{\nu_{\tilde{X}}}} & \mathcal{E}_0^{2(N-d_X),N-d_X}(\nu_{\tilde{X}}, \det^{-1}\nu_{\tilde{X}}) \\
\downarrow{\scriptstyle \vartheta_{p^*T_{X/k}}} & \searrow{\scriptstyle susp} & \downarrow{\scriptstyle \vartheta_{p_\nu^*p^*T_{X/k}}} \\
\mathcal{E}_0^{2d_X,d_X}(p^*T_{X/k}, \det^{-1}p^*T_{X/k}) & \xrightarrow{\vartheta_{\nu_{\tilde{X}}}} & \mathcal{E}_0^{2N,N}(\nu_{\tilde{X}} \oplus p^*T_{X/k}, \det^{-1}p^*T_{X/k} \otimes \det^{-1}\nu_{\tilde{X}}).
\end{array}$$

With $s : \tilde{X} \to p^*T_{X/k}$ the 0-section, this gives us the commutative diagram

$$
\begin{array}{ccc}
\mathcal{E}^{0,0}(\tilde{X}) & \xrightarrow{\vartheta_{\nu_{\tilde{X}}}} & \mathcal{E}_0^{2(N-d_X),N-d_X}(\nu_{\tilde{X}}, \det^{-1}\nu_{\tilde{X}}) \\
\downarrow{\scriptstyle \vartheta_{p^*T_{X/k}}} & \searrow{\scriptstyle susp} & \downarrow{\scriptstyle \vartheta_{p_\nu^* p^*T_{X/k}}} \\
\mathcal{E}_0^{2d_X,d_X}(p^*T_{X/k}, \det^{-1}p^*T_{X/k}) & \xrightarrow{\vartheta_{\nu_{\tilde{X}}}} & \mathcal{E}_0^{2N,N}(\nu_{\tilde{X}} \oplus p^*T_{X/k}, \det^{-1}\nu_{\tilde{X}} \otimes \det^{-1}p^*T_{X/k}) \\
\downarrow{\scriptstyle s^*} & & \downarrow{\scriptstyle i_1^*} \\
\mathcal{E}^{2d_X,d_X}(\tilde{X}, \det^{-1}p^*T_{X/k}) & \xrightarrow{\vartheta_{\nu_{\tilde{X}}}} & \mathcal{E}_0^{2N,N}(\nu_{\tilde{X}}, \det^{-1}p^*T_{X/k} \otimes \det^{-1}\nu_{\tilde{X}}) \\
& \searrow{\scriptstyle \mathrm{Id}} & \uparrow{\scriptstyle \vartheta_{\nu_{\tilde{X}}}} \\
& & \mathcal{E}^{2d_X,d_X}(\tilde{X}, \det^{-1}p^*T_{X/k})
\end{array}
$$

After the identifications

$$\mathcal{E}_0^{2N,N}(\nu_{\tilde{X}}, \det^{-1}\nu_{\tilde{X}} \otimes \det^{-1}p^*T_{X/k}) \cong \mathcal{E}^{2N,N}(\mathrm{Th}(\nu_{\tilde{X}})) \cong \mathcal{E}^{0,0}(\mathrm{Th}(-T_{X/k}))$$

$$p^* : \mathcal{E}^{0,0}(X) \xrightarrow{\sim} \mathcal{E}^{0,0}(\tilde{X}),\ p^* : \mathcal{E}^{2d_X,d_X}(X, \omega_{X/k}) \xrightarrow{\sim} \mathcal{E}^{2d_X,d_X}(\tilde{X}, \det^{-1}p^*T_{X/k})$$

and the identity $\mathcal{E}(\beta_X)^* = i_1^* \circ susp$, this gives

$$\times e^{\mathcal{E}}(T_{X/k}) = \vartheta^{-1}_{-T_{X/k}} \circ \mathcal{E}(\beta_X)^* : \mathcal{E}^{0,0}(X) \to \mathcal{E}^{2d_X,d_X}(X, \omega_{X/k})$$

Moreover, the identification $\mathrm{Th}(-T_{X/k}) = \pi_{X!}(1_X) = (\Sigma^\infty_T X_+)^\vee$ gives the commutative diagram

$$
\begin{array}{ccc}
\mathcal{E}^{2d_X,d_X}(X, \omega_{X/k}) & \xrightarrow{\pi_{X*}} & \mathcal{E}^{0,0}(k) \\
\downarrow{\scriptstyle \vartheta_{-T_{X/k}}} & \nearrow{\scriptstyle \mathcal{E}(\pi_X^*)^*} & \uparrow{\scriptstyle \mathcal{E}(\pi_X^\vee)^*} \\
\mathcal{E}^{0,0}(\mathrm{Th}(-T_{X/k})) & = & \mathcal{E}^{0,0}((\Sigma^\infty_T X_+)^\vee)
\end{array}
$$

Via Lemma 1.2.3, this gives

$$
\begin{aligned}
\pi_{X*}(e^{\mathcal{E}}(T_{X/k})) &= \pi_{X*}((\times e^{\mathcal{E}}(T_{X/k})) \circ (\pi_X^*(u_{\mathcal{E}} : \mathbb{S} \to \mathcal{E}))) \\
&= \mathcal{E}(\pi_X^*)^* \circ (\vartheta_{-T_{X/k}} \circ \times e^{\mathcal{E}}(T_{X/k})) \circ \mathcal{E}(\pi_X)^*(u_{\mathcal{E}} : \mathbb{S} \to \mathcal{E}) \\
&= \mathcal{E}(\pi_X^*)^* \circ \mathcal{E}(\beta_X)^* \circ \mathcal{E}(\pi_X)^*(u_{\mathcal{E}} : \mathbb{S} \to \mathcal{E}) \\
&= \mathcal{E}(\pi_X \circ \beta_X \circ \pi_X^*)^*(u_{\mathcal{E}} : \mathbb{S} \to \mathcal{E}) \\
&= \mathcal{E}(\chi(X/k))^*(u_{\mathcal{E}} : \mathbb{S} \to \mathcal{E}) \\
&= u_{\mathcal{E}*}(\chi(X/k))
\end{aligned}
$$

□

1.3. **Chow-Witt groups.** The twisted Chow-Witt groups are defined using the cohomology of the twisted Milnor-Witt sheaves (see [3, 6] for a somewhat different definition; see also Jean Fasel's lectures [7] for many details concerning the Chow-Witt groups). Sending $u \in F^\times$ to $\langle u \rangle := 1 + \eta[u] \in K_0^{MF}(F)^\times$ extends to a map of sheaves of abelian groups $\mathbb{G}_m \to (\mathcal{K}_0^{MW})^\times$. If $L \to X$ is a line bundle on some $X \in \mathbf{Sm}/k$, we have the principal $\mathbb{G}_m$-bundle $L^\times = L - 0 \to X$ and we can define the sheaf $\mathcal{K}_n^{MW}(L)$ on $X$ by

$$\mathcal{K}_n^{MW}(L) := \mathcal{K}_n^{MW} \times^{\mathbb{G}_m} L^\times$$

Explicitly, if $\lambda$ is a generating section of $L$ over an open subset $U$ of $X$, a section $s$ of $\mathcal{K}_n^{MW}(L)$ over $U$ may be written as $s = x\otimes\lambda$, with the relation $x\otimes u\cdot\lambda = \langle u\rangle\cdot x\otimes\lambda$ for $x \in H^0(U, \mathcal{K}_n^{MW})$, $u \in H^0(U, \mathcal{O}_X^\times)$.

The isomorphism $\mathcal{K}_0^{MW} \to \mathcal{GW}$ extends to an isomorphism $\mathcal{K}_0^{MW}(L) \to \mathcal{GW}(L)$, where $\mathcal{GW}(L)$ is the sheaf of $L$-valued quadratic forms; we have a similar isomorphism $\mathcal{K}_n^{MW}(L) \cong \mathcal{W}(L)$ for all $n < 0$ and an exact sequence

$$0 \to \mathcal{I}(L)^{n+1} \to \mathcal{K}_n^{MW}(L) \to \mathcal{K}_n^M \to 0$$

for $n > 0$, where $\mathcal{I}(L) \subset \mathcal{GW}(L)$ is the sheaf of twisted augmentation ideals, that is, the kernel of the rank map $\mathcal{GW}(L) \to \mathbb{Z}$.

**Definition 1.3.1.** Let $L \to X$ be a line bundle, $X \in \mathbf{Sm}/k$. The $n$th twisted Chow-Witt group $\widetilde{\mathrm{CH}}^n(X; L)$ is defined as $H^n(X, \mathcal{K}_n^{MW}(L))$.

The product structure on $\mathcal{K}_*^{MW}$ gives rise to products

$$\tilde{\mathrm{CH}}^n(X; L) \times \tilde{\mathrm{CH}}^m(X, L') \to \tilde{\mathrm{CH}}^{n+m}(X; L \otimes L')$$

There are canonical isomorphisms

$$\tilde{\mathrm{CH}}^n(X; L \otimes M^{\otimes 2}) \cong \tilde{\mathrm{CH}}^n(X; L)$$

arising from the identity $\langle vu^2\rangle = \langle v\rangle$ for $u, v$ units.

For $f : Y \to X$ in $\mathbf{Sm}/k$, the fact that $\mathcal{K}_n^{MW}$ is a sheaf gives pull-back maps

$$f^* : \tilde{\mathrm{CH}}^n(X, L) \to \tilde{\mathrm{CH}}^n(Y, f^*L)$$

Moreover, for $f : Y \to X$ proper of relative dimensions $d$, there are push-forward maps

$$f_* : \tilde{\mathrm{CH}}^n(Y, \omega_{Y/k} \otimes f^*L) \to \tilde{\mathrm{CH}}^{n-d}(X, \omega_{X/k} \otimes L)$$

The push-forward maps may be described using the *Rost-Schmid complexes* (see [13, §5])

$$\begin{aligned}\mathcal{C}^*_{X,n,L} := \oplus_{x\in X^{(0)}} i_{x*}K_n^{MW}(k(x); L) &\to \oplus_{x\in X^{(1)}} K_{n-1}^{MW}(k(x); \det{}^{-1} \mathfrak{m}_x/\mathfrak{m}_x^2 \otimes L) \to \\ \ldots &\to \oplus \oplus_{x\in X^{(q)}} K_{n-q}^{MW}(k(x); \det{}^{-1} \mathfrak{m}_x/\mathfrak{m}_x^2 \otimes L) \to \ldots\end{aligned}$$

The complex $\mathcal{C}^*_{X,n,L}$ gives a flasque resolution of $\mathcal{K}_n^{MW}(L)$ on $X$ and taking global section gives

$$H^n(X, \mathcal{K}_n^{MW}(L)) \cong H^n(C^*_{X,L,n})$$

where $C^*_{X,L,n} = \Gamma(X, \mathcal{C}^*_{X,L,n})$. For $k$ perfect, this gives generators for $\tilde{\mathrm{CH}}^n(Y; \omega_{Y/k}\otimes f^*L)$ as a subgroup of $\oplus_{y\in Y^{(n)}}\mathrm{GW}(k(y); f^*L\otimes k(y))$. For $x = f(y)$ with $\mathrm{codim}_X x = \mathrm{codim}_Y y - d = n - d$ and $k(x) \to k(y)$ a separable extension, the trace maps

$$\mathrm{Tr}_{k(y)/k(x)} : \mathrm{GW}(k(y); f^*L \otimes k(y)) \to \mathrm{GW}(k(x); L \otimes k(x))$$

induces the map $f_* : \tilde{\mathrm{CH}}^n(Y, \omega_{Y/k} \otimes f^*L) \to \tilde{\mathrm{CH}}^{n-d}(X, \omega_{X/k} \otimes L)$. In general, one needs to handle inseparability issues, which is done by Fasel [6, §5, 6]. For $n = \dim_k Y$, one can simply pass to the perfect closure of $k$, noting that we have assumed the characteristic is odd and that the map on Grothendieck-Witt groups associated to an odd degree extension is always injective.

Similarly, one has a description of the cohomology with supports $H^p_Z(X, \mathcal{K}_n^{MW}(L))$ as the $p$th cohomology of the subcomplex of $C^*_{X,L,n}$ constructed by replacing $\oplus_{x\in X^{(q)}}$ with $\oplus_{x\in X^{(q)}\cap Z}$.

In particular, for $p : V \to X$ a vector bundle of rank $r$ with 0-section $s$ and for $x \in X^{(0)}$ the generic point (assuming $X$ irreducible) taking $1 \in \mathrm{GW}(k(s(x)))$ gives a well-defined class $\theta_V^{CW} \in H^r(V, \mathcal{K}_r^{MW}(\det^{-1} V))$ which in fact gives a twisted Thom class. This gives the cohomology theory

$$X \mapsto H^*(X, \mathcal{K}_*^{MW})$$

the properties one would expect from an SL-orientation.

For instance, we have the Euler class $e^{CW}(V) \in H^r(X, \mathcal{K}_r^{MW}(\det^{-1} V)) = \tilde{\mathrm{CH}}^r(X; \det^{-1} V)$ for $V \to X$ a rank $r$ vector bundle, defined as

$$e^{CW}(V) := s^* \theta_V^{CW}$$

for $s : X \to V$ the 0-section.

**Definition 1.3.2.** Let $\pi_X : X \to \operatorname{Spec} k$ be a smooth projective $k$-scheme. The Chow-Witt Euler characteristic $\chi^{CW}(X/k) \in \tilde{\mathrm{CH}}^0(\operatorname{Spec} k) = \mathrm{GW}(k)$ is defined as

$$\chi^{CW}(X/k) := \pi_{X*} e^{CW}(T_{X/k}).$$

In fact, this cohomology theory is represented by a motivic commutative ring spectrum with an SL-orientation. Let $\mathrm{EM}(\mathcal{K}_n^{MW}, n)$ be the presheaf of spectra on $\mathbf{Sm}/k$ associated to the presheaf $\mathcal{K}_n^{MW}[n]$. There is a canonical isomorphism of presheaves $X \mapsto \mathcal{K}_{n+1}^{MW}(X_+ \wedge \mathbb{G}_m)$ with $\mathcal{K}_n^{MW}$, giving the canonical weak equivalence

$$\epsilon_n : \mathrm{EM}(\mathcal{K}_n^{MW}, n) \to \Omega_T \, \mathrm{EM}(\mathcal{K}_{n+1}^{MW}, n+1)$$

Thus the sequence $(\mathrm{EM}(\mathcal{K}_0^{MW}, 0), \mathrm{EM}(\mathcal{K}_1^{MW}, 1), \ldots)$ with bonding maps $\epsilon_n$ gives rise to a $T$-spectrum $\mathrm{EM}(\mathcal{K}_*^{MW})$ with

$$\mathrm{EM}(\mathcal{K}_*^{MW})^{a+b,b}(X) \cong H^a(X, \mathcal{K}_b^{MW})$$

The Thom classes in Milnor-Witt cohomology described above thus give an SL-orientation for $\mathrm{EM}(\mathcal{K}_*^{MW})$ and one has a canonical isomorphism of $\mathrm{EM}(\mathcal{K}_*^{MW})^{a+b,b}(X; L)$ with the group $H^a(X, \mathcal{K}_b^{MW}(L))$ defined via the twisted Milnor-Witt cohomology. One shows that the resulting push-forward maps in $\mathrm{EM}(\mathcal{K}_*^{MW})$-cohomology for projective morphisms agree with those already defined for twisted Chow-Witt groups by Fasel (see [10, Theorem 5.17]). This gives us

**Theorem 1.3.3.** *Let $X$ be a smooth projective $k$ scheme. Then*

$$\chi(X/k) = \chi^{CW}(X/k)$$

*in* $\mathrm{GW}(k)$

*Proof.* By the motivic Gauss-Bonnet theorem, we have

$$\chi^{CW}(X/k) = u_*^{\mathrm{EM}(\mathcal{K}_*^{MW})}(\chi(X/k))$$

But on $\pi_0(-)_0$, the unit map $\mathbb{S} \to \mathrm{EM}(\mathcal{K}_*^{MW})$ is just Morel's isomorphism

$$\pi_0(\mathbb{S}_k)_0(k) = \mathbb{S}_k^{0,0}(k) \to \mathrm{EM}(\mathcal{K}_*^{MW})^{0,0}(k) = \mathrm{GW}(k)$$

giving us the desired identity. □

*Remark* 1.3.4. A similar argument, using the classical Chow groups instead of the Chow-Witt groups, show that

$$\mathrm{rnk}\chi(X/k) = \chi^{CH}(X/k) := \pi_{X*}(c_{\dim_k X}(T_{X/k})) \in \mathrm{CH}^0(\operatorname{Spec} k) = \mathbb{Z}.$$

Here we represent the Chow groups as the cohomology of the Milnor $K$-sheaves $\mathcal{K}^M$, and use the fact that the unit map $\mathbb{S} \to \mathrm{EM}(\mathcal{K}^M_*)$ induces the rank homomorphism $\mathrm{GW}(k) \to \mathbb{Z} = H^0(\mathrm{Spec}\, k, \mathcal{K}^M_0)$. The $H^*(-, \mathcal{K}^M_*)$ Euler class of a rank $r$ vector bundle $V \to X$ is the usual Chern class $c_r(V) \in H^r(X, \mathcal{K}^M_r) \cong \mathrm{CH}^r(X)$ (Bloch's formula).

1.4. **Hodge cohomology.** Another SL-oriented theory is algebraic $K$-theory, represented by the $T$-spectrum KGL. KGL is oriented, i.e. has Thom classes $\theta_V \in \mathrm{KGL}^{2r,r}(\mathrm{Th}(V))$ for all vector bundles $p : V \to X$, $r = \mathrm{rnk}\, V$. The Thom class is given by the class of the standard Koszul complex

$$\mathrm{Kos}(p^*V^\vee, s^\vee_{can}) : \ \Lambda^* p^* V^\vee : 0 \to \Lambda^r p^* V^\vee \to \ldots \to p^* V^\vee \xrightarrow{s^\vee_{can}} \mathcal{O}_V \to 0$$

associated to the tautological section $s_{can} : V \to p^*V$, and giving a resolution of $s_*\mathcal{O}_X$. Here $\mathrm{Kos}(p^*V^\vee, s^\vee_{can})$ is to be considered as a perfect complex on $V$ with $j^*\mathrm{Kos}(p^*V^\vee, s^\vee_{can})$ acyclic, where $j : V - 0_V \to V$ is the inclusion, i.e., $\mathrm{Kos}(p^*V^\vee, s^\vee_{can})$ defines an element of the relative $K_0$, $K_0(\mathrm{Perf}(V, V - 0_V))$. Algebraic Bott periodicity gives a canonical isomorphism

$$\mathrm{KGL}^{2r,r}(\mathrm{Th}(V)) \cong K_0(\mathrm{Perf}(V, V - 0_V))$$

and $\theta^{\mathrm{KGL}}_V = [\Lambda^* p^* V^\vee]$. Then

$$e^{\mathrm{KGL}}(V) = s^*[\Lambda^* p^* V^\vee] = \sum_{j=0}^{\mathrm{rnk} V} [\Lambda^j V^\vee[j]] = \sum_{j=0}^{\mathrm{rnk} V} (-1)^j [\Lambda^j V^\vee].$$

The pushforward associated to a proper map $f : Y \to X$ of relative dimension $d$,

$$\pi_{X*} : \mathrm{KGL}^{2r,r}(Y) \to \mathrm{KGL}^{2(r-d),r-d}(X),$$

is the usual push-forward

$$\pi_{X*}([\mathcal{E}]) = \sum_{i=0}^{\dim Y} [R^i f_* \mathcal{E}] \in K_0(X) = \mathrm{KGL}^{2(r-d),r-d}(X),$$

for $[\mathcal{E}] \in \mathrm{KGL}^{2r,r}(Y) = K_0(Y)$.

We apply this to $e(T_{X/k})$ for $\pi_X : X \to \mathrm{Spec}\, k$ smooth and projective.

$$e(T_{X/k}) = \sum_{j=0}^{\dim X} (-1)^j [\Omega^j_{X/k}]$$

and

$$\pi_{X*}(e(T_{X/k})) = \sum_{i,j=0}^{\dim X} (-1)^{i+j} [H^i(X, \Omega^j_{X/k})]$$

The unit map $u^{\mathrm{KGL}} : \mathrm{GW}(k) \to K_0(k) = \mathbb{Z}$ is the rank map, so the motivic Gauss-Bonnet theorem gives

**Theorem 1.4.1.** *For $X \in \mathbf{Sm}/k$ projective, we have*

$$\chi^{\acute{e}t}(X) = \sum_{i,j=0}^{\dim X} (-1)^{i+j} \dim_k H^i(X, \Omega^j_{X/k}).$$

Always assuming char$k \neq 2$, this can be refined to give a formula for $\chi(X/k)$ in terms of Hodge cohomology by using hermitian $K$-theory. By work of Panin-Walter [18] and Schlichting-Tripathi [20], hermitian $K$-theory is represented by a motivic commutative ring spectrum $\mathrm{KO} \in \mathrm{SH}(k)$. This is an SL-oriented theory with a direct connection with Schlichting's Grothendieck-Witt groups [19]. More precisely, there are functorial isomorphisms

$$\mathrm{KO}^{2r,r}(X;L) \cong \mathrm{GW}(D_{perf}(X), L[r], can)$$

where $L \to X$ is a line bundle and $\mathrm{GW}(D_{perf}(X), L[r], can)$ is the Grothendieck-Witt group of $L[r]$*-valued quadratic forms on* $D_{perf}(X)$.

**Definition 1.4.2.** Let $L \to X$ is a line bundle be a line bundle. An $L[n]$-valued quadratic form on $C \in D_{perf}(X)$ is a map

$$\phi : C \otimes_{\mathcal{O}_X} C \to L[n]$$

in $D_{perf}(X)$ which is

i. *non-degenerate*: the induced map $C \to \mathcal{RH}om(C, L[n])$ is an isomorphism in $D_{perf}(X)$.

ii. *symmetric*: $\phi \circ \tau = \phi$, where $\tau : C \otimes_{\mathcal{O}_X} C \to C \otimes_{\mathcal{O}_X} C$ is the symmetry isomorphism.

For a rank $r$ vector bundle $p : V \to X$, the Thom class $\theta_V^{\mathrm{KO}} \in \mathrm{KO}^{2r,r}(V; p^* \det^{-1} V)$ is given by the Koszul complex $\mathrm{Kos}(p^*V^\vee, s_{can}^\vee)$, with the quadratic form

$$\phi : \mathrm{Kos}(p^*V^\vee, s_{can}^\vee) \otimes \mathrm{Kos}(p^*V^\vee, s_{can}^\vee) \to p^* \det{}^{-1} V[r] = \Lambda^r V^\vee[r]$$

given by the usual exterior product

$$- \wedge - : \Lambda^i V^\vee \otimes \Lambda^{r-i} V^\vee \to \Lambda^r V^\vee.$$

For $f : Y \to X$ a proper map of relative dimension $d_f$ in $\mathbf{Sm}/k$, the resulting push-forward map

$$f_* : \mathrm{KO}^{2r,r}(Y, \omega_{Y/k} \otimes f^*L) \to \mathrm{KO}^{2r-2d_f, r-d_f}(X, \omega_{X/k} \otimes L)$$

is induced by Grothendieck-Serre duality (see [4], where this is worked out for Witt-theory; the same construction works for the Grothendieck-Witt groups) as follows: Given a quadratic form $\phi : C \otimes C \to \omega_{Y/k} \otimes f^*L[r]$ we have the corresponding isomorphism

$$\tilde{\phi} : C \to \mathcal{RH}om(C, \omega_{Y/k} \otimes f^*L[r]) = \mathcal{RH}om(C, \omega_{Y/X} \otimes f^*(\omega_{X/k} \otimes L[r]))$$

Grothendieck-Serre duality gives the isomorphism

$$Rf_*\mathcal{RH}om(C, \omega_{Y/X} \otimes f^*(\omega_{X/k} \otimes L[r])) \xrightarrow[\sim]{\psi} \mathcal{RH}om(Rf_*C, \omega_{X/k} \otimes L[r - d_f])$$

which gives the isomorphism

$$\psi \circ \tilde{\phi} : Rf_*C \to \mathcal{RH}om(Rf_*C, \omega_{X/k} \otimes L[r - d_f])$$

or

$$Rf_*(\phi) : Rf_*C \otimes Rf_*C \to \omega_{X/k} \otimes L[r - d_f]$$

which one shows is symmetric; explicitly, one has $f_*(C, \phi) = (Rf_*C, Rf_*(\phi))$.

If we apply this to $V = T_{X/k}$, $f = \pi_X : X \to \mathrm{Spec}\, k$ a smooth and proper $k$-scheme, we have the following formula for $\pi_{X*}(e^{\mathrm{KO}}(T_{X/k}))$

$$\pi_{X*}(e^{\mathrm{KO}}(T_{X/k})) = (\oplus_{i,j=0}^{\dim_k X} H^i(X, \Omega^j_{X/k})[j - i], \mathrm{Tr})$$

where

$$\mathrm{Tr} : (\oplus_{i,j=0}^{\dim_k X} H^i(X, \Omega^j_{X/k})[j-i]) \otimes (\oplus_{i,j=0}^{\dim_k X} H^i(X, \Omega^j_{X/k})[j-i]) \to k$$

is the quadratic form in $D^b(k)$ given by the composition

$$H^i(X, \Omega^j_{X/k}) \otimes H^{d_X - i}(X, \Omega^{d_X - j}_{X/k}) \xrightarrow{\cup} H^{d_X}(X, \Omega^{d_X}_{X/k}) \xrightarrow{\mathrm{Tr}} k,\ d_X = \dim_k X.$$

Indeed, for $s : X \to T_{X/k}$ the 0-section, we have

$$e^{\mathrm{KO}}(T_{X/k}) = s^*(\mathrm{Kos}(T_{X/k}), \phi) = (\oplus_{j=0}^{d_X} \Omega^j_{X/k}[j], s^*\phi)$$

with $s^*\phi$ given by the products

$$\Omega^j_{X/k}[j] \otimes \Omega^{d_X - j}_{X/k}[d_X - j] \to \omega_{X/k}[d_X]$$

and thus $\pi_*^{\mathrm{KO}}(e^{\mathrm{KO}}(T_{X/k}))$ is $\oplus_{i,j=0}^{d_X} H^i(X, \Omega^j_{X/k})[j-i])$ with the quadratic form Tr as described above.

Since the unit map $u^{\mathrm{KO}} : \mathbb{S} \to \mathrm{KO}$ induces the identity map

$$\mathrm{GW}(k) \cong \mathbb{S}^{0,0}(k) \xrightarrow{u_*^{\mathrm{KO}}} \mathrm{KO}^{0,0}(k) \cong \mathrm{GW}(k),$$

applying the motivic Gauss-Bonnet theorem gives the following formula proposed by Serre:

**Theorem 1.4.3** ([11, Theorem 8.6]). *Let $X$ be a smooth projective $k$-scheme. Then*

$$\chi(X/k) = (\oplus_{i,j=0}^{\dim_k X} H^i(X, \Omega^j_{X/k})[j-i], Tr).$$

**Corollary 1.4.4.** *For $X$ a smooth and projective $k$-scheme of odd dimension $2n-1$, $\chi(X/k) = m \cdot H$, $H$ the hyperbolic form $x^2 - y^2$, with*

$$m = \sum_{i+j<2n-1} \dim_k H^i(X, \Omega^j_{X/k}) + \sum_{\substack{0 \le i < j \\ i+j = 2n-1}} \dim_k H^i(X, \Omega^j_{X/k})$$

*Proof.* The quadratic form Tr reduces to perfect pairings

$$H^i(X, \Omega^j_{X/k}) \otimes H^{2n-1-i}(X, \Omega^{2n-1-j}_{X/k}) \to k$$

for $i + j < 2n - 1$, or $0 \le i < j$ and $i + j = 2n - 1$, which identifies Tr with the canonical (hyperbolic) form on $V \oplus V^\vee$, where

$$V = \bigoplus_{i+j<2n-1} H^i(X, \Omega^j_{X/k}) \oplus \bigoplus_{\substack{0 \le i < j \\ i+j = 2n-1}} H^i(X, \Omega^j_{X/k})$$

□

## References: Lecture 1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## Lecture 2. Riemann-Hurwitz formulas

We describe methods for computing the Euler characteristics and Euler classes in Chow-Witt theory based on the classical approach of fibering by a Lefschetz pencil. This gives an algebraic version of classical Morse theory, and recovers the calculation of Euler characteristics via Morse theory when looking at an $\mathbb{R}$-realization. Most of these results are taken from [4, §§8-13]

Throughout this lecture we will be working with the Euler classes and Euler characteristics in Chow-Witt theory.

2.1. **Euler class of a dual bundle and tensor product with a line bundle.** For a morphism $f : X \to Y$ one has the associated global section $df$ of $\Omega_{X/k} \otimes f^*\Omega^\vee_{Y/k}$; for $Y$ a curve, this is a twist of $T^\vee_{X/k}$ by a line bundle. The classical Riemann-Hurwitz formula (for $Y$ a curve) is simply a computation of the top Chern class of $T^\vee_{X/k} \otimes f^*\omega_{Y/k}$ in terms of the Chern classes $c_{\dim X}(\Omega_{X/k})$, $c_{\dim X-1}(\Omega_{X/k})$ and $c_1(\omega_{Y/k})$, together with the identity $c_{\dim X}(\Omega_{X/k}) = (-1)^{\dim X} c_{\dim X}(T_{X/k})$. This latter identity in the case of the Chow-Witt valued Euler class becomes

**Theorem 2.1.1.** *Let $V \to X$ be a rank $r$ vector bundle over $X \in \mathbf{Sm}/k$. Then*

$$e(V) = (-\langle -1\rangle)^r e(V^\vee) \in \widetilde{\mathrm{CH}}(X; \det^{-1} V) = \widetilde{\mathrm{CH}}(X; \det^{-1} V^\vee)$$

In case $\det V \cong \mathcal{O}_X$, this is proven by Asok-Fasel [1] using the an interpretation of the Euler class as an obstruction class. Following their approach, we have extended their proof to cover the case of a general $V$ in [4, §11].

This settles the relation of $e(T_{X/k})$ and $e(\Omega_{X/k})$, we now turn to the problem of twisting with a line bundle. The main problem here is the lack of "lower Euler classes" comparable with $c_i(V)$, $2 \le i < \mathrm{rank}(V)$; we can always use $e(\det V)$ as a refinement for $c_1(V)$. In spite of this, we have the following result

**Theorem 2.1.2** ([4, Corollary 10.9])**.** *Let $V \to X$ be a rank $r$ vector bundle on $X \in \mathbf{Sm}/k$ and $L \to X$ a line bundle. Assume that $L \cong M^{\otimes 2}$ for some line bundle $M$ on $X$. Then*

$$e(V \otimes L) = e(V) + \bar{h} \cdot c_1(M) \cdot (\sum_{i=1}^{r} c_{r-i}(V) c_1(L)^{i-1}) \in \widetilde{\mathrm{CH}}^r(X, \det^{-1} V).$$

Note that $\widetilde{\mathrm{CH}}^r(X, \det^{-1} V) \cong \widetilde{\mathrm{CH}}^r(X, \det^{-1}(V \otimes L))$, since $\det^{-1}(V \otimes L)$ and $\det^{-1} V$ differ by the square of some line bundle

We need to explain the notation

$$\bar{h} \cdot c_1(M) \cdot (\sum_{i=1}^{r} c_{r-i}(V) c_1(L)^{i-1}) \in \widetilde{\mathrm{CH}}^r(X, \det^{-1} V).$$

For this, the relation (4) defining the Milnor-Witt ring $K^{MW}_*(F)$

$$0 = \eta \cdot h = \eta \cdot (\langle 1\rangle + \langle -1\rangle)$$

tells us that the map

$$h\times : \mathcal{K}^{MW}_*(L) \to \mathcal{K}^{MW}_*(L)$$

factors through the surjection $\pi : \mathcal{K}^{MW}_*(L) \to \mathcal{K}^{MW}_*(L)/(\eta) = \mathcal{K}^M_*$. We denote the resulting "hyperbolic map" by

$$\bar{h}\cdot : \mathcal{K}^M_* \to \mathcal{K}^{MW}_*(L)$$

and use the same notation for the map in cohomology

$$\bar{h}\cdot : H^a(X, \mathcal{K}^M_b) \to H^a(X, \mathcal{K}^{MW}_b(L)).$$

One easily shows that

$$\bar{h}\cdot c_r(V) = h\cdot e(V) \in \widetilde{\mathrm{CH}}^r(X; \det^{-1} V).$$

for $V \to X$ a rank $r$ vector bundle.

To apply Theorem 2.1.2, we need the notion of local contributions to the Euler class. For this, we recall that for $V \to X$ a rank $r$ vector bundle with 0-section $s_0 : X \to V$, we have

$$e(V) = s_0^*(\theta_V)$$

where $\theta_V \in H^r_{0_V}(V; \mathcal{K}^{MW}_r(\det^{-1} V))$ is the Thom class. Letting $s : X \to V$ be an arbitrary section, it follows from $\mathbb{A}^1$-homotopy invariance that

$$s^*\theta_V = s_0^*\theta_V \in H^r(X; \mathcal{K}^{MW}_r(\det^{-1} V))$$

However, we can remember the supports, giving us the class

$$e_Z(V, s) := s^*\theta_V \in H^r_Z(X; \mathcal{K}^{MW}_r(\det^{-1} V))$$

for any closed subset $Z \supset s^{-1}(0_V)$, i.e. $Z$ containing the 0-locus of $s$.

For $\{s = 0\}$ a union of closed points $\{x_1, \ldots, x_m\}$ we have

$$H^r_{\{x_1,\ldots,x_m\}}(X; \mathcal{K}^{MW}_r(\det^{-1} V)) = \oplus_{i=1}^m H^r_{\{x_i\}}(X; \mathcal{K}^{MW}_r(\det^{-1} V))$$

giving the local contribution $e_{x_i}(V; s) \in H^r_{\{x_i\}}(X; \mathcal{K}^{MW}_r(\det^{-1} V))$ to $e(V)$.

Kass and Wickelgren [3] have given a formula for these local contributions in terms of an explicit quadratic form, originally developed by Eisenbud-Levine and Khimshiashvili in the setting of real algebraic geometry. In the case of a non-degenerate zero of $s$ at $x$ with $r = \dim X$, one has a simple description of $e_x(V; s)$. Let $\lambda_1, \ldots, \lambda_r$ be a basis of sections for $V$ near $x$, giving the local basis $\lambda_1^\vee, \ldots, \lambda_r^\vee$ for $V^\vee$ and let $t_1, \ldots, t_r \in \mathcal{O}_{X,x}$ be local parameters, giving the basis $\partial/\partial t_1, \ldots, \partial/\partial t_r$ for $(\mathfrak{m}/\mathfrak{m}^2)^\vee$. Then $s$ is given by

$$s = \sum_{i=1}^r s_i \cdot \lambda_i$$

with $s_i \in \mathfrak{m}_x = (t_1, \ldots, t_r)$, so

$$s_i = \sum_{j=1}^r a_{ij} t_j,\ a_{ij} \in \mathcal{O}_{X,x}$$

and $s$ has $x$ as a non-degenerate zero exactly when $\det(a_{ij})$ is a unit in $\mathcal{O}_{X,x}$. This gives us the rank one quadratic form $\langle \det(a_{ij}) \rangle \in \mathrm{GW}(k(x))$. The local contribution $e_x(V, s)$ is given by the element

$$\langle \det(a_{ij}) \rangle \otimes \partial/\partial t_1 \wedge \ldots \partial/\partial t_r \otimes \lambda_1^\vee \wedge \ldots \wedge \lambda_r^\vee \in \mathrm{GW}(k(x); \det^{-1} \mathfrak{m}_x/\mathfrak{m}_x^2 \otimes \det^{-1} V),$$

where we use the purity isomorphism

$$H^r_x(X, \mathcal{K}^{MW}_r(\det^{-1} V)) \cong \mathrm{GW}(k(x); \det^{-1}(\mathfrak{m}_x/\mathfrak{m}_x^2) \otimes \det^{-1} V).$$

to identify the local cohomology.

One can also apply the Kass-Wickelgren formula to the case of a "diagonal" section with a higher order zero at $x$. For $s = \sum_i u_i t_i^{n_i} \lambda_i$ with $u_i \in \mathcal{O}_{X,x}^\times$ $n_i > 0$, one has

$$e_x(V, s) = \prod_{i=1}^r (\langle \bar{u}_i \rangle \cdot n_{i\epsilon}) \otimes \partial/\partial t_1 \wedge \ldots \partial/\partial t_r \otimes \lambda_1^\vee \wedge \ldots \wedge \lambda_r^\vee$$

in $\mathrm{GW}(k(x); \det^{-1} \mathfrak{m}_x/\mathfrak{m}_x^2 \otimes \det^{-1} V)$, where $\bar{u}_i \in k(x)^\times$ is the reduction of $u_i$ and for $n > 0$, $n \in \mathbb{Z}$, we define $n_\epsilon = \sum_{i=0}^{n-1} \langle -1 \rangle^i \in \mathrm{GW}(k)$. One has the same formula for $e_x(V, s)$ if one only assumes that

$$s \equiv \sum_{i=1}^r u_i t_i^{n_i} \lambda_i \mod \mathfrak{m}_x \cdot (t_1^{n_1}, \ldots, t_r^{n_r}).$$

**Corollary 2.1.3** (Riemann-Hurwitz formula [4, Corollary 12.3])**.** *Let $f : X \to C$ be a flat morphism, with $X$ and $C$ smooth and projective over $k$ and $C$ a curve. Assume that $\omega_{C/k} \cong M^{\otimes 2}$ for some line bundle $M$ on $C$ and that $df$ has only isolated zeros $x_1, \ldots, x_m$. Then*

$$(-\langle -1 \rangle)^{d_X} \chi(X/k) = \sum_{i=1}^m Tr_{k(x_i)/k} e_{x_i}(\Omega_{X/k} \otimes f^*\omega_{C/k}^{-1}, df) + \frac{1}{2} \deg_k(c_{d_X - 1}(\Omega_{X/k}) \cdot f^* c_1(\omega_{C/k})) \cdot h \in \mathrm{GW}(k).$$

This follows from Theorem 2.1.2 applied to $V = \Omega_{X/k}$, $L = f^*\omega_{C/k}^{-1}$ and Theorem 2.1.1 applied to $V = T_{X/k}$. To make sense of the term $Tr_{k(x_i)/k} e_{x_i}(\Omega_{X/k} \otimes f^*\omega_{C/k}, df)$, we note that, if $k$ is perfect, $e_{x_i}(\Omega_{X/k} \otimes f^*\omega_{C/k}, df)$ lives in

$$\begin{aligned}&\mathrm{GW}(k(x_i); (\det^{-1} \mathfrak{m}_{x_i}/\mathfrak{m}_{x_i}^2 \otimes \omega_{X/k}^{-1} \otimes f^*\omega_{C/k}^{\otimes d_X}) \\ &\qquad = \mathrm{GW}(k(x_i); \omega_{X/k}^{-1} \otimes k(x_i) \otimes (\omega_{X/k}^{-1} \otimes f^* M^{-\otimes 2 d_X})) = \mathrm{GW}(k(x_i)).\end{aligned}$$

The trace is the usual trace map $\mathrm{Tr}_{k(x_i)/k} : \mathrm{GW}(k(x_i)) \to \mathrm{GW}(k)$, as $k(x_i)/k$ is separable. In general, one uses Fasel's pushforward map, as described in [2, §5,6].

Before we apply the Riemann-Hurwitz formula, we give some ideas toward the proof of the twisting theorem 2.1.2.

The first thing to note is that to verify this identity, we are free to pull back to any $Y \in \mathbf{Sm}/k$ such that the pullback map on $\widetilde{\mathrm{CH}}^r$ is injective. We can for instance use a Jouanolou cover and assume that $X$ is affine. We can also use homotopy invariance and replace $e(V)$ and $e(V \otimes L)$ with the pullbacks of the respective Thom classes by any section we chose, not just the 0-section.

We fix a general section $s$ of $L$ with divisor $D \subset X$. Multiplication by $s$ gives the map

$$\phi := \times s : V \to V \otimes L$$

so we compute $e(V \otimes L)$ by $v^*(\phi^* \theta_{V \otimes L})$ for $v : X \to V$ some section. We note that $\phi^{-1}(0_{V \otimes L})$ splits into two pieces

$$\phi^{-1}(0_{V \otimes L}) = \pi_V^{-1}(D) \cup 0_V$$

$\pi_V : V \to X$ the projection, and that $F := \pi_V^{-1}(D) \cap 0_V$ has codimension $r + 1$ on $V$. If $v : X \to V$ is general, then $v^{-1}(F)$ will have codimension $r + 1$ on $X$, so the restriction map $\widetilde{\mathrm{CH}}^r(X,-)\to\widetilde{\mathrm{CH}}^r(X-v^{-1}(F),-)$ is injective and we can safely remove $F$.

Letting $U=V-F$, $\phi^{-1}(0_{V\otimes L})\cap U$ splits into a disjoint union of two pieces

$$\phi^{-1}(0_{V\otimes L})\cap U=\pi_V^{-1}(D)\cap U\amalg 0_V\cap U$$

and thus $\phi^*(\theta_{V\otimes L})$ restricted to $U$ decomposes accordingly

$$j_U^*\phi^*(\theta_{V\otimes L})=\phi_{0_V}^*\theta_{V\otimes L}+\phi_D^*\theta_{V\otimes L}$$

with

$$\phi_{0_V}^*\theta_{V\otimes L}\in H^r_{0_V\cap U}(U,\mathcal{K}_r^{MW}(\det^{-1}V)),\ \phi_D^*\theta_{V\otimes L}\in H^r_{\pi_V^{-1}(D)\cap U}(U,\mathcal{K}_r^{MW}(\det^{-1}(V\otimes L))).$$

Using $\times s$ to identify $\det V$ with $\det V\otimes L$ over $U-\pi^{-1}(D)$ gives

$$\phi_{0_V}^*\theta_{V\otimes L}=j_U^*\theta_V$$

For the other term, we restrict to $U'=U-0_V$, noting that $\pi_V^{-1}(D)\cap U=\pi_V^{-1}(D)\cap U'$. Over $U'$ the tautological section $s':\mathcal{O}_V\to V$ of $V$ gives the rank $r-1$ vector bundle $W:=j_{U'}^*(V)/s'(\mathcal{O}_{U'})$ and the exact sequence

$$0\to j_{U'}^*L\xrightarrow{s'}j_{U'}^*V\otimes L\to W\otimes L\to 0$$

Pulling back again to a Jouanolou cover of $U'$ to split this sequence, and using the multiplicativity of the Thom classes gives the identity

$$\phi_D^*\theta_{V\otimes L}=j_{U'}^*e_D(L,s)\cup e(W\otimes L)$$

Now we use the assumption that $L\cong M^{\otimes 2}$. We can take the section $s$ to be of the form $s=t^{\otimes 2}$ for $t$ a suitably general section of $M$. Using the relation in $K_1^{MW}$:

$$[x^2]=(\langle 1\rangle+\langle -1\rangle)[x]=h\cdot[x]$$

gives

$$e_D(L,s)=e_D(L,t^{\otimes 2})=h\cdot e_D(M)=\bar{h}\cdot c_{1D}(M)$$

and thus

$$\begin{aligned}j_{U'}^*e_D(L,s)\cup e(W\otimes L)&=h\cdot(j_{U'}^*(e_D(M))\cup e(W\otimes L))\\&=\bar{h}\cdot(j_{U'}^*c_{1D}(M)\cup c_{r-1}(W\otimes L))\\&=\bar{h}\cdot((j_{U'}^*c_{1D}(M)\cup\sum_{i=0}^{r-1}c_{r-1-i}(W)\cup j_{U'}^*c_1(L)^i)\end{aligned}$$

As the exact sequence $0\to\mathcal{O}_{U'}\to j_{U'}^*V\to W\to 0$ gives $c_i(W)=j_{U'}^*c_i(V)$ for $i=0,\dots,r-1$, we have

$$\phi_D^*\theta_{V\otimes L}=j_{U'}^*(\bar{h}\cdot c_{1D}(M)\cup\sum_{i=1}^{r}c_{r-i}(V)\cup c_1(L)^{i-1})$$

Noting that $\pi_V^{-1}(D)\cap U'=\pi_V^{-1}(D)\cap U$, pulling back

$$j_U^*\phi^*\theta_{V\otimes L}=\phi_{0_V}^*\theta_{V\otimes L}+\phi_D^*\theta_{V\otimes L}$$

by a general section $v:X\to V$ gives the desired identity in $H^r(X-v^{-1}(F),\mathcal{K}_r^{MW}(\det^{-1}V\otimes L))$, which gives the identity over $X$ since $v^{-1}(F)$ has codimension $\geq r+1$ on $X$.

2.2. **Applications.**

**Theorem 2.2.1** ([4, Theorem 12.7]). *Let $f : X \to C$ be a separable morphism of smooth projective curves $X, C \in \mathbf{Sm}/k$. Suppose that $\omega_{C/k} \cong M^{\otimes 2}$ for some $M \in \mathrm{Pic}(C)$ and let $x_1, \dots, x_m \in X$ be the zeros of $df$. Then*

$$\sum_{i=1}^{m} Tr_{k(x_i)/k} e_{x_i}(\omega_{X/C}, df) \in \mathrm{GW}(k)$$

*is hyperbolic.*

*Proof.* This follows from the fact that $\chi(X/k)$ is hyperbolic, as $X$ has odd dimension 1, and the Riemann-Hurwitz formula 2.1.3. □

*Remark* 2.2.2. We can make this statement more concrete, for example, if $C = \mathbb{P}^1$. We assume that $y_i := f(x_i) \in \mathbb{A}^1 = \operatorname{Spec} k[t] = \mathbb{P}^1 - \{\infty\}$ for all $i = 1, \dots, m$ and that $f$ is tamely ramified. Let $g_i \in k[t]$ be the normalized irreducible polynomial of $y_i$ over $k$ and let $s_i := g_i(t)/g_i'(t) \in \mathfrak{m}_{y_i}$. Then $ds_i = dt$. We have an isomorphism $\omega_{\mathbb{P}^1/k} \cong \mathcal{O}_{\mathbb{P}^1}(-2)$, and we may chose this isomorphism so that $dt$ is the image $\lambda^{\otimes 2}$ for $\lambda$ a generator of $\mathcal{O}_{\mathbb{P}^1}(-1)$ over $\mathbb{A}^1$. This trivializes $f^*\omega_{\mathbb{P}^1/k}$ at all the points $x_i$, giving the isomorphism

$$\mathrm{GW}(k(x_i), \omega_{X/k}^{-1} \otimes f^*\omega_{\mathbb{P}^1/k} \otimes \omega_{X/k}^{-1}) \cong \mathrm{GW}(k(x_i)).$$

explicitly, we write $f^*(s_i) = u_i t_i^{n_i}$, then via this trivialization we have $df = d(f^*(s_i)) = n_i u_i t_i^{n_i - 1} dt_i$, giving the local contribution

$$e_{x_i}(\omega_{X/k} \otimes f^*\omega_{\mathbb{P}^1/k}^{-1}, df) = \langle n_i \bar{u}_i \rangle \cdot (n_i - 1)_\epsilon$$

As

$$\langle n_i \bar{u}_i \rangle \cdot (n_i - 1)_\epsilon = \begin{cases} [(n_i - 1)/2] \cdot h & \text{for } n_i \text{ odd,} \\ \langle n_i \bar{u}_i \rangle + [(n_i - 2)/2] \cdot h & \text{for } n_i \text{ even,} \end{cases}$$

the Riemann-Hurwitz identity gives an identity on the "leading terms" $u_i$:

$$\sum_{n_i \text{ even}} \mathrm{Tr}_{k(x_i)/k} \langle n_i \bar{u}_i \rangle \text{ is hyperbolic in } \mathrm{GW}(k).$$

As another application, we have

**Theorem 2.2.3** ([4, Theorem 13.1]). *Let $X \subset \mathbb{P}_k^{2n+1}$ be a smooth hypersurface defined by an equation of the form $\sum_{i=0}^{2n+1} a_i T_i^m$. Then there are integers $A_{n,m}$ depending only on $n$ and $m$ such that*

$$\chi(X/k) = \begin{cases} A_{n,m} \cdot h + \langle m \rangle & \textit{for } m \textit{ odd,} \\ A_{n,m} \cdot h + \langle m \rangle + \langle -m \prod_{i=0}^{2n+1} a_i \rangle & \textit{for } m \textit{ even.} \end{cases}$$

In fact, as $\mathrm{rank}(\chi(X/k)) = \deg_k c_{2n}(T_{X/k})$ and

$$c_{2n}(T_{X/k}) = i_X^* \left[ \frac{(1+H)^{2n+2}}{1+mH} \right]_{2n},$$

where $H = c_1(\mathcal{O}(1))$ is the hyperplane class and $[-]_{2n}$ means the degree $2n$ component, we have the following formula for $A_{n,m}$:

$$A_{n,m} = \begin{cases} (\deg \left[ \frac{mH \cdot (1+H)^{2n+2}}{1+mH} \right]_{2n+1} - 1)/2 & \text{for } m \text{ odd,} \\ (\deg \left[ \frac{mH \cdot (1+H)^{2n+2}}{1+mH} \right]_{2n+1} - 2)/2 & \text{for } m \text{ even.} \end{cases}$$

The idea of the proof is to use the Lefschetz pencil with base-locus $T_{2n} = T_{2n+1} = 0$. The intersection with $X$ is the hypersurface $Z \subset \mathbb{P}^{2n-1}$ defined by $\sum_{i=0}^{2n-1} a_i T_i^m$. Letting $\tilde{X} \to X$ be the blow-up along $Z$ and $f : \tilde{X} \to \mathbb{P}^1$ the morphism arising from the pencil, we use the blow-up formula to compute

$$\chi(\tilde{X}/k) = \chi(X/k) + \langle -1\rangle \chi(Z/k)$$

The map $f$ is ramified at the closed subscheme $x$ of $X$ defined by $T_0 = \ldots = T_{2n-1} = 0$. We compute the local contribution $e_x(\Omega_{\tilde{X}/k} \otimes f^*\omega^{-1}_{\mathbb{P}^1/k}, df)$ and then use the Riemann-Hurwitz formula to compute $\chi(\tilde{X}/k)$. We then use induction to compute $\chi(Z/k)$, giving the formula for $\chi(X/k)$.

**Corollary 2.2.4.** *Let $Q \subset \mathbb{P}^{2n+1}$ be a smooth quadric hypersurface of even dimension $2n$ defined by the quadratic polynomial $q \in k[T_0, \ldots, T_{2n+1}]$. Then*

$$\chi(Q/k) = n \cdot h + \langle 2\rangle + \langle -2\delta_q\rangle$$

*where $\delta_q$ is the discriminant of $q$; here we use the convention that $\delta_q = \det Q$, where $Q$ is a matrix of the symmetric bilinear form associated to $q$.*

This follows from the well-know topological Euler characteristic $2n + 2$ and the fact that we can diagonalize $q$.

## References: Lecture 2


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## Lecture 3. Pontryagin classes, splitting principles and Becker-Gottlieb transfers

We have already mentioned that the lack of Chow-Witt versions of the "middle" Chern classes make the theory of the Euler class somewhat unwieldy. This can be remedied by passing from the Chow-Witt groups to a simpler theory, namely, (twisted) Witt cohomology $H^*(-, \mathcal{W}(L))$. This is simply the localization of Milnor-Witt cohomology with respect to $\eta$: $\mathcal{W}(L) = \mathcal{K}_*^{MW}(L)[\eta^{-1}]$. Thus, passing to Witt cohomology we have an SL-oriented and $\eta$-invertible theory.

### 3.1. Borel classes, Pontryagin classes and Ananyevskiy's $\mathrm{SL}_2$ splitting principle.

**Definition 3.1.1.** Let $\mathcal{E} \in \mathrm{SH}(k)$ be a motivic commutative ring spectrum. We call $\mathcal{E}$ an *$\eta$-invertible* if $\times\eta : \mathcal{E}^{a,b}(k) \to \mathcal{E}^{a-1,b-1}(k)$ is an isomorphism for all $a, b$.

For an $\eta$-invertible theory, $\times\eta : \mathcal{E}^{a,b}(\mathcal{X}) \to \mathcal{E}^{a-1,b-1}(\mathcal{X})$ is an isomorphism for all $\mathcal{X} \in \mathbf{Spc}_\bullet(k)$ or even $\mathcal{X} \in \mathrm{SH}(k)$.

Ananyevskiy [1] has studied theories $\mathcal{E}$ which are both SL-oriented and $\eta$-invertible. For such theories, the *Pontryagin classes* of vector bundles give a good theory of characteristic classes; we review briefly how these are defined.

Analogous to the notion of an SL-orientation [1, Definition 1] is that of a *symplectic orientation*. A symplectic bundle is a vector bundle $V \to X$ equipped with a non-degenerate alternating form $\omega : V \wedge V \to \mathcal{O}_X$. Then $V$ has even rank, say $2m$ and then $\wedge^m\omega$ trivializes $\det V$, so $V$ is also an $\mathrm{SL}_{2m}$-bundle.

The analog of the projective space $\mathbb{P}^n = \mathrm{Gr}(1, n+1)$ and the tautological line bundle $O_{\mathbb{P}^n}(1)$ in the setting of symplectic bundles is the symplectic projective space $H\mathbb{P}^n \subset \mathrm{Gr}(2, 2n+2)$. Give $V = \mathbb{A}^{2n+2}$ the standard symplectic form

$$\omega_n := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \in M_{2n+2}$$

and let $H\mathbb{P}^n \subset \mathrm{Gr}(2, 2n+2)$ be the open subscheme parametrizing two-planes $\pi \subset \mathbb{A}^{2n+2}$ such that $\omega_n$ restricts to a non-degenerate form on $\pi$. The tautological rank two subbundle $\mathcal{E}_2 \to \mathrm{Gr}(2, 2n+2)$ thus restricts to a rank two symplectic bundle $(O_{H\mathbb{P}^n}(-1) \to H\mathbb{P}^n, \omega)$.

A symplectic orientation for a commutative ring spectrum $\mathcal{E}$ is a functorial choice of classes $\theta^{\mathcal{E}}_{V,\omega} \in \mathcal{E}^{4r,2r}(X)$ for each rank $2r$ symplectic bundle $(V \to X, \omega)$, such that for the "trivial" bundle $(V = \mathcal{O}_X^r \oplus \mathcal{O}_X^r, \omega_r)$, $\theta_{V,\omega}$ corresponds to the unit of $\mathcal{E}$ under the suspension isomorphism and one has a product formula for direct sums [5, §7].

**Definition 3.1.2.** For $\mathcal{E}$ a symplectically oriented motivic ring spectrum, the 1st Borel class $b_1(E_2, \omega) \in \mathcal{E}^{4,2}(X)$ of a rank two symplectic bundle $(E_2 \to X, \omega)$ is the Euler class $s_0^*\theta^{\mathcal{E}}_{E_2,\omega}$.

An SL-oriented $\mathcal{E}$ is also symplectically oriented, and $b_1(E_2, \omega)$ is just the usual Euler class in $\mathcal{E}^{4,2}(X, \det^{-1}(E_2))$, where we use the isomorphism $\omega : E_2 \wedge E_2 \to \mathcal{O}_X$ to trivialize $\det E_2$, giving the identification $\mathcal{E}^{4,2}(X, \det^{-1}(E_2)) \cong \mathcal{E}^{4,2}(X)$.

**Theorem 3.1.3** (Panin-Walter [5, Theorem 8.1])**.** *Let $\mathcal{E}$ be a symplectically oriented commutative motivic ring spectrum. Then $\mathcal{E}^{*,*}(H\mathbb{P}^n)$ is a truncated polynomial ring over $\mathcal{E}^{*,*}(k)$, with generator $b_1(O_{H\mathbb{P}^n}(-1), \omega)$ and relation $b_1^{n+1} = 0$:*

$$\mathcal{E}^{*,*}(H\mathbb{P}^n) = \mathcal{E}^{*,*}(k)[b_1]/(b_1^{n+1})$$

Panin-Walter follow the Grothendieck playbook to construct higher *Borel classes*[1] $b_1(E,\omega),\ldots,b_r(E,\omega)$ for $(E,\omega)$ a rank $2r$ symplectic bundle,

$$b_i(E,\omega)\in\mathcal{E}^{4i,2i}(X)$$

For $V\to X$ a rank $r$ vector bundle, we have the rank $2r$ vector bundle $V\oplus V^\vee$ with canonical symplectic form $\omega_V$:

$$\omega_V(s,s^\vee),(t,t^\vee):=s^\vee(t)-t^\vee(s).$$

**Definition 3.1.4** ([1, Definition 19])**.** Let $\mathcal{E}$ be an $\eta$-invertible, SL-oriented theory and let $V\to X$ be a rank $r$ vector bundle. Define the $i$th Pontryagin class $p_i(V)\in\mathcal{E}^{8i,4i}(X)$, $i=1,\ldots,[r/2]$ by

$$p_i(V):=(-1)^i b_{2i}(V\oplus V^\vee,\omega_V)$$

We set $p_0(V)=1_X$ and $p(V)=\sum_{i=0}^{[r/2]}p_i(V)$.

It turns out that the odd Borel classes of $(V\oplus V^\vee,\omega_V)$ are zero. This gives the Whitney formula: for

$$0\to V'\to V\to V''\to 0$$

an exact sequence of vector bundles on some $X\in\mathbf{Sm}/k$ we have

$$p(V)=p(V')\cdot p(V'').$$

In addition, for $V$ of even rank $r=2m$, one has

$$p_m(V)=e(V)^2\in\mathcal{E}^{8m,4m}(\mathcal{E},\det^{-2}V)=\mathcal{E}^{4m,2m}(\mathcal{E}).$$

We recall that for an odd rank bundle $V$, one has $\eta\cdot e(V)=0$, so $e(V)=0$ in an SL-oriented, $\eta$-invertible theory.

Ananyevskiy proves a number of fundamental results about the Pontryagin classes in an $\eta$-invertible, SL-oriented theory. We mention two of these.

We have the classifying space $\mathrm{BSL}_n$ with universal rank $n$ bundle $\tilde{E}_n\to\mathrm{BSL}_n$.

**Theorem 3.1.5** (Ananyevskiy [1, Theorem 10])**.** *Let $p_i=p_i(\tilde{E}_n)$, $i=1,\ldots,[n/2]$, and for $n=2m$, let $e=e(\tilde{E}_n)$. Let $\mathcal{E}$ be an* SL*-oriented, $\eta$-invertible commutative motivic ring spectrum. Then*

$$\mathcal{E}^{**}(\mathrm{BSL}_n)=\begin{cases}\mathcal{E}^{**}(k)[p_1,\ldots,p_m] & \text{for } n=2m+1\\ \mathcal{E}^{**}(k)[p_1,\ldots,p_m,e]/(p_m-e^2) & \text{for } n=2m\end{cases}$$

**Theorem 3.1.6** (Ananyevskiy [1, Theorem 6, Theorem 10])**.** *Let $i_m:\mathrm{BSL}_2^m\to\mathrm{BSL}_{2m}$ be the map induced by the diagonal inclusion $\mathrm{SL}_2^m\to\mathrm{SL}_{2m}$ and let $\mathcal{E}$ be an* SL*-oriented, $\eta$-invertible commutative motivic ring spectrum. Then*

*1. $\mathcal{E}^{**}(\mathrm{BSL}_2^m)=\mathcal{E}^{**}(\mathrm{BSL}_2)^{\otimes_{\mathcal{E}^{**}}m}=\mathcal{E}^{**}(k)[e_1,\ldots,e_m]$, where $e_i$ is the pullback $p_i^*e(\tilde{E}_2)$ via the projection onto the $i$th factor $p_i:\mathrm{BSL}_2^m\to\mathrm{BSL}_2$.*

*2. The pull-back*

$$i_m^*:\mathcal{E}^{**}(\mathrm{BSL}_{2m})\to\mathcal{E}^{**}(\mathrm{BSL}_2^m)$$

*is injective, and one has*

$$i_m^*p(\tilde{E}_{2m})=\prod_{i=1}^m(1+e_i^2),\ \ i_m^*e(\tilde{E}_{2m})=\prod_{i=1}^m e_i.$$

[1]Unfortunately, these classes are called "Pontryagin classes" in [5], but fortunately, this terminology has been replaced with the more appropriate term "Borel classes" in the literature.

3.2. **Becker-Gottlieb transfers.** We are interested in the following problem: For $\rho : \mathrm{GL}_n \to \mathrm{GL}_m$ a representation, one has the associated functor $S_\rho$ from rank $n$ vector bundles to rank $m$ vector bundles. How can one compute $p(S_\rho E)$ and $e(S_\rho E)$ in terms of $p(E)$ and $e(E)$? A related question: Given bundles $E_1$, $E_2$ what is $p(E_1 \otimes E_2)$ and $e(E_1 \otimes E_2)$ in terms of the Pontryagin and Euler classes of $E_1$ and $E_2$?

For the Chern classes, that is, for characteristic classes in an oriented theory, the answer is given by the classical splitting principle, together with the fact that $\rho$ is determined by its restriction to the maximal torus $T \cong \mathbb{G}_m^n$ of $\mathrm{GL}_n$. This reduces the question of computing these characteristic classes to a problem in symmetric functions for the Weyl group $S_n$ of $\mathrm{GL}_n$. Ananyevskiy's two theorems described above reduce this problem, at least for bundles with trivialized determinant, to the case of rank two bundles. However, as $\mathrm{SL}_2$ has irreducible representations of arbitrary rank (namely, the $k$th symmetric power representation) the $\mathrm{SL}_2$-splitting principle on its own does not give a complete answer.

A first approach to the problem of finding a good splitting principle is furnished by a motivic version of Becker-Gottlieb transfers. In classical homotopy theory, this is the construction of a stable "wrong way" map $p^* : \Sigma^\infty B_+ \to \Sigma^\infty E_+$ for a fiber bundle $p : E \to B$ with fiber $F$ having the homotopy type of a finite CW complex. This transfer map has the property that, at least for oriented theories, $p \circ p^*$ induces multiplication by the Euler characteristic $\chi^{top}(F)$.

In the setting of motivic homotopy theory, Hoyois has constructed a transfer map for $p : E \to B$ smooth and projective. Here we extend this construction to a wider class of maps, using the Grothendieck six-functor formalism on $\mathrm{SH}(-) : \mathbf{Sch}/S \to \mathbf{Tr}$ introduced in § 1.2.

For $f : X \to Y$ in $\mathbf{Sch}/S$ we have the adjoint pairs $f^* \dashv f_*$, $f_! \dashv f^!$. For $f$ smooth, $f^*$ has the left adjoint $f_\#$ and the infinite suspension spectrum $\Sigma^\infty_T X_+ \in \mathrm{SH}(Y)$ is just $f_\#(1_X)$. We write $X/Y$ for $f_\#(1_X)$; the map $f$ induces the map $f/Y : X/Y \to Y/Y = 1_Y$.

Suppose we have $f : E \to B \in \mathbf{Sm}/B$ such that $E/B$ is dualizable with dual $(E/B^\vee, \delta_{E/B}, ev_{E/B})$. We define the transfer map

$$\mathrm{Tr}_{f/B} : 1_B \to E/B$$

in $\mathrm{SH}(B)$ as follows: Upon applying $f_\#$, the diagonal $\Delta_E : E \to E \times_B E$ gives the map

$$\Delta_E/B : E/B \to E \times_B E/B \cong E/B \wedge_B E/B$$

Duality gives the adjunction

$$(-)^{\mathrm{tr}} : \mathrm{Hom}_{\mathrm{SH}(B)}(E/B, E/B \wedge_B E/B) \cong \mathrm{Hom}_{\mathrm{SH}(B)}(E/B^\vee \wedge_B E/B, E/B)$$

so we have the morphism

$$\Delta_E/B^{\mathrm{tr}} : E/B^\vee \wedge_B E/B \to E/B$$

Define $\mathrm{Tr}_{f/B}$ as the composition

$$1_B \xrightarrow{\delta_{E/B}} E/B \wedge_B E/B^\vee \xrightarrow{\tau_{E/B,E/B^\vee}} E/B^\vee \wedge_B E/B \xrightarrow{\Delta_E/B^{\mathrm{tr}}} E/B$$

Applying $f_\#$ gives the maps

$$\mathrm{Tr}_{f/S} : B/S \to E/S,\ f/S : E/S \to B/S$$

Since $E/B$ is dualizable in $\mathrm{SH}(B)$, we have the Euler characteristic $\chi(E/B) \in \mathrm{End}_{\mathrm{SH}(B)}(1_B)$.

**Theorem 3.2.1** (Motivic Becker-Gottlieb transfers [3, Lemma 1.7, Theorem 1.10])**.**
*1. For $f : E \to B$ in $\mathbf{Sm}/B$ with $E/B$ dualizable, we have*

$$f/B \circ Tr(f/B) = \chi(E/B) : 1_B \to 1_B$$

*2. Take $F$ and $B$ in $\mathbf{Sm}/S$ with $F/S$ dualizable in $\mathrm{SH}(S)$. Let $f : E \to B$ be a Nisnevich locally trivial fiber bundle with fiber $F$. Then $E/B$ is dualizable in $\mathrm{SH}(B)$.*
*3. Take $\mathcal{E} \in \mathrm{SH}(S)$ so that $\chi(F/S) : 1_S \to 1_S$ induces an isomorphism on $\mathcal{E}^{0,0}(S)$. Then the map*

$$f/S^* : \mathcal{E}^{**}(B/S) \to \mathcal{E}^{**}(E/S)$$

*is split injective.*

The assertion (1) is a rather easy diagram chase from the definitions. For (2), one uses May's theorem [4] that the Euler characteristic for $\mathrm{SH}(B)$ is additive in exact sequences to reduce to the case of a product, for which the assertion is easy. For (3), it suffices to show that the map $\mathrm{Tr}(E/S) \circ f/S^*$ is an isomorphism. In the case of a product, this is just (1), noting that $\chi(F \times_S B/B) = \pi_B^* \chi(F/S)$. One then makes a Nisnevich descent argument to reduce to this case.

3.3. **Splitting principles.** We would like to apply this to the $\mathcal{E}$-cohomology of $BG$ for $G$ a reductive group. Take for example $G = \mathrm{GL}_n$ and let $i : T \to \mathrm{GL}_n$ be the inclusion of the maximal torus of diagonal matrices. We have the homotopy fiber sequence

$$T\backslash \mathrm{GL}_n \to BT \xrightarrow{Bi} \mathrm{BGL}_n$$

We note that $T\backslash \mathrm{GL}_n$ is $\mathbb{A}^1$ homotopy equivalent to $B\backslash \mathrm{GL}_n$, where $B \supset T$ is a Borel subgroup, giving the full flag manifold $\mathcal{F}l_n = B\backslash \mathrm{GL}_n$. Factoring $\mathcal{F}l_n$ as a tower of projective space bundles and using the projective bundle theorem, this says that for $\mathcal{E}$ an orientable theory, the map

$$Bi^* : \mathcal{E}^{**}(\mathrm{BGL}_n) \to \mathcal{E}^{**}(BT)$$

is injective and in fact identifies $\mathcal{E}^{**}(\mathrm{BGL}_n)$ with the invariants in $\mathcal{E}^{**}(BT)$ for the action of the Weyl group $W(\mathrm{GL}_n) = S_n$.

However, this does not hold for non-oriented theories, even for SL-oriented theories. In fact, if we have an SL-oriented and $\eta$-invertible theory, the map $Bi^*$ maps all the non-trivial elements in $\mathcal{E}^{**}(\mathrm{BGL}_n)$ to zero.

One can still recover a splitting principle that holds for every theory $\mathcal{E}$ by replacing the $W(\mathrm{GL}_n)$-invariants in $\mathcal{E}^{**}(BT)$ with $\mathcal{E}^{**}(BN_T)$, where $N_T$ is the normalizer of $T$ in $\mathrm{GL}_n$. In fact $N_T \cong T \rtimes S_n$, with $S_n \subset \mathrm{GL}_n$ the subgroup of permutation matrices.

We have the following conjecture (formulated by Fabien Morel)

**Conjecture 3.3.1.** *Let $G$ be a split reductive group scheme over a perfect field $k$ with maximal split torus $T$ and $N_T$ the normalizer of $T$. Then*

**Strong form** $\chi(N_T\backslash G) = 1$

**Weak form** $\chi(N_T\backslash G)$ *is invertible in* $\mathrm{GW}(k)$.

It follows using the motivic Becker-Gottlieb transfers that for $G$ special (i.e., $G = \mathrm{GL}_n, \mathrm{SL}_n, \mathrm{Sp}_n$) the above conjecture implies the splitting principle for the normalizer: that $Bi^*_{N_T} : BG \to BN_T$ is injective. Indeed, for $G$ special the bundle

$$N_T\backslash G \to BN_T \to BG$$

is Zariski locally trivial (over the smooth finite type schemes that approximate $BG$) and then one can apply Theorem 3.2.1, assuming the weak form of the conjecture.

We sketch a proof below of:

**Theorem 3.3.2.** *The weak form of the conjecture holds for $G = \mathrm{GL}_n, \mathrm{SL}_n$ over a characteristic zero field, and the strong form holds for $G = \mathrm{GL}_2, \mathrm{SL}_2$ over a perfect field.*

In order to prove the weak form of the conjecture for a given group scheme $G$ over a field of characteristic zero, it suffices to take $G$ over $\mathbb{Q}$. We need to show that $\chi(N_T\backslash G)$ is invertible in $\mathrm{GW}(\mathbb{Q})$; the known structure of $\mathrm{GW}(\mathbb{Q})$ implies that it suffices to show that $\chi(N_T\backslash G)$ has both rank and signature equal to one, in other words, that

$$\chi^{top}((N_T\backslash G)(\mathbb{C})) = 1 = \chi^{top}((N_T\backslash G)(\mathbb{R})).$$

The identity $\chi^{top}((N_T\backslash G)(\mathbb{C}))$ is well-known: Since $\mathbb{C}$ is algebraically closed, we have

$$(N_T\backslash G)(\mathbb{C}) \cong N_T(\mathbb{C})\backslash G(\mathbb{C}) \cong W(G(\mathbb{C}))\backslash (T(\mathbb{C})\backslash G(\mathbb{C}))$$

Now $T(\mathbb{C})\backslash G(\mathbb{C})$ is homotopy equivalent to $B(\mathbb{C})\backslash G(\mathbb{C})$ and the Bruhat decomposition

$$B(\mathbb{C})\backslash G(\mathbb{C}) = \amalg_{w\in W(G(\mathbb{C}))} B(\mathbb{C})\backslash B(\mathbb{C})wB(\mathbb{C})$$

gives a cell decomposition of $B(\mathbb{C})\backslash G(\mathbb{C})$ with $\#W(G(\mathbb{C}))$ even dimensional cells. Thus $\chi(T(\mathbb{C})\backslash G(\mathbb{C})) = \#W(G(\mathbb{C}))$ and since

$$T(\mathbb{C})\backslash G(\mathbb{C}) \to N_T(\mathbb{C})\backslash G(\mathbb{C})$$

is a covering space of degree $\#W(G(\mathbb{C}))$, it follows that

$$\chi^{top}((N_T\backslash G)(\mathbb{C})) = 1.$$

For the computation of $\chi^{top}((N_T\backslash G)(\mathbb{R}))$, we restrict to the case $G = \mathrm{GL}_n$.

$N_T = \mathbb{G}_m^n \rtimes S_n$, so we have

$$N_T\backslash \mathrm{GL}_n = S_n\backslash(\mathbb{G}_m^n\backslash \mathrm{GL}_n)$$

and $\mathbb{G}_m^n\backslash \mathrm{GL}_n$ is the open subscheme of $(\mathbb{P}^{n-1})^n$ parametrizing $n$-tuples of lines through 0 in $\mathbb{A}^n$ which span $\mathbb{A}^n$. This realizes $N_T\backslash \mathrm{GL}_n$ as an open subscheme of $\mathrm{Sym}^n\mathbb{P}^{n-1}$, and gives a decomposition of the $\mathbb{R}$-points $x \in (N_T\backslash \mathrm{GL}_n)(\mathbb{R})$ in terms of the residue fields of the corresponding closed points $y_1, \ldots, y_s$ of $(\mathbb{P}^{n-1})^n$ lying over $x$. This decomposes $(N_T\backslash \mathrm{GL}_n)(\mathbb{R})$ as a disjoint union of open submanifolds

$$(N_T\backslash \mathrm{GL}_n)(\mathbb{R}) = \amalg_{i=0}^n U_i$$

where $U_i$ parametrizes the spanning collections of lines with exactly $i$ real lines. Of course $U_i$ is empty if $n-i$ is odd.

We claim that $\chi^{top}(U_i) = 0$ if $i \geq 2$. Indeed, we can form a finite covering space $\tilde{U}_i \to U_i$ by killing the permutation action on the real lines and choosing an orientation on each real line. Then putting a metric on $\mathbb{C}^n$, we see that $\tilde{U}_i$ is homotopy equivalent to an $SO(i)$-bundle over the space parametrizing the (unordered) collection of the remaining lines. Since $\chi^{top}(SO(i)) = 0$ for $i \geq 2$, we have $\chi^{top}(\tilde{U}_i) = 0$ and hence $\chi^{top}(U_i) = 0$ as well.

Now assume $n = 2m$ is even. $U_0$ parametrizes the unordered collections of $m$ pairs of complex conjugate lines that span $\mathbb{C}^n$, so we may form the degree $m!$ covering space $\tilde{U}_0 \to U_0$ of ordered $m$-tuples of such conjugate pairs. Each pair $\ell, \bar{\ell}$ spans a $\mathbb{A}^2 \subset \mathbb{A}^n$ defined over $\mathbb{R}$, so we may map $\tilde{U}_0$ to the real points of the quotient $(\mathrm{GL}_2)^m \backslash \mathrm{GL}_n$, which is homotopy equivalent to the real points of the partial flag manifold $Fl_m := P_m \backslash \mathrm{GL}_n$, where $P_m$ is the parabolic with Levi subgroup $(\mathrm{GL}_2(\mathbb{C}))^m$. In fact, the map $\tilde{U}_0 \to Fl_m(\mathbb{R})$ is also a homotopy equivalence: given a two-plane $\pi \subset \mathbb{A}^n$ defined over $\mathbb{R}$, the pairs of conjugate lines in $\pi$ are parametrized by the upper half-plane in the Riemann sphere $\mathbb{P}(\pi)$, which is contractible.

We claim that $\chi^{top}(Fl_m(\mathbb{R})) = m!$ for all $m \geq 2$. To compute the Euler characteristic, we use the following fact:

If $X \in \mathbf{Sm}/\mathbb{R}$ is *cellular*, that is, $X$ admits a finite stratification by locally closed subschemes $X_i \cong \mathbb{A}^{n_i}$, then $\chi^{top}(X(\mathbb{R})) = \mathrm{rnk}\, \mathrm{CH}^{even}(X) - \mathrm{rnk}\, \mathrm{CH}^{odd}(X)$.

This follows from [2, Remark 1.11, Proposition 1.14], using the $\mathrm{GW}(\mathbb{R})$-valued Euler characteristic $\chi(X/\mathbb{R})$:
(3.3.1)
i. For $X \in \mathbf{Sm}/\mathbb{R}$, $\chi^{top}(X(\mathbb{R}))$ is equal to the signature of $\chi(X/\mathbb{R})$ [2, Remark 1.11].
ii. For $X \in \mathbf{Sm}/k$ cellular, $\chi(X/k) = \mathrm{rnk}\, \mathrm{CH}^{even} \cdot \langle 1 \rangle + \mathrm{rnk}\, \mathrm{CH}^{odd} \cdot \langle -1 \rangle$ [2, Proposition 1.14].

Alternatively, one can apply the Lefschetz trace formula to complex conjugation $c$ acting on $H^*(X(\mathbb{C}), \mathbb{Q})$, and use the fact that for $X$ cellular, $H^{odd}(X(\mathbb{C}), \mathbb{Q}) = 0$, $H^{2n}(X(\mathbb{C}), \mathbb{Q})$ has dimension $\mathrm{rnk}\, \mathrm{CH}^n(X)$ and $c$ acts by $(-1)^n$, since the cycle class map $\mathrm{cl}^n$ is a $c$-equivariant isomorphism from the $c$-invariant $\mathbb{Q}$ vector space $\mathrm{CH}^n(X) \otimes \mathbb{Q}$ to $H^{2n}(X(\mathbb{C}), \mathbb{Q}(2\pi i)^n) = \mathbb{Q} \cdot \mathrm{cl}^n(\mathrm{CH}^n(X))$.

For $m = 2$, $Fl_2 = \mathrm{Gr}(2, 4)$, which has 4 Schubert cells of even dimension and 2 of odd dimension. Thus $\chi(Fl_2/\mathbb{R}) = 4\langle 1 \rangle + 2\langle -1 \rangle$, which has signature 2, and hence $\chi^{top}(Fl_2(\mathbb{R})) = 2$.

In general, we have the fibration $Fl_m \to Fl_{m-1}$ with fiber $\mathrm{Gr}(2, 2m)$. We can compare the Schubert cells for $\mathrm{Gr}(2, 2m)$ with those for $\mathrm{Gr}(2, 2m-2)$: the "new" ones come from the partitions $(2m-3, i)$, $i = 0, \ldots, 2m-3$ and $(2m-2, i)$, $i = 0, \ldots, 2m-2$. This adds the quadratic form $(2m-1)\langle 1 \rangle + (2m-2)\langle -1 \rangle$ of signature $+1$ to $\chi(\mathrm{Gr}(2, 2m-2))$, which implies that $\chi^{top}(\mathrm{Gr}(2, 2m)(\mathbb{R})) = \chi^{top}(\mathrm{Gr}(2, 2m-2)) + 1$. By induction, this says that $\chi^{top}(\mathrm{Gr}(2, 2m)(\mathbb{R})) = m$, and similarly by induction we find $\chi^{top}(Fl_m) = m \cdot \chi^{top}(Fl_{m-1}) = m!$. Thus $\chi^{top}(\tilde{U}_0) = m!$ and hence $\chi^{top}(U_0) = 1$.

In case $n = 2m + 1$ is odd, we can again choose a metric and fiber $U_1$ over the real projective space $\mathbb{RP}^{2m}$, with fiber over $[\ell] \in \mathbb{RP}^{2m}$ homotopy equivalent to the $U_0$ for the hyperplane perpendicular to the $\mathbb{C}$-span of $\ell$. As $\chi^{top}(\mathbb{RP}^{2m}) = 1$, this yields $\chi^{top}(U_1) = 1$.

We note that $N_T(\mathrm{GL}_n) \backslash \mathrm{GL}_n \cong N_T(\mathrm{SL}_n) \backslash \mathrm{SL}_n$ so the weak form of the conjecture also holds for $G = \mathrm{SL}_n$.

We conclude with a proof of the strong form of the conjecture for $\mathrm{GL}_2$ (or $\mathrm{SL}_2$). We argue as above: $(\mathbb{G}_m)^2\backslash \mathrm{GL}_2 = \mathbb{P}^1 \times \mathbb{P}^1 - \Delta$, where $\Delta \cong \mathbb{P}^1$ is the diagonal, and thus $N_T\backslash \mathrm{GL}_2 = S_2\backslash(\mathbb{P}^1 \times \mathbb{P}^1 - \Delta)$. We have $S_2\backslash\mathbb{P}^1 \times \mathbb{P}^1 = \mathrm{Sym}^2\mathbb{P}^1 \cong \mathbb{P}^2$. Via this isomorphism the quotient map

$$q : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2 = \mathrm{Sym}^2\mathbb{P}^1$$

is given by symmetric functions,

$$q((x_0 : x_1), (y_0 : y_1)) = (x_0y_0 : x_0y_1 + x_1y_0 : x_1y_1)$$

Thus the restriction of $q$ to $\Delta$ is the map $\bar{q}(x_0 : x_1) = (x_0^2 : 2x_0x_1 : x_1^2)$ with image the conic $C \subset \mathbb{P}^2$ defined by $Q := T_1^2 - 4T_0T_2$. Furthermore, this identifies the double cover $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ with $\mathrm{Spec}\,\mathcal{O}_{\mathbb{P}^2}(\sqrt{Q})$, that is, the closed subscheme of $O_{\mathbb{P}^2}(1)$ defined by pulling back the section of $O_{\mathbb{P}^2}(2)$ defined by $Q$ via the squaring map $O_{\mathbb{P}^2}(1) \to O_{\mathbb{P}^2}(2)$.

For $k$ of characteristic $\neq 2$, the map $\bar{q} : \mathbb{P}^1 \to C$ is an isomorphism. If $\mathrm{char} k = 2$, then $C$ is just the line $T_1 = 0$ and $N_T\backslash \mathrm{GL}_2 \cong \mathbb{A}^2$.

In any case, applying (3.3.1)(ii), the identification of $N_T\backslash \mathrm{GL}_2$ with $\mathbb{P}^2 - C \cong \mathbb{P}^2 - \mathbb{P}^1$ gives

$$\chi((N_T\backslash \mathrm{GL}_2)/k) = \chi(\mathbb{P}^2/k) - \chi(\mathbb{P}^1/k) = 2\langle 1\rangle + \langle -1\rangle - (\langle 1\rangle + \langle -1\rangle) = \langle 1\rangle$$

verifying the strong form of the conjecture in this case.

**Proposition 3.3.3.** *For $k$ a perfect field,* $\chi((N_T\backslash \mathrm{GL}_2)/k) = 1$.

## References: Lecture 3

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## Lecture 4. Reduction to the normalizer

In this final lecture, we consider the normalizer $N = N_T \subset \mathrm{SL}_2$ and compute the characteristic classes for all representations $\rho : N \to \mathrm{GL}_n$. In fact, $N = \mathbb{G}_m \rtimes \mathbb{Z}/2$ and all irreducible representations are either one- or two-dimensional. This reduces us to computing the Euler classes of the bundles associated to the two-dimensional representations. The $N_T$-splitting principle reduces the computation of the characteristic classes of bundles formed from a rank 2 $\mathrm{SL}_2$-bundle by applying a representation $\rho : \mathrm{SL}_2 \to \mathrm{GL}_n$ to the case of a bundle arising from an irreducible $N_T$-representation, so knowing the Euler classes of these combined with Ananyevskiy's $\mathrm{SL}_2$-splitting principle gives a complete calculus of characteristic classes of SL (or even GL) vector bundles. Here we give the computation of the Euler classes in Witt cohomology for $N_T$-bundles.

The representation theory of $N$ is closely related to the real representation theory of $O(2)$. The Euler classes for the irreducible representations of $SO(2)$ have been computed, for example by Okonek-Teleman [5], by noting that an $SO(2)$-bundle has a canonical $\mathbb{C}$-structure, and in the rank two case, the Euler class is just the 1st Chern class of the associated $\mathbb{C}$-line bundle. Our method is to mimic this approach; most of this lecture is taken from [3].

### 4.1. $\mathrm{BGL}_n$ and $\mathrm{BSL}_n$.

We mention a result that reduces the study of characteristic classes for arbitrary bundles to the case of bundles with trivial determinant. For $\mathcal{E} \in \mathrm{SH}(k)$, we have a canonical action of the bi-graded presheaf $U \mapsto \mathbb{S}^{*,*}(U)$ on the bi-graded presheaf $U \mapsto \mathcal{E}^{*,*}(U)$; passing from sheaf to presheaf and inverting $\eta$ gives a canonical map of presheaves $[U \mapsto \mathbb{S}^{*,*}(U)] \to [U \mapsto \mathcal{W}(U)]$. We say that $\mathcal{E}^{**}$ is a $\mathcal{W}$-module if the $U \mapsto \mathbb{S}^{*,*}(U)$-action descends to a $U \mapsto \mathcal{W}(U)$-action.

**Theorem 4.1.1** ([3, Theorem 4.1])**.** *Let $\mathcal{E}$ be an* SL*-oriented and $\eta$-invertible theory. We assume in addition that $\mathcal{E}^{**}$ is a $\mathcal{W}$-module. Let $O(1) \to \mathrm{BGL}_n$ be the determinant of the universal bundle $E_n \to \mathrm{BGL}_n$. Then the inclusion $i_n : \mathrm{SL}_n \to \mathrm{GL}_n$ induces an isomorphism*

$$\mathcal{E}^{**}(\mathrm{BGL}_n) \oplus \mathcal{E}^{**}(\mathrm{BGL}_n; O(1)) \xrightarrow[\sim]{Bi_n^*} \mathcal{E}^{**}(\mathrm{BSL}_n)$$

*More precisely, via Ananyevskiy's isomorphism*

$$\mathcal{E}^{**}(\mathrm{BSL}_n) \cong \begin{cases} \mathcal{E}^{**}(k)[p_1, \ldots, p_m] & \text{for } n = 2m+1 \\ \mathcal{E}^{**}(k)[p_1, \ldots, p_m, e]/p_m - e^2 & \text{for } n = 2m \end{cases}$$

*we have*

$$\mathcal{E}^{**}(\mathrm{BGL}_n) \xrightarrow[\sim]{Bi_n^*} \mathcal{E}^{**}(\mathrm{BSL}_n); \quad \mathcal{E}^{**}(\mathrm{BGL}_n; O(1)) = 0$$

*for $n = 2m+1$ odd,*

$$Bi_n^*(\mathcal{E}^{**}(\mathrm{BGL}_n)) = \mathcal{E}^{**}(k)[p_1, \ldots, p_m]$$
$$Bi_n^*(\mathcal{E}^{**}(\mathrm{BGL}_n, O(1))) = \mathcal{E}^{**}(k)[p_1, \ldots, p_m] \cdot e$$

*for $n = 2m$ even.*

The result follows by realizing $\mathrm{BSL}_n \to \mathrm{BGL}_n$ as the $\mathbb{G}_m$-bundle $O(1) - 0_{O(1)} \to \mathrm{BGL}_n$ and analyzing the localization sequence for the diagram

$$\begin{array}{ccccc} \mathrm{BSL}_n = O(1) - 0_{O(1)} & \overset{j}{\hookrightarrow} & O(1) & \overset{s_0}{\hookleftarrow} & \mathrm{BGL}_n \\ & & \wr \uparrow p^* & & \\ & & \mathrm{BGL}_n & & \end{array}$$

4.2. **$\mathcal{W}$-cohomology of $BN_T$.** Via Becker-Gottlieb transfers, we know that the pull-back map for $BN_T \to \mathrm{BSL}_2$ is injective; we give here an explicit computation of $H^*(BN_T, \mathcal{W})$ in terms of $H^*(\mathrm{BSL}_2, \mathcal{W}) = W(k)[e]$.

Recall the description of $N_T\backslash \mathrm{SL}_2$ as $\mathbb{P}^2 - C$ from § 3.3.We use the model $E\,\mathrm{SL}_2 = E\,\mathrm{GL}_2 =$ the colimit of $2 \times N$ matrices of rank two. Then

$$BN_T = N_T\backslash E\,\mathrm{GL}_2 = (N_T\backslash \mathrm{SL}_2) \times^{\mathrm{SL}_2} E\,\mathrm{GL}_2$$

i.e., $BN_T \to \mathrm{BSL}_2 = \mathrm{SL}_2\backslash E\,\mathrm{GL}_2$ is a Zariski locally trivial bundle with fiber $N_T\backslash \mathrm{SL}_2 \cong \mathbb{P}^2 - C$.

Letting $Q = T_1^2 - 4T_0T_2$, then $Q$ is the defining equation for $C$ and the double cover $T\backslash \mathrm{SL}_2 = \mathbb{P}^1 \times \mathbb{P}^1 - \Delta \to \mathbb{P}^2 - C$ is identified with $\mathrm{Spec}_{\,\mathcal{O}_{\mathbb{P}^2 - C}}(\sqrt{Q})$. One checks that $Q$ is $\mathrm{SL}_2$-invariant and thus the quadratic form $\langle q \rangle$ (locally $\langle Q/L^2 \rangle$) gives an $\mathrm{SL}_2$-invariant global section of $\mathcal{W}$ over $\mathbb{P}^2 - C$, which then gives

$$\langle q \rangle \in H^0(BN_T, \mathcal{W})$$

**Proposition 4.2.1.** *Let $p : BN_T \to \mathrm{BSL}_2$ be the projection. Then*

$$p^* : H^n(\mathrm{BSL}_2, \mathcal{W}) \to H^n(BN_T, \mathcal{W})$$

*is an isomorphism for $n \geq 1$. For $n = 0$, we have an exact sequence*

$$0 \to H^0(\mathrm{BSL}_2, \mathcal{W}) = W(k) \xrightarrow{p^*} H^0(BN_T, \mathcal{W}) \to W(k) \to 0$$

*split by the class $\langle q \rangle$. Explicitly*

$$H^*(BN_T, \mathcal{W}) = W(k)[p^*e] \oplus W(k) \cdot \langle q \rangle$$

*Proof.* In invariant terms $N_T\backslash \mathrm{SL}_2 = \mathbb{P}(\mathrm{Sym}^2 F) - \mathbb{P}(F)$, where $F = \mathbb{A}^2$ is the tautological right representation of $\mathrm{SL}_2$. We analyze $H^*(BN_T, \mathcal{W})$ via the localization sequence for

$$(N_T\backslash \mathrm{SL}_2) \times^{\mathrm{SL}_2} E\,\mathrm{GL}_2 \hookrightarrow \mathbb{P}(\mathrm{Sym}^2 F) \times^{\mathrm{SL}_2} E\,\mathrm{GL}_2 \hookleftarrow \mathbb{P}(F) \times^{\mathrm{SL}_2} E\,\mathrm{GL}_2$$

Using $\eta$-invertibility and $\mathbb{A}^1$-homotopy invariance gives

$$\begin{aligned} H^*(\mathbb{P}(F) \times^{\mathrm{SL}_2} E\,\mathrm{GL}_2, \mathcal{W}) &= H^*(\mathbb{A}^2 - \{0\} \times^{\mathrm{SL}_2} E\,\mathrm{GL}_2, \mathcal{W}) \\ &= H^*(\mathrm{SL}_2 \times^{\mathrm{SL}_2} E\,\mathrm{GL}_2, \mathcal{W}) = H^*(\mathrm{Spec}\, k, \mathcal{W}) \end{aligned}$$

so

$$H^n(\mathbb{P}(F) \times^{\mathrm{SL}_2} E\,\mathrm{GL}_2, \mathcal{W}) = \begin{cases} 0 & \text{for } n > 0 \\ W(k) & \text{for } n = 0. \end{cases}$$

Fasel [2, Theorem 11.7] has computed $H^*(\mathbb{P}^2_L, \mathcal{W})$ for a field $L$

$$H^n(\mathbb{P}^2_L, \mathcal{W}) = \begin{cases} 0 & \text{for } n > 0 \\ W(L) & \text{for } n = 0. \end{cases}$$

and then a Leray spectral sequence gives

$$H^*(\mathrm{BSL}_2, \mathcal{W}) \cong H^*(\mathbb{P}(\mathrm{Sym}^2 F) \times^{\mathrm{SL}_2} E\,\mathrm{GL}_2, \mathcal{W})$$

The localization sequence completes the proof. □

4.3. **The bundles $\tilde{O}(m)$.** Let $\sigma \in N_T(k)$ be the matrix

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

For each $m = 1, 2, \ldots$, we have the representation

$$\rho_m = \rho_m^+ : N_T \to \mathrm{GL}_2$$

defined by

$$\rho_m \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} t^m & 0 \\ 0 & t^{-m} \end{pmatrix}; \quad \rho_m(\sigma) = \begin{pmatrix} 0 & 1 \\ (-1)^m & 0 \end{pmatrix}$$

We have as well the representation $\rho_m^-$ with the same values on $T$ and with $\rho_m^-(\sigma) = -\rho_m^+(\sigma)$. We note that for $m$ odd, $\rho_m^\pm$ has values in $\mathrm{SL}_2$, but this is not the case for $m$ even. We have as well the one-dimensional representations $\rho_0 = \mathrm{Id}$ and $\rho_0^-$ with $\rho_0^-(T) = \mathrm{Id}$ and $\rho_0^-(\sigma) = -1$.

This gives us the rank two bundles $\tilde{O}^\pm(m) \to BN_T$, $m > 0$ and the rank one bundle $\gamma = \tilde{O}^-(0) \to BN_T$. We note that $\mathrm{Pic}(BN_T) = \mathbb{Z}/2$ with generator $\gamma$. Our main object is to compute the Euler class of $\tilde{O}^\pm(m)$ for $m > 0$. We note that $e(\tilde{O}^-(m)) = -e(\tilde{O}(m))$ since we have the orientation reversing isomorphism $(x, y) \to (-x, y)$ of $\rho_m$ with $\rho_m^-$. Also $\tilde{O}(1)$ is the tautological bundle, so has Euler class $p^*e$.

Our approach is the following: Let $s^{(m)} : BN_T \to \tilde{O}(m)$ be the zero-section. The map

$$\psi_m : \mathbb{A}^2 \to \mathbb{A}^2; \quad \psi_m(x, y) = (x^m, y^m)$$

is $N_T$-equivariant, if we let $N_T$ act by $\rho_1$ on the domain and by $\rho_m$ on the target. This gives us the map over $BN_T$

$$\psi_m : \tilde{O}(1) \to \tilde{O}(m)$$

and as $\psi_m \circ s^{(1)} = s^{(m)}$. Thus

$$e(\tilde{O}(m)) = s^{(1)*}(\psi_m^*(\theta_{\tilde{O}(m)})$$

so we reduce to calculating

$$\psi_m^*(\theta_{\tilde{O}(m)}) \in H^2_{0_{\tilde{O}(1)}}(\tilde{O}(1), \mathcal{W}) \cong H^0(BN_T, \mathcal{W}) = W(k) \oplus W(k) \cdot \langle q \rangle$$

In fact, it suffices to calculate the image of $\psi_m^*(\theta_{\tilde{O}(m)})$ in $H^2(\tilde{O}(1), \mathcal{W}) = W(k) \cdot p^*e$, so we need only compute $\psi_m^*(\theta_{\tilde{O}(m)})$ modulo the kernel of the surjection

$$H^2_{0_{\tilde{O}(1)}}(\tilde{O}(1), \mathcal{W}) \to H^2(\tilde{O}(1), \mathcal{W})$$

This information is furnished by an explicit computation

**Lemma 4.3.1** ([3, Lemma 5.4]). *The kernel of $H^2_{0_{\tilde{O}(1)}}(\tilde{O}(1), \mathcal{W}) \to H^2(\tilde{O}(1), \mathcal{W})$ is $W(k) \cdot (1 + \langle q \rangle)$.*

Thus, rather than make the computation of $\psi_m^*(\theta_{\tilde{O}(m)})$ in $H^2_{0_{\tilde{O}(1)}}(\tilde{O}(1), \mathcal{W})$, we may instead restrict to the fiber of $\tilde{O}(1)$ over any point $x \in BN_T$ with $\langle q(x)\rangle = \langle -1\rangle = -1$ in $W(k)$, because we have the commutative diagram

$$\begin{array}{ccc} W(k) \oplus W(k)\langle q\rangle & \xrightarrow{i_x^*} & W(k) \\ \| & & \| \\ H^2_{0_{\tilde{O}(1)}}(\tilde{O}(1), \mathcal{W}) & \xrightarrow{i_x^*} & H^2_{0_{\tilde{O}(1)_x}}(\tilde{O}(1)_x, \mathcal{W}) \end{array}$$

and $i_x^*(a + b\cdot\langle q\rangle) = a + b\cdot\langle q(x)\rangle = a - b$.

The map $\psi_m(x) : \tilde{O}(1)_x \to \tilde{O}(m)_x$ induces the map of projective spaces

$$\mathbb{P}(\psi_m(x)) : \mathbb{P}(\tilde{O}(1)_x) \to \mathbb{P}(\tilde{O}(m)_x)$$

For $m$ odd, the bundle $\tilde{O}(m)$ carries a canonical trivialization of $\det \tilde{O}(m)$ so we may replace $\mathbb{P}(\tilde{O}(1)_x)$ and $\mathbb{P}(\tilde{O}(m)_x)$ with any choice of bases $v_{1,1}, v_{2,1}$, $v_{1,m}, v_{2,m}$ such that

$$\det(v_{1,1}, v_{2,1}) = \det(v_{1,m}, v_{2,m})$$

We then have the map

$$\mathbb{P}(\psi_m(x)) : \mathbb{P}^1 \to \mathbb{P}^1$$

and the induced map

$$\psi_m(x)^* : W(k) = H^2_{0_{\tilde{O}(m)_x}}(\tilde{O}(m)_x, \mathcal{W}) \to H^2_{0_{\tilde{O}(1)_x}}(\tilde{O}(1)_x, \mathcal{W}) = W(k)$$

is multiplication by the $\mathbb{A}^1$-degree of $\mathbb{P}(\psi_m(x))$.

We take $k = \mathbb{Q}$ and $x \in BN_T(\mathbb{Q})$ to be the image of

$$\tilde{x} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

For $((x_0 : x_1), (y_0 : y_1)) \in \mathbb{P}^1 \times \mathbb{P}^1 - \Delta$, the image in $\mathbb{P}^2 - C$ is given by $(x_0y_0 : x_0y_1 + x_1y_0 : x_1y_1)$, in other words

$$x = (1 : 0 : 1),\ Q(x) = -4 \equiv -1 \mod \mathbb{Q}^{\times 2},$$

so $\langle q(x)\rangle = \langle -1\rangle = -1 \in W(\mathbb{Q})$.

The basis $v_{1,m}, v_{2,m}$ is chosen to be

$$\begin{pmatrix} v_{1,m} \\ v_{2,m} \end{pmatrix} := \begin{cases} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} & \text{for } m \equiv 1 \mod 4 \\ \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} & \text{for } m \equiv 3 \mod 4 \end{cases}$$

Then $v_{1,m} \wedge v_{2,m} = 2 \cdot e_1 \wedge e_2$, so

$$\left\langle \frac{v_{1,1} \wedge v_{2,1}}{v_{1,m} \wedge v_{2,m}} \right\rangle = 1 \in W(k)$$

One computes

$$\psi_m(x_0v_{1,1} + x_1v_{2,1}) = \begin{cases} \Re(x_0 + ix_1)^m v_{1,m} + \Im(x_0 + ix_1)^m v_{2,m} & \text{for } m \equiv 1 \mod 4 \\ \Re(x_0 - ix_1)^m v_{1,m} + \Im(x_0 - ix_1)^m v_{2,m} & \text{for } m \equiv 3 \mod 4 \end{cases}$$

in other words,

$$\psi_m(x_0 : x_1) = \begin{cases} (\Re(x_0 + ix_1)^m : \Im(x_0 + ix_1)^m) \text{ for } m \equiv 1 \mod 4 \\ (\Re(x_0 - ix_1)^m : \Im(x_0 - ix_1)^m) \text{ for } m \equiv 3 \mod 4 \end{cases}$$

To complete the story (for $m$ odd) we have

**Proposition 4.3.2.** *Let $p > 2$ be prime. For $p \equiv 1 \mod 4$, we have $\deg_{\mathbb{A}^1} \psi_m = p$. For $p \equiv 3 \mod 4$ we have $\deg_{\mathbb{A}^1} \psi_m = -p$.*

*Proof.* Since $(x_0 : x_1) \mapsto (x_0 : -x_1)$ intertwines the formulas for $\psi_m$ with $m \equiv 1 \mod 4$ and $\psi_m$ with $m \equiv 3 \mod 4$, we reduce to showing that the map

$$(x_0 : x_1) \mapsto (\Re(x_0 + ix_1) : \Im(x_0 + ix_1)^p)$$

has $\mathbb{A}^1$-degree $p \in \mathbb{W}(k)$ for all primes $p > 2$. We actually show that the $\mathbb{A}^1$-degree in $\mathrm{GW}(k)$ is $p$.

For this, we compute the $\mathbb{A}^1$-degree as the *Brouwer degree* (see [4, Chapter 2]), that is, we take a regular $k$-rational value $s \in \mathbb{A}^1 = \mathbb{P}^1 - \{(0 : 1)\}$ so that $\psi_m^{-1}(s) \subset \mathbb{A}^1$. We use the parameter $t = x_0/x_1$ and let $g : \mathbb{A}^1 \to \mathbb{P}^1$ be the map

$$g(t) = \frac{\Re(t+i)^p}{\Im(t+i)^p}.$$

Then for each $t \in g^{-1}(s)$ we have the one-dimensional quadratic form $\langle g'(t)\rangle$ and the Brouwer degree is

$$\sum_{t \in g^{-1}(s)} \mathrm{Tr}_{k(t)/k} \langle g'(t)\rangle \in \mathrm{GW}(k).$$

We take $s = (0 : 1)$, giving

$$g^{-1}(s) = \{\cot(\ell\pi/2p) \mid \ell = 1, \ldots, p\}$$

At the point $t_0 = (0 : 1)$ ($\ell = p$) we have $g'(t_0) = p$. The other points are conjugate over $\mathbb{Q}$, and generate the real cyclotomic field $F_{4p} := \mathbb{Q}(\zeta_{4p} + \zeta_{4p}^{-1})$, giving the closed point $t_1 \in \mathbb{A}^1$. One computes that $g'(t_1)$ is a square in $F_{4p}$, so the Brouwer degree is

$$\langle p\rangle + \mathrm{Tr}_{F_{4p}/\mathbb{Q}}(\langle 1\rangle)$$

i.e., $\langle p\rangle$ plus the trace form for $F_{4p}$ over $\mathbb{Q}$.

One can compute this trace form, or rather its invariants: rank, signature, discriminant and Hasse-Witt invariant, to show that $\langle p\rangle + \mathrm{Tr}_{F_{4p}/\mathbb{Q}}(\langle 1\rangle)$ has rank $p$, signature $p$, and trivial discriminant and Hasse-Witt invariant, and therefore in $\mathrm{GW}(\mathbb{Q})$ we have

$$[\langle p\rangle + \mathrm{Tr}_{F_{4p}/\mathbb{Q}}(\langle 1\rangle)] = p \cdot \langle 1\rangle.$$

Alternatively, an integral model for the trace form $\mathrm{Tr}_{F_{4p}/\mathbb{Q}}(\langle 1\rangle)$ has been computed by Bayer and Suarez [1, pg. 222-"The root lattice $\mathbb{A}_{p-1}$"], who show that $\mathrm{Tr}_{F_{4p}/\mathbb{Q}}(\langle 1\rangle)$ is equivalent to the lattice $\mathbb{A}_{p-1} \subset \mathbb{Z}^p$, i.e., the hyperplane $\sum_{i=1}^p x_i = 0$, with quadratic form induced by vector dot product. But then $\langle p\rangle + \mathrm{Tr}_{F_{4p}/\mathbb{Q}}(\langle 1\rangle)$ is the form associated to the lattice $(1, \ldots, 1) + A_{p-1} \subset \mathbb{Z}^p$, and as

$$\mathbb{Q}((1, \ldots, 1) + A_{p-1}) = \mathbb{Q}^p$$

we see that over $\mathbb{Q}$, $\langle p\rangle + \mathrm{Tr}_{F_{4p}/\mathbb{Q}}(\langle 1\rangle)$ is equivalent to a sum of $p$ squares. □

The case $p = 2$ and $m = 4$ are handled much the same way, with the difference being that, $\tilde{O}(2m)$ having non-trivial determinant $\gamma$, $e(\tilde{O}(2m))$ is in $H^2(BN_T, \gamma)$ and hence is not comparable with $p^e$. However, an analogous computation shows that the map $\psi_2 : \tilde{O}(2) \to \tilde{O}(4)$ has degree -2, so

$$e(\tilde{O}(4)) = -2e(\tilde{O}(2))$$

and also, that

$$p_1(\tilde{O}(2)) = e(\tilde{O}(2))^2 = 4p^*e.$$

The upshot is the following. For an integer $m > 0$, write $m = 2^r n$ with $n$ odd and define $\epsilon(m) \in \{\pm 1\}$ by $\epsilon(m) = +1$ if $n \equiv 1 \mod 4$ and $\epsilon(m) = -1$ if $n \equiv 3 \mod 4$. Let $\tilde{e} = e(\tilde{O}(2)) \in H^2(BN_T, \mathcal{W}(\gamma))$.

**Theorem 4.3.3** ([3, Theorem 7.1])**.** *Let $m > 0$ be an integer. Let $k$ be a field of characteristic zero or of characteristic $q$ prime to $m$. Write $m = 2^r n$ with $n$ odd. If $r = 0$, then $e(\tilde{O}(m)) = \epsilon(m) \cdot m \cdot p^*e$. For $r = 1$, then $e(\tilde{O}(m)) = (m/2) \cdot \tilde{e}$ and if $r > 1$, then $e(\tilde{O}(m)) = -(m/2) \cdot \tilde{e}$. Moreover $\tilde{e}^2 = 4p^*e$ and $e(\tilde{O}^-(m)) = -e(\tilde{O}(m))$.*

**Corollary 4.3.4.** *Let $k \geq 1$ be an integer and let $E \to X$ be a rank two bundle over $X \in \mathbf{Sm}/k$. Then for $k = 2m - 1$, we have*

$$e(\mathrm{Sym}^k E) = k!!e(E)^m \in H^{2m}(X, \mathcal{W})$$

*where $k!! := \prod_{i=0}^{[k/2]} k - 2i$. For $k = 2m$, $e(\mathrm{Sym}^k E) = 0$. Moreover, for all $k \geq 1$,*

$$p(\mathrm{Sym}^k E) = \prod_{i=0}^{[k/2]} (1 + (k - 2i)^2 e(E)^2)$$

This follows directly from the theorem, together with the identity

$$\mathrm{Sym}^k \rho_1 \cong \oplus_{i=0}^{[k/2]} \rho_{k-2i}^{(-1)^i}.$$

and the $N_T$ splitting principle.

One can similarly show

**Theorem 4.3.5.** *Let $E_1 \to X$, $E_2 \to Y$ be rank two bundle. Then the bundle $\pi_1^* E_1 \otimes \pi_2^* E_2$ on $X \times Y$ has characteristic classes*

$$e(\pi_1^* E_1 \otimes \pi_2^* E_2) = \pi_1^* e(E_1)^2 - \pi_2^* e(E_2)^2$$

$$p_1(\pi_1^* E_1 \otimes \pi_2^* E_2) = 2(\pi_1^* e(E_1)^2 + \pi_2^* e(E_2)^2)$$

*and*

$$p_2(\pi_1^* E_1 \otimes \pi_2^* E_2) = (\pi_1^* e(E_1)^2 - \pi_2^* e(E_2)^2)^2.$$

## References: Lecture 4


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