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Bloch's higher Chow groups revisited

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Introduction

Bloch defined his higher Chow groups $\mathrm{CH}^q(-,p)$ in [B], with the object of defining an integral cohomology theory which rationally gives the weight–graded pieces $K_p(-)^{(q)}$ of K-theory. For a variety X, the higher Chow group $\mathrm{CH}^q(X,p)$ is defined as the pth homology of the complex $Z^q(X,*)$, which in turn is built out of the codimension q cycles on $X \times \mathbb{A}^p$ for varying p, using the cosimplicial structure on the collection of varieties $\{X \times \mathbb{A}^p \mid p = 0, 1, \ldots\}$. In order to relate $\mathrm{CH}^q(X,p)$ with $K_p(X)$, Bloch used Gillet's construction of Chern classes with values in a Bloch-Ogus twisted duality theory [G]; this requires, among other things, that the complexes $Z^q(X,*)$ satisfy a Mayer-Vietoris property for the Zariski topology, and that they satisfy a contravariant functoriality. Bloch attempted to prove the Mayer-Vietoris property by proving a localization theorem, identifying the cone of the restriction map

$$Z^q(X,*) \to Z^q(U,*),$$

for $U \to X$ a Zariski open subset of X, with the complex $Z^q(X \setminus U,*)[1]$, up to quasi-isomorphism. There is a gap in Bloch's proof, which left open the localization property and the Mayer-Vietoris property for the complexes $Z^q(X,*)$; essentially the same problem leaves a gap in the proof of contravariant functoriality. Recently, Bloch [B3] has provided a new argument which fills the gap in the proof of localization; this, together with a new argument for contravariant functoriality, should allow Bloch's original program for relating $\mathrm{CH}^q(X,p)$ with $K_p(X)$ to go through without further problem.

As part of the argument in [B], Bloch defined a map

$$\mathrm{CH}^q(X,p) \otimes \mathbb{Q} \to K_p(X)^{(q)} \tag{1}$$

for X smooth and quasi-projective over a field, relying on a λ-ring structure on relative K-theory with supports. It turns out that this approach can be followed and extended to show that the map (1) is an isomorphism, without relying on Chern classes (Theorem 3.1). An important new ingredient in this line of argument is the computation of certain relative K_0-groups in terms of the K_0 of an associated iterated double (see Theorem 1.10 and Corollary 1.11). A bit more work then enables us to prove the Mayer-Vietoris property (Theorem 3.3), a weak version of localization (Theorem 3.4) and contravariant functoriality (Corollary 4.9) for the rational complexes $Z^q(X,*)\otimes\mathbb{Q}$. We also construct a product for the rational complexes $Z^q(X,*)\otimes\mathbb{Q}$ and prove the projective bundle formula (Corollary 5.4). The arguments used in [B] then give rational Chern classe

$$c_{q,p} : K_{2q-p}(X) \to \mathrm{CH}^q(X, 2q-p) \otimes \mathbb{Q},$$

satisfying the standard properties.

It turns out that it is somewhat more convenient to work with a modified version of $Z^q(X,*)$, using a cubical structure rather than a simplicial structure. We show that the cubical complexes $Z^q(X,*)^c$ are integrally quasi-isomorphic to the simplicial version $Z^q(X,*)$ (Theorem 4.7), and have a natural external product in the derived category (see §5, especially Theorem 5.2). We also consider the "alternating" complexes $N^q(k)$ defined by Bloch [B2], and used to construct a candidate for a motivic Lie algebra. We show that there is a natural quasi-isomorphism

$$Z^q(\mathrm{Spec}(k),*)^c \otimes \mathbb{Q} \to N^q(k)$$

(Theorem 4.11). The product structures are not quite compatible via this quasi-isomorphism; it is necessary to reverse the order of the product in one of the complexes to get a product-compatible quasi-isomorphism (Corollary 5.5).

The paper is organized as follows: We begin in §1 by proving some extensions of the results of Vorst on K_n-regularity, which we use to prove a basic result on the K_0-regularity of certain iterated doubles. We also recall some basic facts about relative K-theory, and use the K_0-regularity results to compute certain relative K_0 groups in terms of the usual K_0 of an iterated double. In §2 we use, following Bloch, the λ-operations on relative K-theory with supports to give a cycle-theoretic interpretation of certain relative K_0 groups, analogous to the classical Grothendieck-Riemann-Roch theorem relating the rational Chow ring to the rational K_0 for a smooth variety (see Theorem 2.7). In §3, we use this to show that Bloch's map

$$\mathrm{CH}^1(X,p)\otimes\mathbb{Q}\to K_p(X)^{(q)}$$

is an isomorphism for X smooth and quasi-projective. In §4, we relate the cubical complexes with Bloch's simplicial version, and also with his alternating version. In §5 we define products and prove the projective bundle formula for the rational complexes.

As a matter of notation, a *scheme* will alsways mean a separated, Noetherian scheme. For an abelian group A, we denote $A\otimes\mathbb{Q}$ by $A_{\mathbb{Q}}$; for a homological complex C_*, we deonte the cycles in degree p by $Z_p(C_*)$, the boundaries by $B_p(C_*)$ and the homology by $H_p(C_*)$.

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§1. NK and relative K_0

In this section, we give a description of relative K_0, $K_0(X;Y_1,\ldots,Y_n)$, in terms of the K_0 of the so-called iterated double $D(X;Y_1,\ldots,Y_n)$. We begin by extending some of Vorst's results on NK_p of rings to schemes over a ring.

Fix a commutative ring A, and let $\mathbf{Alg}_A$ denote the category of commutative A-algebras, $\mathbf{Ab}$ the category of abelian groups. For a ring R, let $p_R\colon R[T]\to R$ be the R-algebra homomorphism $p_R(T)=0$. For a functor $F\colon\mathbf{Alg}_A\to\mathbf{Ab}$, let $NF\colon\mathbf{Alg}_A\to\mathbf{Ab}$ be the functor

$$NF(R)=\ker[F(p_R)\colon F(R[T])\to F(R)].$$

Define the associated functors N^qF for $q>1$ inductively by

$$N^qF=N(N^{q-1}F).$$

We set $N^0F=F$.

For $R\in\mathbf{Alg}_A$ and $r\in R$, the R-algebra map

$$\begin{gathered}\phi_r\colon R[T]\to R[T]\\ \phi_r(T)=rT\end{gathered}$$

gives rise to the endomorphism $NF(\phi_r)\colon NF(R)\to NF(R)$, thus $NF(R)$ becomes a $\mathbb{Z}[T]$-module with T acting via ϕ_r. Let $NF(R)_{[r]}$ denote the localization $\mathbb{Z}[T,T^{-1}]\otimes_{\mathbb{Z}[T]}NF(R)$. If r is a unit, then the map $NF(R)\to NF(R)_{[r]}$ is an isomorphism; letting R_r denote the localization of R with respect to the powers of r, the natural map

$$NF(R)\to NF(R_r)$$

factors canonically through $N(R)_{[r]}$:

$$\begin{array}{ccc} NF(R) & \to & NF(R)_{[r]} \\ \searrow & & \swarrow \\ & NF(R_r) & \end{array}$$

For elements $r_1, \ldots, r_n$ of R, form the "augmented Čech complex"

$$0 \to NF(R) \xrightarrow{\epsilon} \bigoplus_{1 \le i \le n} NF(R_{r_i}) \to \ldots$$

(1.1)

$$\to \bigoplus_{1 \le i_0 < i_1 < \ldots < i_p \le n} NF(R_{r_{i_0}, r_{i_1}, \ldots, r_{i_p}}) \to \ldots \to NF(R_{r_1, \ldots, r_n}) to 0.$$

where the map

$$\bigoplus_{1 \le i_0 < i_1 < \ldots < i_p \le n} NF(R_{r_{i_0}, r_{i_1}, \ldots, r_{i_p}}) \to \bigoplus_{1 \le i_0 < i_1 < \ldots < i_{p+1} \le n} NF(R_{r_{i_0}, r_{i_1}, \ldots, r_{i_{p+1}}})$$

is given as the direct sum over indices $(1 \le i_0 < i_1 < \ldots < i_{p+1} \le n)$ of the alternating sums:

$$\sum_{j=0}^{p+1} (-1)^j \delta_j \colon \oplus_{j=0}^{p+1} NF(R_{r_{i_0}, \ldots, \widehat{r_{i_j}}, \ldots, r_{i_{p+1}}}) \to NF(R_{r_{i_0}, \ldots, r_{i_{p+1}}}),$$

and where

$$\delta_j \colon NF(R_{r_{i_0}, \ldots, \widehat{r_{i_j}}, \ldots, r_{i_{p+1}}}) \to NF(R_{r_{i_0}, \ldots, r_{i_{p+1}}})$$

is the canonical map. The map ϵ is the direct sum of the canonical maps

$$NF(R) \to NF(R_{r_j}).$$

Lemma 1.1. *Suppose R is a commutative A-algebra, $r_1, \ldots, r_n$ elements of R which generate the unit ideal. Suppose further that the map*

$$NF(R[T]_{r_{i_0}, \ldots, \widehat{r_{i_j}}, \ldots, r_{i_p}})_{[r_{i_j}]} \to NF(R[T]_{r_{i_0}, \ldots, r_{i_p}})$$

is an isomorphism, for each set of indicies $1 \le i_0 < \ldots < i_p \le n$. Then the complex (1.1) is exact. In particular, the map

$$\epsilon \colon NF(R) \to \oplus_{j=1}^n NF(R_{r_j})$$

is injective.

Proof. This is proved in ([V], Theorem 1.2); there the functor F is a functor from $\mathbf{Alg}_{\mathbb{Z}}$ to $\mathbf{Ab}$, but, as the proof uses only the restriction of F to the category $\mathbf{Alg}_R$, the argument works as well in the case of a functor $F \colon \mathbf{Alg}_A \to \mathbf{Ab}$. □

Let X be a scheme. We let P_Z denote the category of locally free sheaves of finite rank on X, and let $K(X)$ denote the space $\Omega \mathrm{BQ} P_Z$; the pth the K-group $K_p(X)$, $p \geq 0$, is thus defined as the homotopy group $\pi_p(K(X))$. Letting $\mathbb{A}^1_X$ denote the affine line over X, and $\mathbb{G}_{mX}$ the open subscheme $\mathbb{A}^1_X \setminus 0_X$, we have the "fundamental exact sequence" for $p \geq 0$

$$0 \to K_{p+1}(X) \to K_{p+1}(\mathbb{A}^1_X) \oplus K_{p+1}(\mathbb{A}^1_X) \to K_{p+1}(\mathbb{G}_{mX}) \to K_p(X) \to 0 \tag{1.2}$$

where the maps are those arising from a spectral sequence computing the K-groups of $\mathbb{P}^1_X$ via the standard cover

$$\mathbb{P}^1_X = \mathbb{A}^1_X \cup \mathbb{A}^1_X.$$

This allows the inductive definition of the K-groups $K_p(X)$ for $p < 0$ by forcing the exactness of

$$K_{p+1}(\mathbb{A}^1_X) \oplus K_{p+1}(\mathbb{A}^1_X) \to K_{p+1}(\mathbb{G}_{mX}) \to K_p(X) \to 0$$

for all p; it then follows (see [*]) that the sequence (1.2) is exact for *all* $p \in \mathbb{Z}$.

Let $i_0 \colon X \to \mathbb{A}^1_X$ be the inclusion as the zero section. Recall the inductive definition of the groups $N^q K_p(X)$ as

$$N^q K_p(X) = \begin{cases} K_p(X) & \text{for q=0,} \\ \ker[i_0^* \colon N^{q-1} K_p(\mathbb{A}^1_X) \to N^{q-1} K_p(X)] & \text{for } q > 0. \end{cases}$$

We recall that a scheme X is K_p*-regular* if $N^q K_p(X) = 0$ for each $q > 0$.

Let $U = \{U_\alpha\}$ be a Zariski open cover of X. Then there is a spectral sequence (see Thomason [T], *)

$$E_1^{p,q} = \oplus_{(\alpha_0, \ldots, \alpha_q)} N^t K_{-p}(U_{\alpha_0} \cap \ldots \cap U_{\alpha_q}) \Rightarrow N^t K_{-p-q}(X). \tag{1.3}$$

The E_2-term is the Čech cohomology with coefficients in the presheaf $N^q K_{-p}$: $H^q_{\text{Čech}}(U, N^t K_{-p})$; the sequence is strongly convergent for finite covers.

For an A-scheme X, and element $f \in A$, we let X_f denote the open subscheme defined by the non-vanishing of f. Let $F_X \colon \mathbf{Alg}_A \to \mathbf{Ab}$ be the functor

$$F_X(R) = K_p(X \otimes_A R);$$

in particular, we have $N^q F(R) = N^q K_p(X \otimes_A R)$. For $f \in A$, we use the notation $N^q K_p(X)_{[f]}$ for $N^q F_X(A)_{[f]}$.

Lemma 1.2. *Let A be a commutative ring, $f \in A$ and X an A-scheme. Suppose we have a covering of X by affine open subsets $U_\alpha = \mathrm{Spec}(A_\alpha)$ such that, for each α, either f is a non-zero divisor in A_α, or f is contained in some minimal prime ideal of A_α. Then the natural map*

$$N^q K_p(X)_{[f]} \to N^q K_p(X_f)$$

is an isomorphism.

Proof. Let B be a commutative ring and suppose $g \in B$ is either a non-zero divisor in B, or is contained in some minimal prime ideal of B. Then Vorst ([V], Lemma 1.4) has shown that the natural map

$$N^q K_p(B)_{[g]} \to N^q K_p(B_g)$$

is an isomorphism (Vorst only proves this for $p \geq 0$, but the general result follows from this and the fundamental exact sequence (1.2)). The general result follows from this and the spectral sequence (1.3). □

Theorem 1.3. *Let A be a commutative ring, X a reduced A-scheme. Suppose we have elements $f_1, \ldots, f_n$ in A generating the unit ideal such that X_{f_j} is K_p-regular for each $j = 1, \ldots, n$. Then X is K_p-regular.*

Proof. Take $q > 0$. Let F be the functor $N^{q-1}F_X$. Since X is reduced, the scheme $X \otimes_A B$ is reduced for all flat A-algebras B, in particular, for all B which are localizations of a polynomial ring $A[T]$. By Lemma 1.1 together with Lemma 1.2, the map

$$N^q K_p(X) \to \oplus_{j=1}^n N^q K_p(X_{f_j})$$

is injective. Since each X_{f_j} is K_p-regular, the groups $N^q K_p(X_{f_j})$ are all zero for all $q > 0$, hence $N^q K_p(X)$ is zero for all $q > 0$, i.e., X is K_p-regular. □

Corollary 1.4. *Let X be a scheme. If X is K_n-regular, then X is K_{n-1}-regular.*

Proof. The exact sequence (1.2) gives the exact sequence for all $p \in \mathbb{Z}$:

$$N^q K_p(\mathbb{A}^1_X) \oplus N^q K_p(\mathbb{A}^1_X) \to N^q K_p(\mathbb{G}_{m\,X}) \to N^q K_{p-1}(X) \to 0.$$

If X is K_p-regular, then $\mathbb{A}^1_X$ is clearly K_p-regular; applying Lemma 1.2, with $A = \mathbb{Z}[t]$, $f = t$, we see that $\mathbb{G}_{m\,X}$ is also K_p-regular. The exact sequence above then shows:

If X is K_p-regular, then X is K_{p-1}-regular,

completing the proof. □

Let X be a scheme, Y a closed subscheme. The *double* of X along Y, $D(X;Y)$, is the scheme making the following square co-Cartesian:

$$\begin{array}{ccc} Y & \xrightarrow{i} & X \\ i\downarrow & & \downarrow r_1 \\ X & \xrightarrow{r_2} & D(X;Y); \end{array}$$

i.e., $D(X;Y)$ is two copies of X glued along Y.

If $X = \mathrm{Spec}(R)$ is affine, and Y is defined by an ideal I, then $D(X;Y)$ is $\mathrm{Spec}(D(R;I))$, where $D(R;I)$ is the subring of $R \times R$ consisting of pairs (r, r') with $r - r' \in I$. If R is Noetherian, then the R-submodule $D(R;I)$ of $R \times R$ is thus a finite R-module, hence $D(R;I)$ is Noetherian if R is. Sending the pair $(R;I)$ to the ring $D(R;I)$ is clearly functorial; thus, as every scheme has an affine open cover, the double $D(X;Y)$ exists for each scheme X and closed subscheme Y.

We have the map

$$p: D(X,Y) \to X$$

splitting the two inclusions $r_i: X \to D(X;Y)$. If Z is a closed subscheme of X, there is a natural identification of $D(Z; Y \cap Z)$ with $p^{-1}(Z)$; we denote the closed subscheme $p^{-1}(Z)$ by $D(Z,Y)$. This allows us to define the iterated double $D(X;Y_1,Y_2)$ inductively as the double of the $D(X;Y_1)$ along $p^{-1}(Y_2)$. The further iterated double $D(X;Y_1,\ldots,Y_n)$ is defined inductively along these lines:

$$D(X;Y_1,\ldots,Y_n) = D(D(X;Y_1,\ldots,Y_{n-1}); D(Y_n;Y_1,\ldots,Y_{n-1})).$$

Suppose we have closed subschemes $Y_1,\ldots,Y_n$ of a scheme X. We form the (opposite) n-cube of subschemes of X, $(X;Y_1,\ldots,Y_n)_*$, by

$$(X;Y_1,\ldots,Y_n)_I = \cap_{i\in I} Y_i$$

for each subset $I \subset \{1,\ldots,n\}$; the map

$$(X;Y_1,\ldots,Y_n)_I \to (X;Y_1,\ldots,Y_n)_J$$

for $J \subset I$ is the natural inclusion. We call the collection of closed subschemes $Y_1,\ldots,Y_n$ *split* if the resulting n-cube is split. We say that $Y_1,\ldots,Y_n$ *define a normal crossing divisor on* X if for each subset I of $\{1,\ldots,n\}$, the subscheme $(X;Y_1,\ldots,Y_n)_I$ is a regular scheme of codimension $|I|$ on X (or is empty).

Lemma 1.5. *Let X be a scheme, Y a closed subscheme. Suppose that the inclusion $i\colon Y \to X$ is split. Then the sequence*

$$0 \to K_0(D(X;Y)) \overset{(r_1^*, r_2^*)}{\to} K_0(X) \oplus K_0(X) \overset{i^* \oplus -i^*}{\to} K_0(Y) \to 0$$

is exact.

Proof. For a scheme Z, let $\mathrm{Iso}P_Z$ the set of isomorphism classes in P_Z; we let $[E]$ denote the isomorphism class of a locally free sheaf. The category $P_{D(X;Y)}$ is equivalent to the category of triples (E, E', ϕ), where E and E' are locally free sheaves on X, and $\phi\colon i^*E \to i^*E'$ is an isomorphism. Since the inclusion i is split, each automorphism ρ of i^*E lifts to an automorphism $\tilde{\rho}$ of E; thus the isomorphism class of (E, E', ϕ) is independent of the choice of isomorphism ϕ. Thus, $\mathrm{Iso}P_{D(X;Y)}$ is the set of pairs $([E], [E'])$ of isomorphism classes of locally free sheaves on X, such that $i^*[E] = i^*[E']$. Using the splitting of i again, this implies that the sequence

$$\mathbb{Z}[\mathrm{Iso}P_{D(X;Y)}] \to \mathbb{Z}[\mathrm{Iso}P_X] \oplus \mathbb{Z}[\mathrm{Iso}P_X] \to \mathbb{Z}[\mathrm{Iso}P_Y] \to 0$$

is exact, and the kernel of the first map is generated by elements of the form

$$([E], [E']) - ([E], [E'']) + ([F], [E'']) - ([F], [E']). \tag{1.4}$$

For a scheme Z, let R_Z denote the kernel of the surjection

$$\mathbb{Z}[\mathrm{Iso}P_Z] \to K_0(Z);$$

i.e., R_Z is the subgroup of $\mathbb{Z}[\mathrm{Iso}P_Z]$ generated by expressions of the form $[E] - [E'] - [E'']$, where $0 \to E' \to E \to E'' \to 0$ is exact. Since i is split, the sequence

$$R_{D(X;Y)} \to R_X \oplus R_X \to R_Y \to 0$$

is exact. On the other hand, for elements $([E], [E'])$, $([E], [E''])$,$([F], [E''])$, $([F], [E'])$ in $\mathrm{Iso}P_{D(X;Y}$ we have the relations in $K_0(D(X;Y))$:

$$\begin{aligned}([E], [E']) + ([F], [E'']) &= ([E \oplus F], [E' \oplus E'']) \\ &= ([E \oplus F], [E'' \oplus E']) \\ &= ([E], [E'']) + ([F], [E']).\end{aligned}$$

Thus, elements of the form (1.4) are contained in $R_{D(X;Y)}$; a diagram chase finishes the proof. □

Theorem 1.6. *Let X be a reduced A-scheme, A a reduced commutative ring, and let $Y_1, \ldots, Y_n$ be subschemes of X, defining a normal crossing divisor on X. Suppose that there are elements $f_1, \ldots, f_k$ of A such that $Y_1 \cap X_{f_j}, \ldots, Y_n \cap X_{f_j}$ is a split collection of closed subschemes of X_{f_j} for each $j = 1, \ldots, k$. Then the iterated double $D(X; Y_1, \ldots, Y_n)$ is K_p-regular for all $p \leq 0$.*

Proof. By Corollary 1.4, we need only consider the case $p = 0$. If we replace X and $Y_1, \ldots, Y_n$ with $\mathbb{A}^q_X$ and $\mathbb{A}^q_{Y_1}, \ldots, \mathbb{A}^q_{Y_n}$, the hypotheses of the theorem remain valid; thus, we need only show that

$$N^1K_0(D(X; Y_1, \ldots, Y_n)) = 0.$$

We have the natural map

$$D(X; Y_1, \ldots, Y_n) \to X;$$

which identifies the iterated double $D(X_f; Y_1 \cap X_f, \ldots, Y_n \cap X_f)$ with $D(X; Y_1, \ldots, Y_n)_f$ for each $f \in A$. By Theorem 1.3, and our hypotheses, we may assume that the collection of subschemes $Y_1, \ldots, Y_n$ is split. The split, normal crossing hypotheses pass to the collection of closed subschemes $Y_1 \cap Y_n, \ldots, Y_{n-1} \cap Y_n$; by induction we may assume that $D(X; Y_1, \ldots, Y_{n-1})$ and $D(Y_n; Y_1 \cap Y_n, \ldots, Y_{n-1} \cap Y_n)$ are K_0-regular. Our hypothesis that the collection of subschemes $Y_1, \ldots, Y_n$ is split implies that the natural inclusion

$$D(Y_n; Y_1 \cap Y_n, \ldots, Y_{n-1} \cap Y_n) \to D(X; Y_1, \ldots, Y_{n-1})$$

is split.

The iterated double $D(X; Y_1, \ldots, Y_n)$ is the same as the double of $D(X; Y_1, \ldots, Y_{n-1})$ along the subscheme $D(Y_n; Y_1 \cap Y_n, \ldots, Y_{n-1} \cap Y_n)$; thus we have the commutative diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
K_0(D(X; Y_1, \ldots, Y_n)) & \to & K_0(D(\mathbb{A}^1_X; \mathbb{A}^1_{Y_1}, \ldots, \mathbb{A}^1_{Y_n})) \\
\downarrow & & \downarrow \\
K_0(D(X; Y_1, \ldots, Y_{n-1})) & & K_0(D(\mathbb{A}^1_X; \mathbb{A}^1_{Y_1}, \ldots, \mathbb{A}^1_{Y_{n-1}})) \\
\oplus & \to & \oplus \\
K_0(D(X; Y_1, \ldots, Y_{n-1})) & & K_0(D(\mathbb{A}^1_X; \mathbb{A}^1_{Y_1}, \ldots, \mathbb{A}^1_{Y_{n-1}})) \\
\downarrow & & \downarrow \\
K_0(D(Y_n; Y_1 \cap Y_n, \ldots, Y_{n-1} \cap Y_n)) & \to & K_0(D(\mathbb{A}^1_{Y_n}; \mathbb{A}^1_{Y_1 \cap Y_n}, \ldots, \mathbb{A}^1_{Y_{n-1} \cap Y_n})) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

By Lemma 1.3, the columns above are exact; since $D(X; Y_1, \ldots, Y_{n-1})$ and $D(Y_n; Y_1 \cap Y_n, \ldots, Y_{n-1} \cap Y_n)$ are K_0-regular, and we have natural isomorphisms

$$\begin{aligned}
D(\mathbb{A}^1_X; \mathbb{A}^1_{Y_1}, \ldots, \mathbb{A}^1_{Y_{n-1}}) &\to \mathbb{A}^1_{D(X; Y_1, \ldots, Y_{n-1})} \\
D(\mathbb{A}^1_{Y_n}; \mathbb{A}^1_{Y_1 \cap Y_n}, \ldots, \mathbb{A}^1_{Y_{n-1} \cap Y_n}) &\to \mathbb{A}^1_{D(Y_n; Y_1 \cap Y_n, \ldots, Y_{n-1} \cap Y_n)},
\end{aligned}$$

the last two horizontal arrows are isomorphisms, hence the first horizontal arrow is an isomorphism. Thus $N^1 K_0(D(X; Y_1, \ldots, Y_n)) = 0$, completing the proof. □

For a scheme X, let $K^B(X)$ denote the (possibly non-connective) spectrum defined by Thomason in [*] with $\pi_n(K^B(X)) = K_n(X)$, for $n \in \mathbb{Z}$. If X is regular, all negative homotopy groups vanish. We also will consider the spectrum $KH(X)$ defined by Weibel [W*]; the nth homotopy group of $KH(X)$ is denoted $KH_n(X)$. We recall from [W*] that there is a natural map

$$K^B(X) \to KH(X),$$

and a spectral sequence

(1.5) $$E_1^{p.q} = N^{-p} K_{-q}(X) \Rightarrow KH_{-p-q}(X).$$

In particular, (Thm*.* of [W*]), if X is K_p-regular for all $p \leq n$, then the map

$$K_p(X) \to KH_p(X)$$

is an isomorphism for all $p \leq n$. In addition, the "homotopy K-groups of X", $KH_n(X)$, satisfy:

KH-1) (Homotopy) the map

$$KH_n(X) \to KH_n(\mathbb{A}^1_X)$$

is an isomorphism.

KH-2) (Excision) Let $\phi\colon A \to B$ be a map of commutative rings, I an ideal of A such that $I = \phi(I)B$. Then, letting $KH(A, I)$ and $KH(B, I)$ denote the respective homotopy fibers of the maps

$$KH(A) \to KH(A/I)$$
$$KH(B) \to KH(B/I)$$

the map $KH(A, I) \to KH(B, I)$ induced by ϕ is a weak equivalence.

KH-3) (Mayer-Vietoris for closed subschemes) If $X = Y \cup Z$, with Y and Z closed subschemes of X, then

$$KH(X) \to KH(Y) \times KH(Z) \to KH(Y \cap Z)$$

is a homotopy fiber sequence.

KH-4) (Mayer-Vietoris for open subschemes) If $X = U \cup V$, with U and V open subschemes of X, then

$$KH(X) \to KH(U) \times KH(V) \to KH(U \cap V).$$

is a homotopy fiber sequence.

We now recall some basic facts about relative K-theory. To define relative K-theory in the needed generality, we use the language of n-cubes. The n-cube $<n>$ is the category associated to the set of subsets of $\{1, \ldots, n\}$, ordered under inclusion, i.e., the objects of $<n>$ are the subsets I of $\{1, \ldots, n\}$, and there is a unique morphism $\iota_{I\subset J}\colon I \to J$ if and only if $I \subset J$. If $\mathcal{C}$ is a category, we have the category of n-cubes in $\mathcal{C}$, $C(<n>)$, being the category of functors from $<n>$ to $\mathcal{C}$, e.g., n-cubes of sets, schemes, topological spaces, etc. The *split n-cube* is the category $<n>_{spl}$, gotten by adjoining to $<n>$ morphisms $\rho_{I\subset J}\colon J \to I$ if $I \subset J$, with

$$\rho_{I\subset J} \circ \iota_{I\subset J} = \mathrm{id}_I$$
$$\rho_{I\subset J} \circ \rho_{J\subset K} = \rho_{I\subset K}$$

A functor from $<n>_{spl}$ to $\mathcal{C}$ is called a *split n-cube*, and an extension of $F\colon <n> \to C$ to $F_{spl}\colon <n>_{spl} \to C$ is a *splitting* of F. We note that sending I to its complement I^c defines isomorphisms $<n> \to <n>^{op}$ and $<n>_{spl} \to <n>^{op}_{spl}$; we often define an n-cube or a split n-cube on the opposite category via these isomorphisms.

If $X\colon <n> \to C$ is an n-cube in $\mathcal{C}$, we form the map of $(n-1)$-cubes

$$X^{\pm}\colon X^+ \to X^-$$

by taking

$$X^+_I = X_I; \quad X^-_I = X_{I\cup\{n\}}; X^{\pm}_I = X(I \subset I \cup \{n\}).$$

This determines a functor from the category of n-cubes in $\mathcal{C}$ to the category of maps of $(n-1)$-cubes in $\mathcal{C}$. If $X\colon <n> \to \mathbf{Top}^*$ is an n-cube of pointed spaces, let $\mathrm{Fib}(X)\colon <n-1> \to \mathbf{Top}^*$ be the $(n-1)$-cube defined by setting $\mathrm{Fib}(X)_I$ equal to the homotopy fiber of the map

$$X^{\pm}_I\colon X^+_I \to X^-_I.$$

This gives the functor

$$\mathrm{Fib}\colon \mathbf{Top}^*(<n>) \to \mathbf{Top}^*(<n-1>);$$

iterating Fib n times defines the *iterated homotopy fiber* functor

$$\mathrm{Fib}^n\colon \mathbf{Top}^*(<n>) \to \mathbf{Top}^*;$$

we call $\mathrm{Fib}^n(X)$ the *iterated homotopy fiber* of X. A similar construction defines the iterated homotopy fiber of an n-cube of spectra.

Let X be a scheme, and $Y_1, \ldots, Y_n$ subschemes. Applying the functor $K(-)$ to the (opposite) n-cube $(X; Y_1, \ldots, Y_n)_*$ gives the n-cube of spaces $K(X; Y_1, \ldots, Y_n)_*$ with

$$K(X; Y_1, \ldots, Y_n)_I = K(\cap_{i\in I} Y_i).$$

Let $K(X; Y_1, \ldots, Y_n)$ denote the iterated homotopy fiber over this n-cube of spaces. $K(X; Y_1, \ldots, Y_n)$ is a model for the K-theory of X relative to $Y_1, \ldots, Y_n$ and the relative K-groups are given by

$$K_p(X; Y_1, \ldots, Y_n) = \pi_p(K(X; Y_1, \ldots, Y_n)).$$

Applying the functors $K^B(-)$ and $KH(-)$ to $(X; Y_1, \ldots, Y_n)_*$ and taking iterated homotopy fibers defines the relative spectra $K^B(X; Y_1, \ldots, Y_n)$ and $KH(X; Y_1, \ldots, Y_n)$; denote the nth homotopy groups, $n \in \mathbb{Z}$, by $K^B_n(X; Y_1, \ldots, Y_n)$ and $KH_n(X; Y_1, \ldots, Y_n)$, resp. We have the natural map

$$K^B(X; Y_1, \ldots, Y_n) \to KH(X; Y_1, \ldots, Y_n)$$

and a natural isomorphism

$$K_n(X; Y_1, \ldots, Y_n) \to K^B_n(X; Y_1, \ldots, Y_n)$$

for $n \geq 0$. If all the subschemes $Y_I := \cap_{i\in I} Y_i$ are regular, then

$$K^B(X; Y_1, \ldots, Y_n) \to KH(X; Y_1, \ldots, Y_n)$$

is a weak equivalence.

Let $D = D(X; Y_1, \ldots, Y_n)$, with X reduced. As a topological space, D is quotient of the disjoint union of 2^n copies of X:

$$D = \coprod_{I\in <n>} X/\equiv$$

where x in the copy of X indexed by I is identified with x in the copy of X indexed by J if $I \subset J$ and x is in $Y_{I\setminus J}$. We denote the copy of X indexed by $I \subset \{1, \ldots, n\}$ by X_I, and let $i_I\colon X_I \to D$ denote the inclusion. Let $D_1, \ldots, D_n$ be the reduced closed subschemes of D,

$$D_j = \cup_{I \text{ with } j\in I} X_I$$

Then $D_j \cap X_\emptyset = Y_j$ (scheme-theoretically) for each $j = 1, \ldots, n$, so the inclusion $i_\emptyset$ defines the maps

$$\begin{aligned} i^*_\emptyset\colon K(D; D_1, \ldots, D_n) &\to K(X; Y_1, \ldots, Y_n)\\ i^*_\emptyset\colon K^B(D; D_1, \ldots, D_n) &\to K^B(X; Y_1, \ldots, Y_n)\\ i^*_\emptyset\colon KH(D; D_1, \ldots, D_n) &\to KH(X; Y_1, \ldots, Y_n)\end{aligned}$$

If Z is a closed subscheme of X, the iterated double $D(Z; Y_1 \cap Z, \ldots, Y_n \cap Z)$ is naturally a closed subscheme of D; we denote this closed subscheme of D by $D(Z; Y_1, \ldots, Y_n)$.

Lemma 1.8. *Let Z be a scheme, $W_1, \ldots, W_n$ closed subschemes. Then the map*

$$i_\emptyset^*: KH(D(Z; W_n); D(W_1; W_n), \ldots, D(W_{n-1}; W_n), D_1) \to KH(Z; W_1, \ldots, W_n)$$

is a weak equivalence.

Proof. We may suppose Z is affine; the general case follows by taking an affine open cover of Z, noting that $D(Z; W_n)$ is a finite Z-scheme, and using Mayer-Vietoris (KH-4) for the resulting open covers of Z and $D(Z; W_n)$.

The spectra $KH(D(Z; W_n); D(W_1; W_n), \ldots, D(W_{n-1}; W_n), D_1)$ and $KH(Z; W_1, \ldots, W_n)$ are the iterated homotopy fibers over the n-cubes of spectra:

$$\begin{aligned} I &\mapsto KH(D(Z; W_n); D(W_1; W_n), \ldots, D(W_{n-1}; W_n), D_1)_I \\ I &\mapsto KH(Z; W_1, \ldots, W_n)_I \end{aligned}$$

The map $i_\emptyset$ thus gives the map of n-cubes of spectra

$$i_\emptyset^*: KH(D(Z; W); D(W_1; W_n), \ldots, D(W_{n-1}; W_n), D_1)_* \to KH(Z; W_1, \ldots, W_n)_*$$

whence the commutative square of $(n-1)$-cubes

$$(1.6)\qquad \begin{array}{ccc} KH(D(Z; W); D(W_1; W_n), \ldots, D(W_{n-1}; W_n), D_1)^+_* & \xrightarrow{i_\emptyset^{*+}} & KH(Z; W_1, \ldots, W_n)^+_* \\ \downarrow & & \downarrow \\ KH(D(Z; W); D(W_1; W_n), \ldots, D(W_{n-1}; W_n), D_1)^-_* & \xrightarrow{i_\emptyset^{*-}} & KH(Z; W_1, \ldots, W_n)^-_*. \end{array}$$

For each $I \subset \{1, \ldots, n-1\}$, we have

$$\begin{aligned} (D(W_1; W_n), \ldots, D(W_{n-1}; W_n), D_1)_I &= D(W_I, W_n) \\ (D(W_1; W_n), \ldots, D(W_{n-1}; W_n), D_1)_{I \cup \{n\}} &= D(W_I, W_n) \cap D_1 \end{aligned}$$

Taking $* = I$ in (1.6) thus gives the commutative square

$$(1.7)\qquad \begin{array}{ccc} KH(D(W_I, W_n)) & \to & KH(W_I) \\ \downarrow & & \downarrow \\ KH(D(W_I, W_n) \cap D_1) & \to & KH(W_I \cap W_n). \end{array}$$

Since Z is affine, so are W_I and W_n; thus, (1.7) is gotten by applying the functor KH to the diagram of rings

$$\begin{array}{ccc} D(R; I) & \xrightarrow{p_0} & R \\ p_1 \downarrow & & \downarrow p \\ R & \xrightarrow{p} & R/I \end{array}$$

Here, $W_I = \mathrm{Spec}(R)$, and the subscheme $W_I \cap W_n$ of W_I is defined by the ideal I; the maps p_0 and p_1 are the maps

$$p_0(r, r') = r; \quad p_1(r, r') = r',$$

and $p: R \to R/I$ is the quotient map. Since p_1 is surjective with kernel $(I, 0)$, we may apply excision to the square (1.7), and conclude that the induced map

$$(1.8)_I \qquad KH(D(W_I, W_n); W_I) \to KH(W_I; W_I \cap W_n)$$

is a weak equivalence. As the iterated homotopy fiber over an n-cube of spectra X is formed by first taking the $(n-1)$-cube of homotopy fibers $\mathrm{Fib}(X)$ of the map $X^\pm: X^+ \to X^-$, and then taking the iterated homotopy fiber over the $(n-1)$-cube $\mathrm{Fib}(X)$, the weak equivalences $(1.8)_I$ for $I \subset \{1, \ldots, n-1\}$, together with the Queztelcoatl lemma, imply that $i_\emptyset^*$ is a weak equivalence, as desired. □

Proposition 1.9. *Let X be a scheme, $Y_1, \dots, Y_n$ closed subschemes. Then the map*

$$i_\emptyset^*\colon KH(D(X;Y_1,\dots,Y_n);D_1,\dots,D_n) \to KH(X;Y_1,\dots,Y_n)$$

is a weak equivalence.

Proof. Repeatedly applying Lemma 1.8, we have the weak equivalences

$$\begin{aligned}
&KH(D(X;Y_1,\dots,Y_n);D_1,\dots,D_n)\\
&\quad\to KH(D(X;Y_1,\dots,Y_{n-1});D_1,\dots,D_{n-1},D(Y_n;Y_1,\dots,Y_{n-1}))\\
&\quad\to KH(D(X;Y_1,\dots,Y_{n-2});D_1,\dots,D_{n-2},D(Y_{n-1};Y_1,\dots,Y_{n-2}),D(Y_n;Y_1,\dots,Y_{n-2}))\\
&\quad\cdot\\
&\quad\cdot\\
&\quad\cdot\\
&\quad\to KH(X;Y_1,\dots,Y_n).
\end{aligned}$$

This proves the result. □

Theorem 1.10. *Let X be a scheme, $Y_1, \dots, Y_n$ closed subschemes. Suppose that*

i) For each $I \subset \{1,\dots,n\}$ the scheme Y_I is regular.

ii) The iterated double $D(X;Y_1,\dots,Y_n)$ is K_m-regular.

Then the map

$$i_\emptyset^*\colon K_m^B(D(X;Y_1,\dots,Y_n);D_1,\dots,D_n) \to K_m^B(X;Y_1,\dots,Y_n)$$

is an isomorphism. If $m \geq 0$, then the map

$$i_\emptyset^*\colon K_m(D(X;Y_1,\dots,Y_n);D_1,\dots,D_n) \to K_m(X;Y_1,\dots,Y_n)$$

is an isomorphism.

Proof. Under the assumption (i), the map

$$K^B(Y_I) \to KH(Y_I)$$

is a weak equivalence for each $I \subset \{1,\dots,n\}$. Thus, the natural map

$$K^B(X;Y_1,\dots,Y_n) \to KH(X;Y_1,\dots,Y_n)$$

is a weak equivalence. Under the assumption of K_m-regularity, it follows from Corollary 1.4 and the spectral sequence (1.5) that the natural map

$$K_m^B(D(X;Y_1,\dots,Y_n)) \to KH_m(D(X;Y_1,\dots,Y_n))$$

is an isomorphism.

The (opposite) n-cube of schemes $(D(X;Y_1,\dots,Y_n);D_1,\dots,D_n)_*$ is split; thus there are natural projections

$$\begin{aligned}
K_m^B(D(X;Y_1,\dots,Y_n)) &\to K_m^B(D(X;Y_1,\dots,Y_n);D_1,\dots,D_n)\\
KH_m(D(X;Y_1,\dots,Y_n)) &\to KH_m(D(X;Y_1,\dots,Y_n);D_1,\dots,D_n)
\end{aligned}$$

making the diagram

$$\begin{array}{ccc} K_m^B(D(X;Y_1,\ldots,Y_n)) & \to & K_m^B(D(X;Y_1,\ldots,Y_n);D_1,\ldots,D_n) \\ \downarrow & & \downarrow \\ KH_m(D(X;Y_1,\ldots,Y_n)) & \to & KH_m(D(X;Y_1,\ldots,Y_n);D_1,\ldots,D_n) \end{array}$$

commute. Thus, the natural map

$$K_m^B(D(X;Y_1,\ldots,Y_n);D_1,\ldots,D_n) \to KH_m(D(X;Y_1,\ldots,Y_n);D_1,\ldots,D_n)$$

is an isomorphism as well. From the commutative diagram

$$\begin{array}{ccc} K_m^B(D(X;Y_1,\ldots,Y_n);D_1,\ldots,D_n) & \to & K_m^B(X;Y_1,\ldots,Y_n) \\ \downarrow & & \downarrow \\ KH_m(D(X;Y_1,\ldots,Y_n);D_1,\ldots,D_n) & \to & KH_m(X;Y_1,\ldots,Y_n) \end{array}$$

we see that

$$K_m^B(D(X;Y_1,\ldots,Y_n);D_1,\ldots,D_n) \to K_m^B(X;Y_1,\ldots,Y_n)$$

is an isomorphism, completing the proof of the first assertion. The second follows from the fact that

$$K_m^B(D(X;Y_1,\ldots,Y_n);D_1,\ldots,D_n) = K_m(D(X;Y_1,\ldots,Y_n);D_1,\ldots,D_n)$$
$$K_m^B(X;Y_1,\ldots,Y_n) = K_m(X;Y_1,\ldots,Y_n)$$

for all $m \geq 0$. $\square$

If $D_1,\ldots,D_n$ are codimension one reduced subschemes, intersecting properly, let D be the divisor $D_1+\ldots+D_n$. We often write $K(X;D)$ for $K(X;D_1,\ldots,D_n)$, etc.

§2 Relative cycles and relative K_0

We use Bloch's idea of a relative cycle to give a cycle-theoretic interpretation of the relative K_0. We start with a discussion of relative K-theory with supports, and the functorial λ-operations on these groups.

If $X = \mathrm{Spec}(R)$ is an affine scheme, Hiller [H] and Kratzer [K] have defined λ-operations $\lambda^k\colon K_p(X) \to K_p(X)$, satisfying the special λ-ring identities, by giving maps

$$\lambda^k_n\colon BGL_n(X)^+ \to BGL(X)^+$$

which are stable, up to homotopy, in n.

Let Y be a scheme, and U an open subscheme; let Z be the complement $Y\backslash U$. Define the space $K^Z(Y)$ as the homotopy fiber of the restriction map $K(Y) \to K(U)$. Similarly, if we have closed subschemes $D_1,\ldots D_n$ of Y, define $K^Z(Y;D_1,\ldots,D_n)$ as the homotopy fiber of the restriction map $K(Y;D_1,\ldots,D_n) \to K(U;U\cap D_1,\ldots,U\cap D_n)$. The group $K^Z_p(Y) := \pi_p(K^Z(Y))$ is the pth K-group of Y *with supports along* Z; the group $K^Z_p(Y;D_1,\ldots,D_n) := \pi_p(K^Z(Y;D_1,\ldots,D_n))$ is the pth K-group of Y with supports along Z, relative to $D_1,\ldots,D_n$.

Suppose that X is a regular scheme over a field. Then, following Gillet [G], we have the following sheaf-theoretic description of $K_p(X)$. Form the sheaf $\mathcal{K}_X$ of simplicial sets on X associated to the pre-sheaf $V \mapsto BGL^+(\Gamma(V,\mathcal{O}_V)) \times \mathbb{Z}$. Then there is a natural isomorphism $K_p(X) \to \mathbb{H}^{-p}(X,\mathcal{K}_X)$. We have as well the sheaves of simplicial sets $\mathcal{K}_{X,n}$ gotten by using BGL^+_n instead of BGL^+; the stability results of Suslin [S] show that $\mathbb{H}^{-p}(X,\mathcal{K}_{X,n}) = K_p(X)$ for all n sufficiently large.

Soulé [So] has given λ-operations on the sheaf level, $\lambda^k_n\colon \mathcal{K}_{X,n} \to \mathcal{K}_X$, which satisfy the special λ-ring identities in the closed model category of simplicial sheaves on the big Zariski sité over X, and are stable, in the model category, in n. This then gives functorial λ-operations λ^k on the groups $K^Z_p(X)$, satisfying the special λ-ring identities. These operations agree with the λ-operations of Hiller and Kratzer on $K_p(X)$ when X is affine.

Grayson [Gr1] has another approach to the construction of λ-operations, which gives functorial operations for an arbitrary scheme, and agrees with the operations of Soulé or with those of Hiller-Kratzer when defined. It is not known, however, whether Grayson's λ-operations satisfy the special λ-ring identities. We now give a brief sketch of Grayson's construction.

If P is an exact category, Grayson and Gillet [GG] have constructed a functorial simplicial set $GG(P)$ whose geometric realization is naturally homotopy equivalent to ΩBQP. Grayson constructs the λ-operation λ^k as a simplicial map from a certain subdivision of $GG(P)$ to a certain other subdivision. This gives the operation λ^k on the geometric realization of $GG(P)$, functorial in the category $\mathcal{P}$. Grayson has shown that these operations agree with those defined by Hiller and Kratzer in the case $P = P_X$ for X affine; this implies that they agree with the operations of Soulé in the regular case. In any case, we may apply the construction of Grayson to any iterated homotopy fiber as above, giving functorial λ-operations on the relative groups with supports $K^Z_p(X;D_1,\ldots,D_n)$, which agree with the operations defined by Hiller-Kratzer or Soulé, when the latter operations are defined. In particular, this defines functorial Adams operations ψ^k on $K^Z_p(X;D_1,\ldots,D_n)$, although the standard properties of the Adams operations are only known in the cases discussed by Hiller-Kratzer, or by Soulé. Additionally, Grayson [Gr2] has defined a delooping of ψ^k; in particular, the operations ψ^k on $K^Z_p(X;D_1,\ldots,D_n)$ are group homomorphisms for all $p \geq 0$.

We fix an integer $k > 1$, and let $K^Z_p(X;D_1,\ldots,D_n)^{(q)}$ denote the k^q-characteristic subspace of ψ^k acting on $K^Z_p(X;D_1,\ldots,D_n)\otimes\mathbb{Q}$; i.e., the set of $v \in K^Z_p(X;D_1,\ldots,D_n)\otimes\mathbb{Q}$ such that

$$(\psi^k - k^q\cdot\mathrm{id})^N(v) = 0$$

for some $N > 0$.

Lemma 2.1. *If X is regular and $D_1+\ldots+D_n$ is a reduced normal crossing divisor, we have the functorial finite direct sum decomposition*

$$K^Z_p(X;D_1,\ldots,D_n)\otimes\mathbb{Q} = \oplus_q K^Z_p(X;D_1,\ldots,D_n)^{(q)},$$

In addition, there is an integer N such that $K_p^Z(X; D_1, \ldots, D_n)^{(q)}$ is the subspace for which $(\psi^k - k^q \cdot \mathrm{id})^N = 0$.

Proof. Let V be a $\mathbb{Q}$-vector spaces with an endomorphism L, and suppose we have an L-stable flag

$$0 = V_0 \subset V_1 \subset \ldots \subset V_n = V$$

in V. Suppose further that each quotient $W_i := V_i/V_{i-1}$ breaks up into a finite direct sum of subspaces

$$W_i = \oplus_q W_i^{(q)},$$

where L acts on $W_i^{(q)}$ by multiplication by k^q. Then one easily sees that V is a finite direct sum of subspaces $V^{(q)}$, where $V^{(q)}$ is the subspace of V on which $(L - k^q \cdot \mathrm{id})^n = 0$. Thus the finite direct sum decomposition

$$V = \oplus_q V^{(q)}$$

is functorial on the full subcategory of the category of $\mathbb{Q}[L]$-modules consisting of those $\mathbb{Q}[L]$-modules with finite filtration as above.

By considering the various long exact localization and relativization sequences associated with Z, X and $D_1, \ldots, D_n$, we see that each group $K_p^Z(X; D_1, \ldots, D_n) \otimes \mathbb{Q}$ has a ψ^k-stable filtration with successive quotients being ψ^k-subquotients of ψ^k-modules of the form $K_q(Y) \otimes \mathbb{Q}$, where Y is a regular scheme. Thus, the considerations of the previous paragraph prove the lemma. □

In the general setting, we have only the functorial subspaces

$$K_p^Z(X; D_1, \ldots, D_n)^{(q)} \subset K_p^Z(X; D_1, \ldots, D_n) \otimes \mathbb{Q}.$$

Let X be a regular k-scheme, and s a finite set of closed subsets of X. Let $Z^d(X)$ denote the group of codimension d cycles on X, $Z^d(X)_s$ the subgroup of $Z^d(X)$ consisting of cycles which intersect S properly for each $S \in s$. If $D_1, \ldots, D_n$ are distinct locally principal closed subschemes of X, and I is a subset of $\{1, \ldots, n\}$, let $D_I = \cap_{i \in I} D_i$. Let D be the divisor $D_1 + \ldots + D_n$, let $s(D) = \{D_I \mid I \subset \{1, \ldots, n\}\}$, and $s(D) \cap s$ the set of closed subsets $D_I \cap S$, for $I \subset \{1, \ldots, n\}$ and $S \in s$, together with the subsets D_I, $I \subset \{1, \ldots, n\}$. Let $Z^d(X; D)_s$ be the subgroup of $Z^d(X)_{s(D) \cap s}$ consisting of cycles Z with $Z \cdot D = 0$. Bloch [B] has defined a homomorphism

$$cyc^d \colon Z^d(X; D)_s \to K_0(X; D)^{(d)},$$

which now describe.

If W is a closed subset of X, let $Z^d(X; D)^W$ denote the subgroup of $Z^d(X; D)$ consisting of cycles supported on W.

If $W \subset T$ are closed subsets of X, let $i_{W,T*} \colon K_p^W(X; D)^{(q)} \to K_p^T(X; D)^{(q)}$ be the natural map. Similarly, suppose we have $W \subset Y \subset X$, where Y is a regular closed subscheme of X, of pure codimension c, with Y intersecting each D_I properly. The natural maps

$$K(Y \cap D_I) \to K^{Y \cap D_I}(D_I); \quad K(Y \cap D_I \backslash W) \to K^{Y \cap D_I \backslash W}(D_I \backslash W)$$

followed by the natural maps

$$K^{Y \cap D_I}(D_I) \to K(D_I); \quad K^{Y \cap D_I \backslash W}(D_I \backslash W) \to K(D_I \backslash W)$$

defines the map

$$p^W_{Y\subset X}\colon K^W(Y;Y\cap D)\to K^W(X;D).$$

Composing $p^W_{Y\subset X}$ with the inclusion of the summand $K^W_p(Y;Y\cap D)^{(q-c)}$ in $K^W_p(Y;Y\cap D)$ and the projection of $K^W_p(X;D)$ to the summand $K^W_p(X;D)^{(q)}$ defines the map

$$p^W_{Y\subset X}\colon K^W_p(Y;Y\cap D)^{(q-c)}\to K^W_p(X;D)^{(q)}.$$

Similarly, the inclusions $W\subset T$ and $Y\subset X$ induce the maps

$$i_{W,T*}\colon Z^d(X;D)^W\to Z^d(X;D)^T;\quad p^W_{Y\subset X}\colon Z^{d-c}(Y;Y\cap D)^W\to Z^d(X;D)^W.$$

Lemma 2.2. *Let W be a pure codimension d closed subset of X, such that each irreducible component of W intersects each D_I properly. Then*

i) There is an isomorphism

$$cyc^W\colon Z^d(X;D)^W\otimes\mathbb{Q}\to K^W_0(X;D)^{(d)},$$

functorial for pull-back by flat maps $X'\to X$.

ii) If W' is another pure codimension d closed subset of X with $W\subset W'$, and Z is in $Z^d(X;D)^W\otimes\mathbb{Q}$, then

$$i_{W,W'*}(cyc^W(Z))=cyc^{W'}(Z).$$

iii) Suppose $W\subset Y\subset X$, where Y is a regular codimension c closed subscheme of X such that Y intersects each D_I properly. Then the diagram

$$\begin{array}{ccc} Z^{d-c}(Y;D\cap Y)^W\otimes\mathbb{Q} & \overset{cyc^W}{\to} & K^W_0(Y;D\cap Y)^{(d-c)}\\ p^W_{Y\subset X}\downarrow & & \downarrow p^W_{Y\subset X}\\ Z^d(X;D)^W\otimes\mathbb{Q} & \overset{cyc^W}{\to} & K^W_0(X;D)^{(d)}\end{array}$$

commutes.

Proof. (following Bloch) We have $D=D_1+\ldots+D_n$, with each D_j regular. We first show, by induction on n, that

$$K^W_a(X;D)^{(b)}=0;\quad \text{for } a>0, b\le d. \tag{2.1}$$

Suppose first that $n=d=0$; we may then suppose $W=X$. If F is a field, Soulé [So] has shown that

$$K_s(F)^{(q)}=0\quad \text{for } s>0, q\le 0. \tag{2.2}$$

Let X^p denote the set of codimension p points of X. Since X is regular over a field, we have the Quillen spectral sequence

$$E^{p,q}_1=\oplus_{x\in X^p}K_{-q}(k(x))^{(b-p)}\Rightarrow K_{-p-q}(X)^{(b)}. \tag{2.3}$$

By (2.2), this proves (2.1) for $n = 0$, $W = X$. Now suppose W is regular of codimension d. By the Riemann-Roch theorem [G], and Quillen's localization theorem ([Q] Theorem *.*) the weak equivalence $K(W) \to K^W(X)$ implies that the map

$$p^W_{W\subset X}: K_a(W)^{(a)} \cong K^W_a(X)^{(a+d)} \tag{2.4}$$

is an isomorphism. This proves (2.1) in this case. If W is an arbitrary closed subset of X of pure codimension d, let W' be a closed subset of W such that $W\backslash W'$ is regular, and W' has pure codimension $d+1$. By downward induction on d (starting with $d = \dim(X)+1$ we may assume that (2.1) is true for W'. Then (2.1) for W follows from the exact localization sequence

$$\ldots \to K^{W'}_a(X)^{(b)} \to K^W_a(X)^{(b)} \to K^{W\backslash W'}_a(X\backslash W')^{(b)} \to \ldots$$

This completes the proof of (2.1) for $n = 0$. The general case follows by induction and the exact relativization sequence

$$\ldots \to K^{W\cap D_n}_{a+1}(D_n, D_n\cap D_1,\ldots,D_n\cap D_{n-1})^{(b)} \to K^W_a(X, D_1,\ldots,D_n)^{(b)} \to K^W_a(X, D_1,\ldots,D_{n-1})^{(b)} \to \ldots$$

We now prove the statement of the lemma, proceeding by induction on n. For $n = 0$, we use (2.4) to give the isomorphism

$$p^W_{W\subset X}: K_0(W)^{(0)} \cong K^W_0(X)^{(d)}, \tag{2.5}$$

in case W is regular. Using the spectral sequence (2.3) (with $X = W$), we see that the restriction map

$$K_0(W)^{(0)} \to K_0(k(W))^{(0)} \tag{2.6}$$

is an isomorphism. As $K_0(k(W))^{(0)} = K_0(k(W))\otimes\mathbb{Q}$ is the $\mathbb{Q}$-vector space on the irreducible components of W, the inverse of the isomorphism (2.6) composed with the isomorphism (2.5) defines the isomorphism

$$cyc^W: Z^0(W)\otimes\mathbb{Q} \to K^W_0(X)^{(d)}.$$

If W is an arbitrary closed subset of codimension d, let $W'\subset W$ be a closed subset of codimension $d+1$ on X such that $W\backslash W'$ is regular. Then the spectral sequence (2.3) implies the map

$$K^W_0(X)^{(d)} \to K^{W\backslash W'}_0(X\backslash W')^{(d)}$$

is an isomorphism. As $Z^0(W)\to Z^0(W\backslash W')$ is also an isomorphism, the map $cyc^{W\backslash W'}$ induces the isomorphism

$$cyc^W: Z^0(W)\otimes\mathbb{Q} \to K^W_0(X)^{(d)}.$$

in this case as well. Let $T\supset W$ be a closed subset of X, of pure codimension d. The compatibility

$$i_{W,T*}\circ cyc^W = cyc^T\circ i_{W,T*} \tag{2.7}$$

is obvious if W is a connected component of T; in general, we may remove a closed subset of T of codimension $d+1$ on X to reduce the proof of (2.7) to this case.

If Y is a regular closed codimension c subset of X, and $W \subset Y \subset X$ is a regular closed codimension d closed subset of X, we have the homotopy commutative diagram

$$\begin{array}{ccc} K(W) & \rightarrow & K^W(Y) \\ & \searrow \quad \swarrow & \\ & K^W(X) & . \end{array}$$

This gives the compatibility

$$p^W_{Y\subset X} \circ cyc^W = cyc^W \circ p^W_{Y\subset X} \tag{2.8}$$

in this case; for W an arbitrary closed codimension d closed subset, the compatibility (2.8) follows by localization as above.

In addition, Serre's intersection multiplicity formula shows that, for A a closed regular subscheme of X, intersecting each component of W properly, we have the commutative digram

$$\begin{array}{ccc} Z^d(X)^W \otimes \mathbb{Q} & \overset{cyc^W}{\rightarrow} & K_0^W(X)^{(d)} \\ \cdot A \downarrow & & \downarrow i_A^* \\ Z^d(A)^{W\cap A} \otimes \mathbb{Q} & \overset{cyc^{W\cap A}}{\rightarrow} & K_0^{W\cap A}(A)^{(d)}. \end{array}$$

For general n, we have the subscheme $D_{i,n} = D_i \cap D_n$ of D_n. We have the long exact relativization sequence

$$\ldots \rightarrow K_1^{W\cap D_n}(D_n, D_{1,n}, \ldots, D_{n-1,n})^{(d)} \rightarrow K_0^W(X, D_1, \ldots, D_n)^{(d)} \rightarrow K_0^W(X, D_1, \ldots, D_{n-1})^{(d)}$$

$$\rightarrow K_0^{W\cap D_n}(D_n, D_{1,n}, \ldots, D_{n-1,n})^{(d)}.$$

Since $K_1^{W\cap D_n}(D_n, D_{1,n}, \ldots, D_{n-1,n})^{(d)} = 0$, we have the exact sequence

$$0 \rightarrow K_0^W(X, D_1, \ldots, D_n)^{(d)} \rightarrow K_0^W(X, D_1, \ldots, D_{n-1})^{(d)} \rightarrow K_0^{W\cap D_n}(D_n, D_{1,n}, \ldots, D_{n-1,n})^{(d)}.$$

This in turn gives the commutative ladder with exact rows

$$\begin{array}{ccccccc} 0 & \rightarrow & Z^d(X;D)^W \otimes \mathbb{Q} & \rightarrow & Z^d(X;D-D_n)^W \otimes \mathbb{Q} & \overset{\cdot D_n}{\rightarrow} & Z^d(D_n;(D-D_n)\cdot D_n)^{W\cap D_n} \otimes \mathbb{Q} \\ & & cyc^W \downarrow & & cyc^W \downarrow & & cyc^{W\cap D_n} \downarrow \\ 0 & \rightarrow & K_0^W(X, D_1, \ldots, D_n)^{(d)} & \rightarrow & K_0^W(X, D_1, \ldots, D_{n-1})^{(d)} & \rightarrow & K_0^{W\cap D_n}(D_n, D_{1,n}, \ldots, D_{n-1,n})^{(d)}. \end{array}$$

The lemma now follows by induction and the five lemma. □

Let s be a finite set of closed subsets of X. Let $K_0^d(X;D)_s^{(d)}$ denote the direct limit of the groups $K_0^W(X;D)^{(d)}$, as W ranges over pure codimension d closed subsets of X which intersect each D_I properly and intersect each $D_I \cap S$ properly for each $S \in s$. From Lemma 2.2, we have the isomorphism

$$cyc^d \colon Z^d(X;D)_s \otimes \mathbb{Q} \rightarrow K_0^d(X;D)_s^{(d)}.$$

We now investigate the natural map $K_0^d(X;D)_s^{(d)} \rightarrow K_0(X;D)^{(d)}$.

Theorem 2.3. *Suppose X is a regular, quasi-projective scheme over an infinite field, and the divisor $D = Y_1 + \ldots + Y_n$ is a reduced normal crossing divisor. Supppose further that X is an A-scheme for some ring A, and there are elements $f_1, \ldots, f_n$ of A, generating the unit ideal, such that, for each $f = f_i$, the collection of closed subschemes $Y_{1f}, \ldots, Y_{nf}$ of X_f is split. Let s be a finite collection of closed subsets of X. Then the map*

$$K_0^d(X;D)_s^{(d)} \to K_0(X;D)^{(d)}$$

is surjective.

Proof. We may suppose X is irreducible. Let T be the iterated double

$$T := D(X; Y_1, \ldots, Y_n)$$

We recall that T has 2^n irreducible components, each isomorphic to X; as in section 1, we index the components of T by the subsets I of $\{1, \ldots, n\}$, and let $T_1, \ldots, T_n$ denote the closed subschemes

$$T_j = \cup_{I \text{ with } j \in I} X_I.$$

Via this indexing we have the inclusion

$$i_\emptyset : X \to T; \quad i_\emptyset^*(T_j) = Y_j, \ j = 1, \ldots, n.$$

By Theorem 1.6 and Theorem 1.10, the map

$$i_\emptyset^* : K_0(T; T_1, \ldots, T_n) \to K_0(X; Y_1, \ldots, Y_n)$$

is an isomorphism. The map $i_\emptyset^*$ is therefore an isomorphism of ψ^k-modules.

The group $(\mathbb{Z}/2)^n$ acts on T: for each $i = 1, \ldots, n$, we may view T as the double

$$T = D(D(X; Y_1, \ldots, \hat{Y}_i, \ldots, Y_n); D(X; Y_1, \ldots, \hat{Y}_i, \ldots, Y_n)) \tag{2.9}$$

We then have the involution

$$\tau_i : T \to T$$

gotten by exchanging the two copies of $D(X; Y_1, \ldots, \hat{Y}_i, \ldots, Y_n)$ in the above representation of T. Similarly, the representation (2.9) of T defines the ith inclusion

$$\iota_i : D(X; Y_1, \ldots, \hat{Y}_i, \ldots, Y_n) \to T$$

identifying $D(X; Y_1, \ldots, \hat{Y}_i, \ldots, Y_n)$ with T_i, and the ith projection

$$\pi_i : T \to D(X; Y_1, \ldots, \hat{Y}_i, \ldots, Y_n)$$

The inclusion

$$K_0(T; T_1, \ldots, T_n) \to K_0(T)$$

is then split by the projection operator

$$\sigma = \sum_{i=1}^{n} (\mathrm{id} - \pi_i^* \circ \iota_i^*).$$

Similarly, if W is a closed subset of T, invariant under the automorphisms τ_i, we have the splitting of the map

$$K_0^W(T; T_1, \ldots, T_n) \to K_p^W(T),$$

with splitting σ^W defined by the same formula as above i.e., we have the commutative diagram

$$\begin{array}{ccc} K_p^W(T) & \stackrel{\sigma^W}{\to} & K_0^W(T; T_1, \ldots, T_n) \\ \downarrow & & \downarrow \\ K_p(T) & \stackrel{\sigma}{\to} & K_0(T; T_1, \ldots, T_n). \end{array}$$

By Grothendieck [G], $K_0(T)$ is a special λ-ring; as $K_0(T; T_1, \ldots, T_n)$ is a λ-summand of $K_0(T)$, it follows that $K_0(T; T_1, \ldots, T_n)$ is a special λ-ring (without identity) as well.

We recall the result of Fulton [F]: Let Z be a quasi-projective scheme over a field k, and let η be an element of $K_0(Z)$. Then there is a map $f\colon Z \to H$, where H is a homogeneous space for GL_n/k, for some n, H is proper over Spec(k), and there is an element ρ of $K_0(H)$ with $f^*(\rho) = \eta$.

Let then η be an element of $K_0(T; T_1, \ldots, T_n)^{(d)} = K_0(X; Y_1, \ldots, Y_n)^{(d)}$. Consider η as an element of $K_0(T)^{(d)}$. Take $f\colon Y \to H$ and $\rho \in K_0(H) \otimes \mathbb{Q}$ as above, so that $f^*(\rho) = \eta$ in $K_0(T)^{(d)}$. We may project ρ to $\rho^{(d)} \in K_0(H)^{(d)}$; since $K_0(T; T_1, \ldots, T_n)$ is a special λ-ring, the projection on this subspace is thus functorial, and we have

$$f^*(\rho^{(d)}) = \eta.$$

On the other hand, using the Riemann-Roch theorem on the smooth variety H, there is a pure codimension d closed subset Z of H and an element χ of $K_0^Z(H)$ with image $\rho^{(d)}$ in $K_0(H) \otimes \mathbb{Q}$.

For $S \in s$, let $T(S)$ denote the subscheme $D(S, Y_1, \ldots, Y_n)$ of T. We now apply the tranversality result of Kleiman [Kl], which states that there is an element g of $GL_n(k)$ such that $f^{-1}(gZ)$ is pure codimension d on T and intersects $X_{I_1} \cap \ldots \cap X_{I_t} \cap T(S)$ of T properly, for each collection of indices $I_1, \ldots, I_t$, $I_j \subset \{1, \ldots, n\}$, and each closed subset $S \in s$. Additionally, $GL_n(k)$ acts trivially on $K_0(H)$, so we may assume $g = \mathrm{id}$, after changing notation. Let W be a pure codimension d closed subset of T containing $f^{-1}(Z)$, intersecting each $X_{I_1} \cap \ldots \cap X_{I_t} \cap T(S)$ properly and invariant under all the τ_i, $i = 1, \ldots, n$. Let γ be the element $\sigma(f^*(\chi))$ of $K_0^W(T)$. Then γ is in $K_0^W(T; T_1, \ldots, T_n)$ and has image η in $K_0(T; T_1, \ldots, T_n) \otimes \mathbb{Q}$. Let $W' = i_\emptyset^*(W)$ and let $\beta = i_\emptyset^*(\gamma)$,

$$\beta \in K_0^{W'}(X; Y_1, \ldots, Y_n).$$

Then β goes to η in $K_0(X; Y_1, \ldots, Y_n) \otimes \mathbb{Q}$. By Lemma 2.1, we have the functorial finite direct sum decomposition

$$K_0^{W'}(X; D) = \oplus_q K_0^{W'}(X; D)^{(q)}.$$

Let α be the projection of β to the factor $K_0^{W'}(X; D)^{(d)}$; then α has image η in $K_0(X; D) \otimes \mathbb{Q}$, proving the theorem. □

Let $cyc\colon \mathcal{Z}^d(X; D) \otimes \mathbb{Q} \to K_0(X; D)^{(d)}$ be the composition of the map $cyc^d\colon \mathcal{Z}^d(X; D) \otimes \mathbb{Q} \to K_0^d(X; D)^{(d)}$ and the natural map $K_0^d(X; D)^{(d)} \to K_0^d(X; D)^{(d)}$

Corollary 2.4. *Suppose X is a regular, quasi-projective scheme over an infinite field, and the divisor $D = D_1 + \ldots + D_n$ is a reduced normal crossing divisor. Suppose further that X is an A-scheme for some reduced ring A, and there are elements $f_1, \ldots, f_n$ of A, generating the unit ideal, such that, for each $f = f_i$, the collection of closed subschemes $D_{1f}, \ldots, D_{nf}$ of X_f is split. Let s be a finite collection of closed subsets of X. Then the map*

$$cyc\colon \mathcal{Z}^d(X;D)_s \otimes \mathbb{Q} \to K_0(X;D)^{(d)}$$

is surjective.

Proof. This follows directly from Lemma 2.2 and Theorem 2.3. □

We now investigate the kernel of the map *cyc*. For a set s of closed subsets of X, let $s \times \mathbb{A}^1$ denote the set of closed subsets $S \times \mathbb{A}^1$ of $X \times \mathbb{A}^1$. We have the group $Z^d(X \times \mathbb{A}^1; D \times \mathbb{A}^1 + X \times 1)_{s \times \mathbb{A}^1}$ and the subgroup $Z^d(X \times \mathbb{A}^1; D \times \mathbb{A}^1 + X \times 1)_{X \times 0 \cup s \times \mathbb{A}^1}$ consisting of cycles which intersect $X \times 0$ properly. This gives the map

$$Z^d(X \times \mathbb{A}^1; D \times \mathbb{A}^1 + X \times 1)_{X \times 0 \cup s \times \mathbb{A}^1} \to \mathcal{Z}^d(X;D)_s$$

by identifying X with $X \times 0$ and intersecting a cycle in $Z^d(X \times \mathbb{A}^1; D \times \mathbb{A}^1 + X \times 1)$ with $X \times 0$. We let $\mathrm{CH}^d(X;D)_s$ denote the quotient group $Z^d(X;D)_s/\mathrm{Im}(\mathcal{Z}^d(X \times \mathbb{A}^1; D \times \mathbb{A}^1 + X \times 1)_{X \times 0} \cup s \times \mathbb{A}^1)$.

Lemma 2.5. *The map*

$$cyc\colon Z^d(X;D)_s \otimes \mathbb{Q} \to K_0(X;D)^{(d)}$$

descends to a map

$$cyc\colon \mathrm{CH}^d(X;D)_s \otimes \mathbb{Q} \to K_0(X;D)^{(d)}$$

Proof. We have the commutative diagram

$$\begin{array}{ccc} Z^d(X \times \mathbb{A}^1; D \times \mathbb{A}^1 + X \times 1)_{X \times 0} \otimes \mathbb{Q} & \stackrel{cyc}{\to} & K_0(X \times \mathbb{A}^1; D \times \mathbb{A}^1 + X \times 1)^{(d)} \\ \cdot(X \times 0) \downarrow & & \downarrow i^*_{X \times 0} \\ \mathcal{Z}^d(X;D) \otimes \mathbb{Q} & \stackrel{cyc}{\to} & K_0(X;D)^{(d)}. \end{array}$$

We have as well the exact relativization sequence

$$\ldots \to K_{p+1}(X \times \mathbb{A}^1; D \times \mathbb{A}^1) \to K_{p+1}(X \times 1; D \times 1)$$
$$\to K_p(X \times \mathbb{A}^1; D \times \mathbb{A}^1 + X \times 1) \to K_p(X \times \mathbb{A}^1; D \times \mathbb{A}^1)$$
$$\to K_p(X \times 1; D \times 1) \to \ldots;$$

since the maps

$$K_p(X \times \mathbb{A}^1; D \times \mathbb{A}^1) \to K_p(X \times 1; D \times 1)$$

are all isomorphisms by the homotopy property for the K-groups of regular schemes, the groups

$$K_p(X \times \mathbb{A}^1; D \times \mathbb{A}^1 + X \times 1)$$

are all zero. Thus the composition

$$cyc \circ (- \cdot X \times 0)\colon \mathcal{Z}^d_{X \times 0}(X \times \mathbb{A}^1; D \times \mathbb{A}^1 + X \times 1) \otimes \mathbb{Q} \to K_0(X;D)^{(d)}$$

is the zero map, proving the lemma. □

Let U be an open subset of X, W the complement $X \backslash U$. Using the model BQP_- for $\Omega^{-1}K(-)$, we form the spaces

$$\Omega^{-1}K(X \times \mathbb{A}^1; D \times \mathbb{A}^1 + X \times 1 + X \times 0),$$
$$\Omega^{-1}K(X \times \mathbb{A}^1; D \times \mathbb{A}^1 + X \times 1 + U \times 0),$$
$$\Omega^{-1}K^W(X; D)$$

and

$$\Omega^{-1}K(U; D);$$

$U \times 0$ is not closed, but we define $\Omega^{-1}K(X \times \mathbb{A}^1, D \times \mathbb{A}^1 + X \times 1 + U \times 0)$ as the homotopy fiber of the map

$$\Omega^{-1}K(X \times \mathbb{A}^1, D \times \mathbb{A}^1 + X \times 1) \to \Omega^{-1}K(U \times 0, D \times 0).$$

By the Quetzelcoatl lemma, the homotopy fiber of the map

$$\Omega^{-1}K(X \times \mathbb{A}^1, D \times \mathbb{A}^1 + X \times 1 + X \times 0) \to$$
$$\Omega^{-1}K(X \times \mathbb{A}^1; D \times \mathbb{A}^1 + X \times 1 + U \times 0)$$

is the homotopy fiber of $* \to \Omega^{-1}K^W(X; D)$, i.e., $K^W(X; D)$. This gives us the homotopy commutative diagram

$$\begin{array}{ccc} K^W(X;D) & = & K^W(X;D) \\ \downarrow & & \downarrow \\ K(X;D) & \to & \Omega^{-1}K(X \times \mathbb{A}^1; D \times \mathbb{A}^1 + X \times 1 + X \times 0) \\ \downarrow & & \downarrow \\ K(U;D) & \to & \Omega^{-1}K(X \times \mathbb{A}^1; D \times \mathbb{A}^1 + X \times 1 + U \times 0) \end{array} \tag{2.10}$$

where the columns are homotopy fiber sequences.

Let $E = D_1 \times \mathbb{A}^1 + \ldots + D_n \times \mathbb{A}^1 + X \times 1 + X \times 0$. Let T be a closed subset of $X \times \mathbb{A}^1$ such that T intersects each E_I properly, let $W \times 0 = T \cap X \times 0$ and let $U = X \backslash W$. Since $T \cap U \times 0 = \emptyset$, we have a canonical lifting of the map

$$\Omega^{-1}K^T(X \times \mathbb{A}^1; D \times \mathbb{A}^1 + X \times 1) \to \Omega^{-1}K(X \times \mathbb{A}^1; D \times \mathbb{A}^1 + X \times 1)$$

to a map

$$\phi: \Omega^{-1}K^T(X \times \mathbb{A}^1; D \times \mathbb{A}^1 + X \times 1) \to \Omega^{-1}K(X \times \mathbb{A}^1; D \times \mathbb{A}^1 + X \times 1 + U \times 0).$$

Additionally, the space $\Omega^{-1}K(X \times \mathbb{A}^1; D \times \mathbb{A}^1 + X \times 1)$ is contractible, hence the horizontal arrows in (2.10) are homotopy equivalences.

Lemma 2.6. *Let η be an element of $K_0^T(X \times \mathbb{A}^1; D \times \mathbb{A}^1 + X \times 1)$, and let $\tau \in K_1(U; D)$ be the element going to $\phi(\eta)$ under the natural map $K_1(U; D) \to K_0(X \times \mathbb{A}^1; D \times \mathbb{A}^1 + X \times 1 + U \times 0)$ given by the diagram (2.10). Let*

$$\delta: K(U, D) \to K_0^W(X; D)$$

be the boundary map in the long exact localization sequence, and let

$$i_0^*: K_0^T(X \times \mathbb{A}^1; D \times \mathbb{A}^1 + X \times 1) \to K_0^W(X; D)$$

denote the intersection pullback by the zero-section $i_0: X \to X \times \mathbb{A}^1$*. Then*

$$\delta(\tau) = i_0^*(\eta).$$

Proof. Let

$$\delta': \pi_1(\Omega^{-1}K(X \times \mathbb{A}^1; D \times \mathbb{A}^1 + X \times 1 + U \times 0)) \to K_0^W(X; D)$$

be the boundary map coming from the second column in (2.10). Then $\delta(\tau) = \delta'(\phi(\eta))$, by the homotopy commutativity of (2.10). The relevant relativization sequences gives the commutative ladder

$$(2.11)\qquad \begin{array}{ccc} K^W(X,D) & = & K^W(X,D) \\ \downarrow & & \downarrow \\ \Omega^{-1}K^T(X \times \mathbb{A}^1; D \times \mathbb{A}^1 + X \times 1 + X \times 0) & \to & \Omega^{-1}K(X \times \mathbb{A}^1; D \times \mathbb{A}^1 + X \times 1 + X \times 0) \\ \downarrow & & \downarrow \\ \Omega^{-1}K^T(X \times \mathbb{A}^1; D \times \mathbb{A}^1 + X \times 1 + U \times 0) & \to & \Omega^{-1}K(X \times \mathbb{A}^1; D \times \mathbb{A}^1 + X \times 1 + U \times 0) \\ \downarrow & & \downarrow \\ \Omega^{-1}K^W(X, D_1, \ldots, D_n) & = & \Omega^{-1}K^W(X; D) \end{array}$$

where the columns are homotopy fiber sequences. This shows that $\delta'(\phi(\eta)) = i_0^*(\eta)$, proving the lemma. □

Theorem 2.7. *Let X be a regular, quasi-projective scheme over an infinite field, and $D = D_1 + \ldots + D_n$ a reduced normal crossing divisor. Let s be a finite set of closed subsets of X. Suppose that*

- *i) X is an A-scheme for some reduced ring A, and there are elements $f_1, \ldots, f_n$ of A, generating the unit ideal, such that, for each $f = f_i$, the collection of closed subschemes $D_{1f}, \ldots, D_{nf}$ of X_f is split.*
- *ii) Let W' be a closed subset of X of pure codimension d, such that W' intersects each Y_I and each $Y_I \cap S$, $S \in s$, properly. Then there is a closed, pure codimension d subset W of X, containing W', such that W intersects each Y_I and each $Y_I \cap S$ properly, and, for each $f = f_i$, the collection of closed subschemes $D_{1f} \backslash W', \ldots, D_{nf} \backslash W$ of $X_f \backslash W$ is split.*

  Then the map

$$cyc: CH^d(X; D)_s \otimes \mathbb{Q} \to K_0(X; D)^{(d)}$$

is an isomorphism.

Proof. Surjectivity follows from Corollary 2.4. Let then Z be in $Z^d(X; D)_s \otimes \mathbb{Q}$ and suppose $cyc(Z) = 0$. Let W be the support of Z and let $U = X \backslash W$. We may suppose that W satisfies the conditions of (ii) above. We have the localization sequence

$$\ldots \to K_1(U; D)^{(d)} \xrightarrow{\delta} K_0^W(X; D)^{(d)} \to K_0(X; D)^{(d)}$$

so there is an element τ of $K_1(U; D_1 \cap U, \ldots, D_n \cap U)^{(d)}$ with $\delta(\tau) = cyc^W(Z)$. We have the isomorphism

$$K_0(U \times \mathbb{A}^1; D \times \mathbb{A}^1 + U \times 1 + U \times 0)^{(d)} \to K_1(U; D)^{(d)};$$

let $\tilde{\eta}$ be the element of $K_0(U \times \mathbb{A}^1; D \times \mathbb{A}^1 + U \times 1 + U \times 0)^{(d)}$ corresponding to τ. Let $E = D_1 \times \mathbb{A}^1 + \ldots, D_n \times \mathbb{A}^1 + X \times 1 + X \times 0$, $E_U = E \cap U$. Note that (X, E) and (U, E_U) both satisfy the splitting conditions of Corollary 2.4; indeed, we need only replace the ring A with the ring $A[x]$, and the elements $f_1, \ldots, f_n$ of A with the elements $xf_1, \ldots, xf_n, (x-1)f_1, \ldots, (x-1)f_n$ of A. By Corollary 2.4, there is a pure codimension d closed subset T_U of $U \times \mathbb{A}^1$, intersecting each E_{UI} and each $E_{UI} \cap S \times \mathbb{A}^1$ properly, and an element η_U of $K_0^{T_U}(U \times \mathbb{A}^1; E_U)^{(d)}$ mapping to $\tilde{\eta}$ under the natural map. By Lemma 2.2, there is a cycle $\tilde{Z}_U$ in $Z^{T_U}(U \times \mathbb{A}^1; E_U) \otimes \mathbb{Q}$ with $cyc^{T_U}(\tilde{Z}_U) = \eta_U$.

Let T be the closure of T_U in $X \times \mathbb{A}^1$. We claim that T intersects each component of E_I and $E_I \cap S \times \mathbb{A}^1$ properly. Indeed, each E_I is either of the form $D_J \times \mathbb{A}^1$, $D_J \times 0$ or $D_J \times 1$, for some J. Additionally we have

$$T \cap E_I \subset ((W \times \mathbb{A}^1) \cap E_I) \cup (\overline{T_U \cap E_{UI}}).$$

Since T_U intersects E_{UI} properly, the term $\overline{T_U \cap E_{UI}}$ has the proper dimension. Since W intersects each D_J properly on X, it follows that $W \times \mathbb{A}^1$ intersects $D_J \times \mathbb{A}^1$, $D_J \times 0$ and $D_J \times 1$ properly on $X \times \mathbb{A}^1$. Thus the term $(W \times \mathbb{A}^1) \cap E_I$ has the proper dimension as well, proving our claim for E_I; the proof for $E_I \cap S \times \mathbb{A}^1$ is similar. In particular, we have $Z^T(X \times \mathbb{A}^1) = \mathcal{Z}^T(X \times \mathbb{A}^1)_{E \cup s \times \mathbb{A}^1}$

Let $i_0 \colon X \to X \times \mathbb{A}^1$, $i_1 \colon X \to X \times \mathbb{A}^1$ be the inclusions as the zero-section and the one-section. Let $\tilde{Z} \in Z^T(X \times \mathbb{A}^1)_{E \cup s \times \mathbb{A}^1}$ be the closure of $\tilde{Z}_U$. Let $\tilde{Z}_1 = \tilde{Z} \cdot (X \times 1)$. As $\tilde{Z}_U \cdot U \times 1 = 0$, it follows that $\tilde{Z}_1$ has support contained in W. Replacing $\tilde{Z}$ with $\tilde{Z} - \tilde{Z}_1 \times \mathbb{A}^1$, and changing notation, we have $\tilde{Z} \cdot (X \times 1) = 0$ and $\tilde{Z}_U = \tilde{Z} \cap U$.

Let i be an integer, $0 \leq i \leq n-1$. Since

$$\begin{aligned}\tilde{Z} \cdot (D_i^0 \times \mathbb{A}^1) \cap U &= \tilde{Z}_U \cdot (D_i \times \mathbb{A}^1)\\ &= 0,\end{aligned}$$

it follows that $\tilde{Z} \cdot (D_i^0 \times \mathbb{A}^1) = Z_i^0 \times \mathbb{A}^1$, for some cycle Z_i^0 supported on W. Thus

$$\begin{aligned}0 &= (\tilde{Z} \cdot X \times 1) \cdot (D_i^0 \times \mathbb{A}^1)\\ &= (\tilde{Z} \cdot (D_i^0 \times \mathbb{A}^1)) \cdot (X \times 1)\\ &= (Z_i^0 \times \mathbb{A}^1) \cdot (X \times 1)\\ &= Z_i^0.\end{aligned}$$

Similarly, $\tilde{Z} \cdot (D_i^1 \times \mathbb{A}^1) = 0$, hence $\tilde{Z}$ is in the subgroup $Z^T(X \times \mathbb{A}^1; D \times \mathbb{A}^1 + X \times 1)_{X \times 0, s \times \mathbb{A}^1} \otimes \mathbb{Q}$ of $Z^T(X \times \mathbb{A}^1)_{E \cup s \times \mathbb{A}^1} \otimes \mathbb{Q}$. Let $\eta = cyc^T(\tilde{Z}) \in K_0^T(X \times \mathbb{A}^1; D_1 \times \mathbb{A}^1, \ldots, D_n \times \mathbb{A}^1, X \times 1)^{(d)}$. By Lemma 2.6, we have

$$\begin{aligned}cyc^W(\tilde{Z} \cdot (X \times 0)) &= i_0^*(cyc(\tilde{Z}))\\ &= i_0^*(\eta)\\ &= \delta(\tau)\\ &= cyc^W(Z).\end{aligned}$$

Since cyc^W is an isomorphism, we see that

$$\tilde{Z} \cdot (X \times 0) = Z,$$

so $Z = 0$ in $\mathrm{CH}^d(X; D)_s \otimes \mathbb{Q}$, proving injectivity. □

§3 Relative cycles and K_p

Following Bloch [B], we give a cycle-theoretic description of the rational higher K-groups of a regular, quasi-projective scheme over a field. We use a "cubical" version rather than a simplicial version for reason's which will become apparent. We define an isomorphism of the cubically defined groups with Bloch's simplicial version in the next section.

Let X be a k-scheme, s a finite set of closed subsets of X. Let $\square^n = \mathbb{A}^n$. Let D_i^1 be the subscheme $x_i = 1$, D_i^0 the subscheme $x_i = 0$, and D_i the subscheme $x_i(x_i - 1) = 0$. Let $\partial\square^n$ be divisor $D_1 + \ldots + D_n$, and let $\partial^+\square^n$ be the divisor $\partial\square^n - D_n^0$. If s is a finite set of closed subsets of X, and $E = E_1 + \ldots E_t$ is a reduced divisor on a k-scheme Y, we let $s \times s(E)$ denote the set of closed subsets $\{S \times E_I \mid S \in s, I \subset \{1, \ldots, t\}\}$ of $X \times Y$. By a *face* of $X \times \partial\square^p$, we mean a irreducible component of an intersection of some of the divisors $X \times D_i$, $i = 1, \ldots, p$; we also consider $X \times \square^p$ as a face of $X \times \partial\square^p$.

Let $Z^q(X, n)_s^c$ be the group

$$Z^q(X, n)_s^c = Z^q(X \times \square^n; X \times \partial^+\square^n)_{s \times s(\partial^+\square^n) \cup X \times D_n^0}.$$

Intersection with the face D_n^0 defines map $d_n\colon Z^q(X, n)_s^c \to Z^q(X, n-1)_s^c$. Since

$$\begin{aligned} d_{n-1} \circ d_n(Z) &= D_{n-1}^0 \cdot (D_n^0 \cdot Z) \\ &= D_n^0 \cdot (D_{n-1}^0 \cdot Z) \\ &= 0, \end{aligned}$$

we have the complex $(Z^q(X, *)_s^c, d)$

$$\ldots \xrightarrow{d_{n+1}} Z^q(X, n)_s^c \xrightarrow{d_n} \ldots \xrightarrow{d_0} Z^q(X, 0)_s^c.$$

By definition, we have

$$H_p(\mathcal{Z}^q(X, *)^c) = \mathrm{CH}^q(X \times \square^p; X \times \partial\square^p).$$

We define $\mathrm{CH}^q(X, p)^c$ to be $H_p(Z^q(X, *)^c)$.

Theorem 3.1. *Let X be a smooth, quasi-projective k-scheme, s a finite set of closed subsets of X. Then the map $cyc\colon \mathrm{CH}^q(X \times \square^p; X \times \partial\square^p)_{s \times \square^p} \otimes \mathbb{Q} \to K_0(X \times \square^p; X \times \partial\square^p)^{(q)}$ defines an isomorphism*

$$cyc_{q,p}\colon \mathrm{CH}^q(X, p)_s^c \otimes \mathbb{Q} \to K_p(X)^{(q)}.$$

Proof. Using the homotopy property of K-theory of regular schemes, there is a natural homotopy equivalence

$$K(X \times \square^p; X \times \partial\square^p) \to \Omega^p(K(X))$$

giving the isomorphism

$$K_0(X \times \square^p; X \times \partial\square^p)^{(q)} \to K_p(X)^{(q)}.$$

Suppose we have verified the hypotheses of Theorem 2.7 for the normal crossing divisor $D = X \times \partial\square^p = D_1 + \ldots + D_p$ on $X \times \square^p$; then the map $cyc\colon \mathrm{CH}^q(X \times \square^p; X \times \partial\square^p)_{s \times \square^p} \otimes \mathbb{Q} \to K_0(X \times \square^p; X \times \partial\square^p)^{(q)}$ is an isomorphism, proving the theorem.

We now proceed to verify the hypotheses of Theorem 2.7. Let $A = k[x_1, \ldots, x_p]$. For each $I \subset \{1, \ldots, p\}$, let f_I be the element of A defined by

$$f_I = \prod_{i \in I} x_i \times \prod_{i \notin I} (x_i - 1),$$

and let $v_I = \cap_{i \in I}(x_i = 0) \cap \cap_{i \notin I} x_i = 1)$. Then, for each I, v_I is a closed point of $\square^p$ (with coordinates either 0 or 1), and the divisor $f_I = 0$ is the union of components of $\partial\square^p$ passing through v_I. Thus, the n-cubes $(\square^p_{f_I}; D_{1f_I}, \ldots, D_{pf_I})$ for different I are all isomorphic; for $I = \{1, \ldots, n\}$, this n-cube is the collection of coordinate hyperplanes $x_i = 0$ in the open subscheme $x_i \neq 1$ of $\square^p$. In particular, the collection $\{f_I \mid I \subset \{1, \ldots, n\}\}$ generate the unit ideal in A. Additionally, the n-cube $(\square^p_{f_I}; D_{1f_I}, \ldots, D_{pf_I})$ is a split n-cube; for $I = \{1, \ldots, n\}$, the splitting generated by the linear projections

$$\pi_i^0 \colon \square^p \to D_i^0$$
$$\pi_i^0(t_1, \ldots, t_p) = (t_1, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_p).$$

This verifies the condition (i) in Theorem 2.7.

For condition (ii), let π_i^1 be the linear projection

$$\pi_i^1 \colon \square^p \to D_i^1$$
$$\pi_i^1(t_1, \ldots, t_p) = (t_1, \ldots, t_{i-1}, 1, t_{i+1}, \ldots, t_p).$$

Let W' be a pure codimension d closed subset of $X \times \square^p$, intersecting each face of $\partial\square^p$ properly. From the condition it follows that for each i, the closed subsets W_i^0 and W_i^1 defined by

$$W_i^0 = (\pi_i^0)^{-1}(W' \cap X \times D_i^0); \quad W_i^1 = (\pi_i^1)^{-1}(W' \cap X \times D_i^1)$$

are pure codimension d on $X \times \square^p$, and intersect each face of $X \times \partial\square^p$ properly. Indeed, for a face F of $X \times \partial\square^p$, the projection $\pi_i^0(F)$ is again face of $X \times \partial\square^p$, and is contained in D_i^0. We have

$$\begin{aligned} \mathrm{codim}_F(W_i^0 \cap F) &= \mathrm{codim}_{\pi_i^0(F)}((W' \cap D_i^0) \cap \pi_i^0(F)) \\ &= \mathrm{codim}_{\pi_i^0(F)}((W' \cap \pi_i^0(F)) \\ &\geq d \end{aligned}$$

The computation for W_i^1 is similar. Thus, letting W be the closed subset of $X \times \square^p$,

$$W = W' \cup \cup_{i=1}^p W_i^0 \cup \cup_{i=1}^p W_i^1$$

W has pure codimension d on $X \times \partial\square^p$, and intersects each face of $X \times \partial\square^p$ properly. By construction, the linear projections π_i^0 and π_i^1 map $X \times \square^p \backslash W$ into $D_i^0 \backslash W$ and $D_i^1 \backslash W$, respectively. Thus the n-cube

$$((X \times \square^p \backslash W)_{f_I}; (D_1 \backslash W)_{f_I}), \ldots, (D_p \backslash W)_{f_I})$$

is split for each $I \in \{1, \ldots, p\}$, verifying condition (ii). This completes the proof of the theorem. $\square$

For a scheme X, the space BQP_X gives the canonical delooping of the space $K(X)$. If we have closed subschemes $Y_1, \ldots, Y_n$, this gives the canonical delooping of the iterated homotopy fiber $K(X; Y_1, \ldots, Y_n)$; denote this delooping by $\Omega^{-1}K(X; Y_1, \ldots, Y_n)$. We let $BQP^q_X(n)$ denote the connected component of the base point in $\Omega^{-1}K(X \times \square^n; X \times \partial\square^n)$ and let $BQP^q_X(n+1)^+$ denote the connected component of the base point in $\Omega^{-1}K(X \times \square^{n+1}; X \times \partial^+\square^{n+1})$

Corollary 3.2. *Let s be a finite set of closed subsets of X. Then the map*

$$Z^q(-,*)^c_s \otimes \mathbb{Q} \to Z^q(-,*)^c \otimes \mathbb{Q}$$

is a quasi-isomorphism.

Proof. We have the commutative diagram

$$\begin{array}{ccc} H_p(Z^q(-,*)^c_s \otimes \mathbb{Q}) & \to & H_p(Z^q(-,*)^c \otimes \mathbb{Q}) \\ cyc^{q,p} \searrow & & \swarrow cyc^{q,p} \\ & K_p(X)^{(q)} & \end{array}$$

As the maps $cyc^{q,p}$ are isomorphisms for all p by Theorem 3.1, the map

$$Z^q(-,*)^c_s \otimes \mathbb{Q} \to Z^q(-,*)^c \otimes \mathbb{Q}$$

is a quasi-isomorphism, as desired. □

Theorem 3.3. *The complexes $Z^q(-,*)^c \otimes \mathbb{Q}$ satisfy the Mayer-Vietoris axiom for the Zariski topology, i.e., if U and V are open subsets of X with $X = U \cup V$, then the natural map*

$$Z^q(X,*)^c \otimes \mathbb{Q} \to \mathrm{Cone}(Z^q(U,*)^c \otimes \mathbb{Q} \oplus Z^q(V,*)^c \otimes \mathbb{Q} \to Z^q(U \cap V,*)^c \otimes \mathbb{Q})[-1]$$

is a quasi-isomorphism.

Proof. Let $\mathcal{C}$ denote the complex

$$\mathrm{Cone}(Z^q(U,*)^c \otimes \mathbb{Q} \oplus Z^q(V,*)^c \otimes \mathbb{Q} \to Z^q(U \cap V,*)^c \otimes \mathbb{Q})[-1].$$

We first show how the isomorphism $cyc\colon H_p(Z^q(X,p)^c \otimes \mathbb{Q}) \to K_p(X)^{(q)}$ extends to a map $cyc\colon H_p(C) \to K_p(X)^{(q)}$.

Let F^q be the iterated homotopy fiber over the square

$$\begin{array}{ccc} BQP^q_U(n+1)^+ \times BQP^q_V(n+1)^+ & \to & BQP^q_U(n) \times BQP^q_V(n) \\ \downarrow & & \downarrow \\ BQP^q_{U\cap V}(n+1)^+ & \to & BQP^q_{U\cap V}(n). \end{array} \tag{3.2}$$

As each term in this square can be functorially delooped, the homotopy groups of F^q are all abelian groups, including π_0.

Let π^q_{1*} denote the complex of abelian groups associated to the double complex

$$\begin{array}{ccc} \pi_1(BQP^q_U(n+1)^+) \oplus \pi_1(BQP^q_V(n+1)^+) & \to & \pi_1(BQP^q_U(n)) \oplus \pi_1(BQP^q_V(n)) \\ \downarrow & & \downarrow \\ \pi_1(BQP^q_{U\cap V}(n+1)^+) & \to & \pi_1(BQP^q_{U\cap V}(n)), \end{array}$$

with differential decreasing degree and with $\pi_1(BQP^q_{U\cap V}(n))$ in degree -1. The long exact fibration sequences associated to the square (3.2) then give the following exact sequence describing $\pi_0(F^q)$:

$$\pi_2(BQP^q_{U\cap V}(n)) \to \pi_0(F^q) \to H_0(\pi^q_{1*}) \to 0. \tag{3.3}$$

The Adams operation ψ^k acts on the square (3.2), inducing an action on the homology $H_0(\pi^q_{1*})$ and a functorial finite decomposition

$$H_0(\pi^q_{1*}) \otimes \mathbb{Q} = \oplus_a H_0(\pi^q_{1*})^{(a)};$$

there is also an action on $\pi_0(F^q)$, but this latter action may conceivably be non-additive. On the other hand, the maps cyc^q induces an isomorphism of the square

$$\begin{array}{ccc} Z^q(U,p+1)\otimes\mathbb{Q}\oplus Z^q(V,p+1)\otimes\mathbb{Q} & \to & Z^q(U,p+1)\otimes\mathbb{Q} \\ \downarrow & & \downarrow \\ Z_p(Z^q(U,*))\otimes\mathbb{Q}\oplus Z_p(Z^q(V,*))\otimes\mathbb{Q} & \to & Z_p(Z^q(U,*))\otimes\mathbb{Q} \end{array} \tag{3.4}$$

to the square $(\pi^q_{1*}\otimes\mathbb{Q})^{(q)}$. Letting $Tot(3.4)$ denote the total (homological) complex of the square (3.4), with $Z_p(Z^q(U,*))\otimes\mathbb{Q}$ in degree -1, the map cyc^q thus gives a map

$$H_0(cyc^q)_0\colon H_0(Tot(3.4)) \to H_0(\pi^q_{1*})^{(q)}.$$

Composing this with the surjection $Z_p(C) \to H_0(Tot(3.4))$ gives the map $\overline{Z_p(cyc^q)}\colon Z_p(C) \to H_0(\pi^q_{1*})^{(q)}$.

Let $F = F^0$. The spaces $BQP_U(p+1)^+$, $BQP_V(p+1)^+$ and $BQP_{U\cap V}(p+1)^+$ are all contractible, hence we have the homotopy equivalence

$$F \to \Omega\mathrm{Fib}(BQP_U(p)\times BQP_V(p) \to BQP_{U\cap V}(p)),$$

compatible with the ψ^k-action. By the Mayer-Vietoris property for the functor $K(-)$, this gives the homotopy equivalence

$$F \to K(X\times\square^p; X\times\partial\square^p),$$

compatible with the ψ^k-action; similarly, the exact sequence (3.3) for $q=0$ gives the commutative diagram of abelian groups

$$\begin{array}{ccc} \pi_2(BQP^0_{U\cap V}(n)) & \to & \pi_0(F^0) \\ \downarrow & & \downarrow \\ K_1(U\cap V\times\square^p; U\cap V\times\partial\square^p) & \to & K_0(X\times\square^p; X\times\partial\square^p); \end{array} \tag{3.5}$$

here the map

$$K_1(U\cap V\times\square^p; U\cap V\times\partial\square^p) \to K_0(X\times\square^p; X\times\partial\square^p)$$

arises from the Mayer-Vietoris sequence for the covering $\{U\times\square^p, V\times\square^p\}$ of $X\times\square^p$. The maps in (3.5) are compatible with the ψ^k-action and the vertical maps are isomorphisms; in particular, the ψ^k-action on $\pi_0(F^0)$ is additive

Let $p^q\colon \pi_0(F^q) \to K_0(X\times\square^p; X\times\partial\square^p)^{(q)}$ be the composition

$$\pi_0(F^q) \to \pi_0(F^0) \to K_0(X\times\square^p; X\times\partial\square^p) \to K_0(X\times\square^p; X\times\partial\square^p)^{(q)},$$

where the first map in induced by the map $F^q \to F^0$, the second comes from the square (3.5) and the third is the projection of $K_0(X \times \square^p; X \times \partial\square^p)$ onto the summand $K_0(X \times \square^p; X \times \partial\square^p)^{(q)}$. Suppose we have an element η of $\pi_2(BQP^q_{U\cap V}(n))$ with image $h \in \pi_0(F^q)$ under the map in (3.3). Then $p^q(h)$ can be gotten by applying the composition of maps

$$\begin{aligned} \pi_2(BQP^q_{U\cap V}(n)) &\to \pi_2(BQP^0_{U\cap V}(n)) \\ &\to K_0(X \times \square^p; X \times \partial\square^p) \\ &\to K_0(X \times \square^p; X \times \partial\square^p)^{(q)} \end{aligned}$$

to the element η. As this composition is the same as the composition

$$\begin{aligned} \pi_2(BQP^q_{U\cap V}(n)) &\to \pi_2(BQP^q_{U\cap V}(n))^{(q)} \\ &\to \pi_2(BQP^0_{U\cap V}(n))^{(q)} \\ &\to K_0(X \times \square^p; X \times \partial\square^p)^{(q)} \end{aligned}$$

and as $\pi_2(BQP^q_{U\cap V}(n))^{(q)} = 0$ by (2.1) in proof of Lemma 2.2, we see that $p^q(h) = 0$. Thus the map p^q factors through the quotient $H_0(\pi^q_{1*})$ of $\pi_0(F^q)$, and we may define the map

$$Z_p(cyc^q)\colon Z_p(C) \to K_0(X \times \square^p; X \times \partial\square^p)(q)$$

by setting

$$Z_p(cyc^q)(\alpha) = p^q(h), \quad \alpha \in Z_p(C),$$

where $h \in \pi_0(F^q) \otimes \mathbb{Q}$ is any lifting of $\overline{Z_p(cyc^q)}(\alpha) \in H_0(\pi^q_{1*})^{(q)}$ via the sequence (3.3). One checks easily that this is indeed an extension of the map

$$cyc^q\colon Z_p(Z(X,*) \otimes \mathbb{Q}) \to K_0(X \times \square^p; X \times \partial\square^p)^{(q)}.$$

Using the argument of Theorem 2.7, we see that $Z_p(cyc^q)$ descends to the map

$$H_p(cyc)\colon H_p(C) \otimes \mathbb{Q} \to K_0(X \times \square^p; X \times \partial\square^p)(q) = K_p(X)(q).$$

We have the commutative diagram

$$\begin{array}{ccc} H_{p+1}(Z^q(U,*)) \otimes \mathbb{Q} \oplus H_{p+1}(Z^q(V,*)) \otimes \mathbb{Q} & \overset{cyc\oplus cyc}{\to} & K_{p+1}(U)(q) \oplus K_{p+1}(V)(q) \\ \downarrow & & \downarrow \\ H_{p+1}(Z^q(U\cap V,*)) \otimes \mathbb{Q} & \overset{cyc}{\to} & K_p(U\cap V)(q) \\ \downarrow & & \downarrow \\ H_p(C) \otimes \mathbb{Q} & \overset{H_p(cyc)}{\to} & K_p(X)(q) \\ \downarrow & & \downarrow \\ H_p(Z^q(U,*)) \otimes \mathbb{Q} \oplus H_p(Z^q(U,*)) \otimes \mathbb{Q} & \overset{cyc\oplus cyc}{\to} & K_p(U)(q) \oplus K_{p+1}(V)(q); \end{array}$$

thus $H_p(cyc)$ is an isomorphism by the five lemma. $\square$

.

For W a closed subset of X, let $j\colon X\backslash W \to X$ be the inclusion of the complement, and let $Z^q_W(X,*)^c$ denote the complex

$$\mathrm{Cone}(j^*\colon Z^q(X,*)^c \to Z^q(X\backslash W,*)^c)[-1].$$

If W is a closed subscheme of pure codimension d, we have the natural map

$$i_{W*}\colon Z^{q-d}(W,*)^c \to Z^q_W(X,*)^c.$$

We let $\mathrm{CH}^q_W(X,p) = H_p(Z^q_W(X,*)^c)$.

Theorem 3.4. *Let X be a regular, quasi-projective k-scheme, $i\colon W \to X$ a closed subscheme, $j\colon U \to X$ the inclusion of the complement $U = X\backslash W$. Then there are natural isomorphisms*

$$cyc^W_{q,p}\colon \mathrm{CH}^q_W(X,p)\otimes\mathbb{Q} \to K^W_p(X)^{(q)}$$

giving the commutative diagram

$$\begin{array}{ccccccccc}
\ldots\to & CH^q(U,p+1)\otimes\mathbb{Q} & \to & CH^q_W(X,p)\otimes\mathbb{Q} & \to & CH^q(X,p)\otimes\mathbb{Q} & \to & CH^q(U,p)\otimes\mathbb{Q} & \to\ldots\\
 & cyc_{q,p+1}\downarrow & & cyc^W_{q,p}\downarrow & & cyc_{q,p}\downarrow & & cyc_{q,p}\downarrow & \\
\ldots\to & K_{p+1}(U)^{(q)} & \to & K^W_p(X)^{(q)} & \to & K_p(X)^{(q)} & \to & K_p(U)^{(q)} & \to\ldots.
\end{array}$$

In addition, if W is regular, of pure codimension d on X, then the map

$$i_{W*}\otimes\mathbb{Q}\colon Z^{q-d}(W,*)^c\otimes\mathbb{Q} \to Z^q_W(X,*)^c\otimes\mathbb{Q}.$$

is a quasi-isomorphism.

Proof. The construction of the map $cyc^W_{q,p}$ is similar to that of the map $H_p(cyc)$ in Theorem 3.2. We give a sketch of the construction.

Let $U = X\backslash W$. Let G^q be the iterated homotopy fiber over the commutative square

$$\begin{array}{ccc}
BQP^q_X(n+1)^+ & \to & BQP^q_X(n)\\
\downarrow & & \downarrow\\
BQP^q_U(n+1)^+ & \to & BQP^q_U(n).
\end{array} \tag{3.6}$$

By considering the square of abelian groups gotten by applying the functor π_1 to the square (3.6) for q and for $q=0$ as in the proof of Theorem 3.2, we arrive at definition of the map $cyc^W_{q,p}$.

In addition, if W is regular and pure codimension d on X, we have the commutative diagram

$$\begin{array}{ccc}
\mathrm{CH}^{q-d}(W,p)\otimes\mathbb{Q} & \xrightarrow{i_{W*}} & \mathrm{CH}^q_W(X,p)\otimes\mathbb{Q}\\
cyc_{q-d,p}\downarrow & & \downarrow cyc^W_{q,p}\\
K_p(W)^{(q-d)} & \xrightarrow{i_{W*}} & K^W_p(X)^{(q)}.
\end{array}$$

Since $cyc_{q-d,p}$, $cyc^W_{q,p}$ and $i_{W*}\colon K_p(W)^{(q-d)} \to K^W_p(X)^{(q)}$ are isomorphisms, the map

$$i_{W*}\colon \mathrm{CH}^{q-d}(W,p)\otimes\mathbb{Q} \to \mathrm{CH}^q_W(X,p)\otimes\mathbb{Q}$$

is an isomorphism as well, proving the second assertion. □

§4 Cubes and simplices

In this section, we show that the higher Chow groups defined via cubes agrees with Bloch's higher Chow groups defined via simplices. To do this we first prove the weak moving lemma and the homotopy property for the cubical complexes $Z^q(X,*)^c_s$. The proofs are essentially the same as Bloch's proofs of the analogous properties for the simplicialy defined complexes $Z^q(X,*)_s$, only rather easier, as the cubical structure allows us to circumvent the necessity of taking subdivisions, as is required in the simpicial version. For this reason, we will be rather sketchy in our proofs, refering for the most part to Bloch's argument for details. We then use the homotopy property for both complexes to define the desired quasi-isomorphism. From this we derive the contravarient functoriality of the cubical complexes. We also consider the $\mathbb{Q}$-complexes Bloch has defined by using *alternating* cycles on $X \times \square^n$, and we show that these complexes are quasi-isomorphic to $Z^q(X,*)^c \otimes \mathbb{Q}$

We note that the complexes $Z^q(X,*)^c_s$ are contravariantly functorial for flat maps, and covariantly functorial (with approriate shift in codimension) for proper maps. If K is a finite field extension of k, X_K the extension of X to a scheme over K, and $\pi\colon X_K \to X$ the projection, then

$$\pi_* \circ \pi^* = \times[K:k] \tag{4.1}$$

.

Let $i_{W_n^X}\colon W_n^X \to X \times \square^{n+1} \times \mathbb{P}^1$ be the subvariety of $X \times \square^{n+1} \times \mathbb{P}^1 = X \times \mathrm{Spec}(k[x_1,\ldots,x_{n+1}]) \times \mathrm{Proj}(k[T_0,T_1])$ defined by the equation

$$T_0(1-x_n)(1-x_{n+1}) = T_0 - T_1.$$

Let $\pi_n\colon W_n^X \to X \times \square^n$ be the map defined by $\pi_n(x,x_1,\ldots,x_{n+1},(t_O:t_1)) = (x,x_1,\ldots,x_n)$. Let $p_n\colon X \times \square^{n+1} \times \mathbb{P}^1 \to X \times \square^{n+1}$ be the projection. For a cycle $Z \in Z^q(X \times \square^n)$, we let $W_n^X(Z) = p_{n*}(i_{W_n^X*}(id \times \pi^*(Z))$.

Lemma 4.1. *For $Z \in Z^q(X \times \square^n)_{s(X \times \partial\square^n)}$, the cycle $W_n^X(Z)$ is in $Z^q(X \times \square^{n+1})_{s(X \times \partial\square^{n+1})}$. In addition, we have*

$$W_n^X(Z) \cdot (x_{n+1} = 0) = Z = W_n^X(Z) \cdot (x_n = 0). \tag{4.2}$$

In this last formula, we identify the locus $x_n = 0$ with $X \times \square^n$ by sending x_{n+1} to x_n.

Proof. The reader will easily verify the following properties of the subvariety W_n^X:

(1) W_n^X is regular and flat over $X \times \square^n$ with 1-dimensional fibers.

(2) Let Δ^0 be the intersection of the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$ with $\mathbb{A}^1 \times \mathbb{P}^1$. Then

$$W_n^X \cdot (x_{n+1} = 0) = \Delta^0 = W_n^X \cdot (x_n = 0).$$

(3) For $i = n, n+1$,

$$W_n^X \cdot X \times (x_i = 1) \times \mathbb{P}^1 = X \times (x_i = 1) \times (1:1).$$

The lemma follows from the properties (1)-(3), and the associativity and commutativity of intersection product. □

We suppose we have an algebraic group G and an action of G on X. Let K be an extension field of k, and let $\psi: \mathbb{A}^1_K \to G_K$ be a morphism with $\psi(1) = id$. Let $\phi: X_K \times \mathbb{A}^1_K \to X_K \times \mathbb{A}^1_K$ be the isomorphism

$$\phi(x,t) = (\psi(t) \cdot x, t).$$

We define the map $h_n: Z^q(X \times \square^n) \to Z^q(X_K \times \square^{n+1})$ by

$$h_n(Z) = Z \times \mathbb{A}^1 - \phi(Z \times \mathbb{A}^1) - W_n^X(dZ \times \mathbb{A}^1) + W_n^{X_K}(\phi(dZ \times \mathbb{A}^1)).$$

Lemma 4.2. *Let X be a k-scheme, with finite collections $y = \{Y_1, \ldots /y_n\}$, $s = \{S_1, \ldots, S_m\}$ of closed subsets of X. Suppose $G \cdot Y = X$ for each $Y \in y$, and that $\psi(x)$ is k-generic for each $x \in \mathbb{A}^1(\bar{k})$. Then, for each $Z \in Z^q_s(X, n)^c$, $h_n(Z)$ is in $Z^q(X_K, n+1)^c_s$ and $\psi(0)(Z)$ is in $Z^q(X_K, n+1)^c_{y \cup s}$. In addition,*

$$dh_n(Z) = Z - \psi(0)(Z) - dZ \times \mathbb{A}^1 + \phi(dZ \times \mathbb{A}^1).$$

Proof. Let Z be in $Z^q(X,n)^c_s$. Arguing as in the proof of Lemma(2.2) of [B] shows that $\psi(0)(Z)$ is in $Z^q(X_K, n+1)^c_{y \cup s}$, that $Z \times \mathbb{A}^1$ and $\phi(Z \times \mathbb{A}^1)$ are in $Z^q(X \times \square^{n+1})_{s(X \times \partial\square^{n+1})}$, and that $dZ \times \mathbb{A}^1$ and $\phi(dZ \times \mathbb{A}^1)$ are in $Z^q(X \times \square^n)_{s(X \times \partial\square^n)}$. In addition, the cycles $Z \times \mathbb{A}^1$, $\phi(Z \times \mathbb{A}^1)$ $dZ \times \mathbb{A}^1$ and $\phi(dZ \times \mathbb{A}^1)$ intersect each $S_i \times D_J$ properly, where D is either $X \times \partial\square^n$ or $X \times \partial\square^{n+1}$, and J is any appropriate index. We have

$$\begin{gathered}(Z \times \mathbb{A}^1 - \phi(Z \times \mathbb{A}^1)) \cdot (x_{n+1} = 1) = 0 \\ (Z \times \mathbb{A}^1 - \phi(Z \times \mathbb{A}^1)) \cdot (x_{n+1} = 0) = Z - \psi(0)(Z) \\ (Z \times \mathbb{A}^1 - \phi(Z \times \mathbb{A}^1)) \cdot (x_n = 1) = 0 \\ (Z \times \mathbb{A}^1 - \phi(Z \times \mathbb{A}^1)) \cdot (x_n = 0) = (dZ \times \mathbb{A}^1 - \phi(dZ \times \mathbb{A}^1))\end{gathered}$$

and all other intersections $(Z \times \mathbb{A}^1 - \phi(Z \times \mathbb{A}^1)) \cdot (x_i = 0, 1)$ are zero. Applying Lemma 4.1, we see that $h_n(Z)$ is in $Z^q(X \times \square^{n+1})_{s(X \times \partial\square^{n+1})}$, intersecting each $S_i \times D_J$ properly. It follows from formula (4.2) that $h_n(Z) \cdot (x_i = 0, 1)$ is zero for $i = 1, \ldots, n$, and $h_n(Z) \cdot (x_{n+1} = 1) = 0$ as well. Thus $h_n(Z)$ is in $Z^q(X_K, n+1)^c_s$. The formula for $dh_n(Z)$ follows directly from the definition of h_n, the intersection computations made above, and formula (4.2). □

Lemma 4.3. *Suppose $G \cdot Y = X$ for each $Y \in y$, and that $\psi(x)$ is k-generic for each $x \in \mathbb{A}^1(\bar{k})$. Let $\pi: X_K \to X$ be the natural projection. Then the map*

$$\bar{\pi}^*: Z^q(X, *)^c_s / Z^q(X, *)^c_{y \cup s} \to Z^q(X_K, *)^c_s / Z^q(X_K, *)^c_{y \cup s}$$

is null-homotopic. If K is a pure transcendental extension of k, then the inclusion

$$Z^q(X, *)^c_{y \cup s} \subset Z^q(X, *)^c_s$$

is a quasi-isomorphism.

Proof. For the first assertion, the maps h_n define a null-homotopy. For the second, if k is finite, we may find an infinite, algebraic, pure p-power extension k_p, for each prime integer p. If we prove the assertion for k_p and k_q with $p \neq q$, the result then follows for k, using the formula (4.1). We therefore assume k is infinite. Thus, if $T_1, \ldots, T_r$ are in $Z^q(X_K, p)^c_{y \cup s}$, $K = k(t_1, \ldots, t_m)$, we can find an open subset U of $\mathbb{A}^m_k$ such that the T_i are the restriction to the generic point of cycles T_i in $Z^q(X \times U, p)^c_{y \cup s}$, for $i = 1, \ldots, r$. We may then find a k-point $x \in U$ and form the specialization $sp_x(T_i) := i_x^*(T_i)$, arriving at the cycles $sp_x(T_i) \in Z^q(X, p)^c_{y \cup s}$. We have a similar specialization for $Z^q(X_K, p)^c_s$.

It suffices to show that $Z^q(X, *)^c_s / Z^q(X, *)^c_{y \cup s}$ is acyclic. Since the map $\bar{\pi}^*$ is null-homotopic, it suffices to show that $\bar{\pi}^*$ is injective on homology. If $\bar{\pi}^*(Z) = dW$, then we may specialize to get $Z = sp_x(dW) = d(sp_x(W))$, proving injectivity. □

Proposition 4.4. *Let X be a k-scheme, with a finite collection $s = \{S_1, \ldots, S_m\}$ of closed subsets of X. Let $y = \{X \times H_1, \ldots, X \times H_r\}$, where H_i is a closed subset of $\mathbb{A}^n$, $i = 1, \ldots, r$, $n > 0$. Then the inclusion*

$$Z^q(X \times \mathbb{A}^n, *)^c_{y \cup p_1^*(s)} \subset Z^q(X \times \mathbb{A}^n, *)^c_{p_1^*(s)}$$

is a quasi-isomorphism.

Proof. Let $G = \mathbb{A}^n/k$, acting on $\mathbb{A}^n$ by translation. Let $t_1, \ldots, t_n, u_1, \ldots, u_n$ be transcendental over k, and map $\mathbb{A}^1_K$ to G_K by sending x to $(t_1 + xu_1, \ldots, t_n + xu_n)$. Applying Lemma 4.3 proves the proposition. □

We now can prove the homotopy property for the complexes $Z^q(X, *)^c_s$. The proof follows the method of Bloch in [B].

Theorem 4.5. Suppose X is a k-scheme. Let $s = \{S_1, \ldots, S_n\}$ be a finite collection of closed subsets of X. Let $p: X \times \mathbb{A}^n \to X$ be the projection. Then the map

$$p_1^*: Z^q(X, *)^c_s \to Z^q(X \times \mathbb{A}^n, *)^c_{p_1^*(s)}$$

is a quasi-isomorphism.

Proof. By induction, we need only consider the case $n = 1$. Let P be a finite set of k-points of $\mathbb{A}^1$. By Proposition 4.4, the inclusion

$$Z^q(X \times \mathbb{A}^1, *)^c_{X \times P \cup p_1^*(s)} \subset Z^q(X \times \mathbb{A}^1, *)^c_{p_1^*(s)}$$

is a quasi-isomorphism. Next, let $i_0: X \to X \times \mathbb{A}^1$, $i_1: X \to X \times \mathbb{A}^1$ the zero-section and the one-section. We claim that the two maps

$$Z^q(X \times \mathbb{A}^1, *)^c_{X \times \{0,1\} \cup p_1^*(s)} \underset{i_1^*}{\overset{i_0^*}{\rightrightarrows}} Z^q(X, *)^c_s$$

are homotopic. Indeed, identify $X \times \mathbb{A}^1 \times \square^n$ with $X \times \square^{n+1}$ by sending $(x, t, x_1, \ldots, x_n)$ to $(x, x_1, \ldots, x_n, t)$. Let $H_n: Z^q(X \times \mathbb{A}^1, n)^c_{X \times \{0,1\}} \to Z^q(X, n+1)^c$ be defined by

$$H_n(Z) = Z - i_1^*(Z) \times \mathbb{A}^1 - W^X_n(dZ) + W^X_{n-1}(i_1^*(dZ)) \times \mathbb{A}^1.$$

By Lemma 4.2, H_n does in fact define a map $Z^q(X \times \mathbb{A}^1, n)^c_{X \times \{0,1\} \cup p_1^*(s)} \to Z^q(X, n+1)^c_s$. We also have

$$dH_n(Z) = i_0^*(Z) - i_1^*(Z) - dZ + i_1^*(dZ) \times \mathbb{A}^1,$$

so

$$(dH_n + H_{n-1}d)(Z) = i_0^*(Z) - i_1^*(Z) - dZ + i_1^*(dZ) \times \mathbb{A}^1 + dZ - i_1^*(dZ) \times \mathbb{A}^1$$

$$= i_0^*(Z) - i_1^*(Z),$$

giving the desired homotopy.

Finally, let $\tau\colon \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ be the multiplication map $\tau(x, y) = xy$. τ is flat, hence $\tau^*\colon Z^q(X \times \mathbb{A}^1, *)^c \to Z^q(X \times \mathbb{A}^1 \times \mathbb{A}^1, *)^c$ is defined. Consider the diagram (we omit the subscripts s etc. for clarity)

$$
\begin{array}{ccccc}
Z^q(X,*)^c & \xrightarrow{p_1^*} & Z^q(X \times \mathbb{A}^1, *)^c & \xrightarrow{\tau^*} & Z^q(X \times \mathbb{A}^1 \times \mathbb{A}^1, *)^c \\
p_1^* \searrow & & \nearrow q.iso & & \uparrow q.iso \\
& Z^q(X \times \mathbb{A}^1, *)^c_{X \times \{0,1\}} & & \xrightarrow{\tau^*} & Z^q(X \times \mathbb{A}^1 \times \mathbb{A}^1, *)^c_{X \times \mathbb{A}^1 \times \{0,1\}} \\
& & & & i_0^* \downarrow \quad \downarrow i_1^* \\
& & & & Z^q(X \times \mathbb{A}^1, *)^c
\end{array}
$$

For Z in $Z^q(X \times \mathbb{A}^1, *)^c_{X \times \{0,1\}}$, we have

$$i_1^* \tau^*(Z) = Z, \quad i_0^* \tau^*(Z) = p_1^* i_0^*(Z);$$

since $i_1^* = i_0^*$ on homology, the map p_1^* is surjective on homology. Since $i_0^* p_1^*(Z) = Z$, p_1^* is injective on homology, proving the theorem. □

Let $\Delta^n = \mathrm{Spec}(k[t_0, \ldots, t_n]/\sum_i t_i - 1)$. Let $\delta_n^i\colon \Delta^{n-1} \to \Delta^n$, $\sigma_n^i\colon \Delta^n \to \Delta^{n-1}$ be the morphisms with

$$\delta_n^{i*}(t_j) = \begin{cases} t_j & \text{if } j < i \\ 0 & \text{if } j = i \\ t_{j-1} & \text{if } j > i \end{cases}$$

$$\sigma_n^{i*}(t_j) = \begin{cases} t_j & \text{if } j < i \\ t_i + t_{i+1} & \text{if } j = i \\ t_{j-1} & \text{if } j > i \end{cases}$$

This forms the co-simplicial scheme $X \times \Delta^\bullet$. Let $\partial\Delta^n$ be the reduced normal crossing divisor $(t_0 = 0) + (t_1 = 0) + \ldots + (t_n = 0)$. Form the simplicial abelian group $Z_s^q(X \times \Delta^\bullet)$ with n-simplices

$$Z_s^q(X \times \Delta^\bullet)_n = Z^q(X \times \Delta^n)_{s \times s(\partial\Delta^n) \cup s(X \times \partial\Delta^n)}$$

and with boundary and degeneracy maps induced by δ_n^{i*} and σ_n^{i*}. Let $Z_s^q(X, *)$ be the normalized chain complex of $Z_s^q(X \times \Delta^\bullet)$. Bloch's higher Chow groups, $\mathrm{CH}^q(X, p)$ are defined by

$$\mathrm{CH}^q(X, p) = H_p(Z^q(X, *)).$$

Bloch has shown that the complexes $gZ^q(X, *)$ are contravariantly functorial for flat maps, covariantly functorial for proper maps and that

(1) (Theorem 2.1 of [B]) Let X be a scheme over k, s a finite set of closed subsets of X. The pull-back

$$p_1^*\colon Z^q(X, *)_s \to Z^q(X \times \mathbb{A}^n, *)_{s \times \mathbb{A}^n}$$

is a quasi-isomorphism.

(2) (Theorem *.* of [B]) Let X be a scheme over k, s and y finite sets of closed subsets of X, K an extension field of k. Suppose $G \cdot Y = X$ for each $Y \in y$, and that $\psi(x)$ is k-generic for each $x \in \mathbb{A}^1(\bar{k})$ (notation as above). Let $\pi\colon X_K \to X$ be the natural projection. Then the map

$$\bar{\pi}^*\colon Z^q(X, *)_s / Z^q(X, *)_{y \cup s} \to Z^q(X_K, *)_s / Z^q(X_K, *)_{y \cup s}$$

is null-homotopic. If K is a pure transcendental extension of k, then the inclusion

$$Z^q(X,*)_{y\cup s} \subset Z^q(X,*)_s$$

is a quasi-isomorphism.

Let $Z^q(X,m,n)_s$ be the subgroup of $Z^q(X\times\square^m\times\Delta^n)_{s(X\times(\partial\square^m\times\Delta^n+\square^m\times\partial\Delta^n))}$ consisting of cycles Z such that

$$\begin{aligned} Z\cdot(X\times(x_i=0)\times\Delta^n)=0 \quad &\text{for } i=1,\ldots,n-1\\ Z\cdot(X\times(x_i=1)\times\Delta^n)=0 \quad &\text{for } i=1,\ldots,n\\ Z\cdot(X\times\square^m\times(t_i=0))=0 \quad &\text{for } i=1,\ldots,n; \end{aligned}$$

we also assume the cycle Z intersects each $S_i\times D_I\times\Delta^j$ properly, where D_I is a face of $\square^m$ and Δ^j is a face of Δ^n. Let $d'\colon\mathcal{Z}^q(X,m,n)_s\to\mathcal{Z}^q(X,m-1,n)_s$ be the map $Z\mapsto Z\cdot(X\times(x_m=0)\times\Delta^n)$, and let $d''\colon\mathcal{Z}^q_s(X,m,n)\to\mathcal{Z}^q_s(X,m,n-1)$ be the map $Z\mapsto Z\cdot(X\times\square^m\times(t_0=0))$. This gives us a double complex $(\mathcal{Z}^q_s(X,m,n),d',d'')$; we let Tot_* be the associated total complex with differential $d=d'+(-1)^m d''$ on $\mathcal{Z}^q_s(X,m,n)$. We have the two augmentations $\epsilon'\colon Tot_*\to\mathcal{Z}^q(X,*)^c_s$ and $\epsilon''\colon Tot_*\to\mathcal{Z}^q_s(X,*)$.

Lemma 4.6. *The complexes $(Z^q_s(X,m,*),d'')$ and $(Z^q_s(X,*,n),d')$ are acyclic for $n,m\geq 1$.*

Proof. Let $(I\to C_{*,I})$ be an n-cube of homological complexes. We consider $C_{*,*}$ as an $(n+1)$-dimensional complex, and let $Tot(C_{*,*})$ denote the associated total complex, with $C_{0,\emptyset}$ in degree zero. Let $C^0_{*,\emptyset}$ denote the intersection of the kernels of the maps

$$C_{*,\emptyset}\to C_{*,\{i\}}\quad i=1,\ldots,n.$$

Then we have the natural map $C^0_{*,\emptyset}\to Tot(C_{*,*})$ which is a quasi-isomorphism if all the maps

$$C_{*,I}\to C_{*,I\cup\{i\}}\quad i=1,\ldots,n, I\subset\{1,\ldots,n\}$$

are surjective.

For $I\subset\{1,\ldots,n\}$, we let Δ_I denote the face of Δ^n defined by $t_i=0$ for $i\in I$. We apply the above considerations to the n-cube of complexes $C_{*,I}$

$$I\mapsto\mathcal{Z}^q_{s\times\Delta_I\cup X\times\partial\Delta_I)}(X\times\Delta_I,*)^c.$$

Since the inclusion maps $\Delta_{I\cup\{i\}}\to\Delta_I$ are split by linear projections $\Delta_I\to\Delta_{I\cup\{i\}}$, all the maps in the above n-cube are surjective. Thus we have the quasi-isomorphism $C^0_{*,\emptyset}\to Tot(C_{*,*})$. The homotopy property Proposition 4.4, together with the weak moving lemma Lemma 4.3 imply that $Tot(C_{*,*})$ is acyclic for $n\geq 1$. As $C^0_{*,\emptyset}=(\mathcal{Z}^q_s(X,*,n),d')$, we have proved this half of the lemma. The proof of the other half is similar (using properties (1) and (2) above instead of Lemma 4.3 and Proposition 4.5), and is left to the reader. □

Theorem 4.7. *Let X be a scheme over k, $s=\{S_1,\ldots,S_m\}$ a finite collection of closed subsets of X. Then there is a natural quasi-isomorphism*

$$\mathcal{Z}^q(X,*)^c_s\to\mathcal{Z}^q_s(X,*).$$

Proof. Consider the (homological) spectral sequence

$$E^1_{a,b}=H_b(\mathcal{Z}^q_s(X,a,*))\Rightarrow H_{a+b}(Tot_*).$$

By Lemma 4.6, the spectral sequence degenerates at E^1, and the augmentation $\epsilon''\colon Tot_*\to\mathcal{Z}^q_s(X,*)$ is a quasi-isomorphism. Similarly, the augmentation $\epsilon'\colon Tot_*\to\mathcal{Z}^q(X,*)^c_s$ is a quasi-isomorphism. Thus $\epsilon''\circ{\epsilon'}^{-1}$ is the desired quasi-isomorphism. □

Corollary 4.8. *Let X be a regular quasi-projective scheme over k, $s = \{S_1, \ldots, S_m\}$ a finite collection of closed subsets of X. Then the inclusion*

$$Z^q(X,*)_s \otimes \mathbb{Q} \to Z^q(X,*) \otimes \mathbb{Q}.$$

is a quasi-isomorphism.

Proof. By Theorem 4.7 we have a commutative diagram, with the vertical arrows quasi-isomorphisms

$$\begin{array}{ccc} Z^q(X,*)^c_s \otimes \mathbb{Q} & \to & Z^q(X,*)^c \otimes \mathbb{Q} \\ \downarrow & & \downarrow \\ \mathcal{Z}^q_s(X,*) \otimes \mathbb{Q} & \to & \mathcal{Z}^q(X,*) \otimes \mathbb{Q}. \end{array}$$

By Corollory 3.2, the top horizontal arrow is a quasi-isomorphism, hence the bottom horizontal arrow is a quasi-isomorphism as well. ☐

Corollary 4.9. *The assignments*

$$X \mapsto Z^q(X,*) \otimes \mathbb{Q}$$
$$X \mapsto Z^q(X,*)^c \otimes \mathbb{Q}$$

extend to a contravarient functor from the category of smooth quasi-projective k-schemes to the derived category $D_+(\mathbf{Ab})$ of homological complexes which are zero in sufficiently large negative degree.

Proof. If $f: Y \to X$ is a morphism of quasi-projective k-schemes, with X smooth, let $S_i = \{x \in X \mid \dim f^{-1} \geq i\}$, and let $s = s(f) = \{S_0, S_1, \ldots, S_N = \emptyset\}$. One checks (as in [B], proof of Theorem 4.1) that $f^{-1}(Z)$ is defined for each cycle in $Z^q(X,*)^c_s$. Let $i_s: Z^q(X,*)^c_s \otimes \mathbb{Q} \to \mathcal{Z}^q(X,*)^c \otimes \mathbb{Q}$ be the inclusion, and let $f^*: Z^q(X,*)^c \otimes \mathbb{Q} \to Z^q(Y,*)^c \otimes \mathbb{Q}$ be the composition in $D_+(\mathbf{Ab})$

$$Z^q(X,*)^c \otimes \mathbb{Q} \xrightarrow{i_s^{-1}} \mathcal{Z}^q(X,*)^c_s \otimes \mathbb{Q} \xrightarrow{f^*} Z^q(Y,*)^c \otimes \mathbb{Q}.$$

If y is any other set of closed subsets of X such that $f^*: Z^q_y(X,*)^c \to Z^q(Y,*)^c$ is defined, then, the commutativity of the diagram of inclusions

$$\begin{array}{ccc} Z^q_{s\cup y}(X,*)^c & \xrightarrow{i_{s,s\cup y}} & \mathcal{Z}^q(X,*)^c_s \\ i_{y,s\cup y} \downarrow & \searrow i_{s\cup y} & \downarrow i_s \\ Z^q_y(X,*)^c & \xrightarrow{i_y} & Z^q(X,*)^c \end{array}$$

shows that $f^* \circ i_s^{-1} = f^* \circ i_{s\cup y}^{-1} = f^* \circ i_y^{-1}$. This gives the functoriality $f^* \circ g^* = (g \circ f)^*$ for composable maps f and g, completing the proof for the cubical complexes $Z^q(X,*)^c$. The proof for the complexes $Z^q(X,*)$ is the same. ☐

Notation. Let $f: Y \to X$ be a morphism of quasi-projective k-schemes, with X smooth, and let $s(f)$ be the set of closed subsets of X given in the proof of Cor. 4.8. We set $Z^q(X,*)^c_f = Z^q(X,*)^c_{s(f)}$.

Bloch [B2] has defined $\mathbb{Q}$-complexes $N^q(X)_*$; for $X = \mathrm{Spec}(k)$, Bloch has defined products

$$\cup: N^q(k)_* \otimes N^{q'}(k)_* \to N^{q+q'}(k)_*$$

making the homology $\oplus_{p,q} H_p(N^q(k)_*)$ into a bi-graded ring (graded commutative in the p-grading, commutative in the q-grading). We conclude this section by defining quasi-isomorphisms

$$Alt^q\colon \mathcal{Z}^q(X,*)\otimes\mathbb{Q} \to \mathcal{N}^q(X)_*.$$

After we define products on the complexes $Z^q(-,*)\otimes\mathbb{Q}$ in the next section, we will show how Alt^* is compatible with the products when $X = \mathrm{Spec}(k)$ (actually, the two ring structures are opposites of each other).

Let F_p be the subgroup of the the group of k-automorphisms of $\square^p$ generated by the permutations $(x_1,\ldots,x_p)\mapsto(x_{\sigma 1},\ldots,x_{\sigma p})$, $\sigma\in\Sigma_n$, and the map $\tau(x_1,x_2,\ldots,x_p) = (1-x_1,x_2,\ldots,x_p)$. F_p is the semi-direct product $(\mathbb{Z}/2)^p\times\Sigma_p$ with σ_p acting on $(\mathbb{Z}/2)^p$ by permuting the factors. In particular, the homomorphism $sgn\colon\Sigma_p\to\{\pm1\}$ and sum $(\mathbb{Z}/2)^p\to\mathbb{Z}/2$ extend uniquely to the homomorphism $sgn\colon F_p\to\{\pm1\}$. Let Alt_p be the central idempotent in the rational group ring $\mathbb{Q}[F_p]$:

$$Alt_p = \frac{1}{|F_p|}\sum_{\nu\in F_p}(-1)^{sgn(\nu)}\nu.$$

F_p acts on $Z^q(X\times\square^p)_{X\times\partial\square^p}$ in the obvious way; the group $N^q(X)_p$ is defined by

$$N^q(X)_p = Alt_p(Z^q(X\times\square^p)_{X\times\partial\square^p}\otimes\mathbb{Q})\subset Z^q(X\times\square^p)_{X\times\partial\square^p}\otimes\mathbb{Q}.$$

Sending Z to $2p(Z\cdot(x_p=0))$ defines the map

$$d_p\colon N^q(X)_p\to N^q(X)_{p-1}$$

giving the complex $(N^q(k)_*,d)$. The product

$$\cup\colon N^q(k)_*\otimes N^{q'}(k)_*\to N^{q+q'}(k)_*.$$

is defined by $Z\cup W = Alt_{p+p'}(Z\times W)$ for $Z\in N^q(k)_p$, $W\in N^{q'}(k)_{p'}$.

We now define a projection

$$\pi_p\colon Z^q(X\times\square^p)_{X\times\partial\square^p}\to Z^q(X,p)^c.$$

in two steps: $\pi_p = q_2\circ q_1$. To define q_1, let $i_j\colon\square^{p-1}\to\square^p$ be the inclusion

$$i_j(x_1,\ldots,x_{p-1}) = (x_1,\ldots,x_{j-1},1,x_j,\ldots,x_{p-1}),$$

$j=1,\ldots,p$, and let $p_j\colon\square^p\to\square^{p-1}$ be the projection

$$p_j(x_1,\ldots,x_p) = (x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_p).$$

For $Z\in\mathcal{Z}^q(X\times\square^p)_{X\times\partial\square^p}$, define $q_1(Z)$ to be the cycle $Z-\sum_{j=1}^p p_j^*(i_j^*(Z))$. This defines

$$q_1\colon Z^q(X\times\square^p)_{X\times\partial\square^p}\to Z^q(X\times\square^p;(x_1=1)+(x_2=1)+\ldots+(x_p=1))_{X\times\partial\square^p}.$$

Then

$$q_1(Z)\cdot(x_j=1)=0 \quad j=1,\ldots,p$$
$$q_1(Z)\cdot(x_j=0)=Z\cdot(x_j=0)-Z\cdot(x_j=1) \quad j=1,\ldots,p$$

To define q_2, we let $\tau_j \in \Sigma_p$ be the permutation

$$\tau_j(i)=\begin{cases} i & \text{if } i<j \\ i-1 & \text{if } i>j \\ p & \text{if } i=j. \end{cases}$$

and let $\rho_j\colon (x_j=0)\to\square^{p-1}$ be the isomorphism

$$\rho_j(x_1,\ldots,x_{j-1},0,x_{j+1},\ldots,x_p)=(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_p).$$

For $Z\in Z^q(X\times\square^p)_{X\times\partial\square^p}$, let $W_p^j(Z)$ be the cycle $\tau_j^*(W_p(\rho_j(Z\cdot(x_j=0))))$. Define

$$q_2\colon Z^q(X\times\square^p)_{X\times\partial\square^p}\to Z^q(X\times\square^p)_{X\times\partial\square^p}$$

by

$$q_2(Z)=Z-\sum_{j=1}^{p-1}W_p^j(Z).$$

By Lemma 4.1, we have

$$q_2(Z)\cdot(x_j=1)=Z\cdot(x_j=1) \quad j=1,\ldots,p$$

$$q_2(Z)\cdot(x_j=0)=0 \quad j=1,\ldots,p-1$$

$$q_2(Z)\cdot(x_p=0)=Z\cdot(x_p=0)-\sum_{j=1}^{p-1}Z\cdot(x_j=0)$$

Letting $\pi_p=q_2\circ q_1$, we have defined the desired projection.

We form the complex $Z^q(X,*)^{Alt}$ by $Z^q(X,*)^{Alt}=Z^q(X\times\square^p)_{X\times\partial\square^p}$, with

$$d_p\colon Z^q(X,p)^{Alt}\to Z^q(X,p-1)^{Alt}$$

the map

$$d_p(Z)=\sum_{\rho\in\Sigma_p}(-1)^{sgn(\rho)}\rho_*(Z)\cdot[(x_p=0)-(x_p=1)].$$

Then the inclusions

$$i\colon Z^q(X,*)^c\to Z^q(X,*)^{Alt}, \quad j\colon N^q(X)_*\subset Z^q(X,*)^{Alt}$$

are maps of complexes, as is the projection

$$\pi\colon Z^q(X,*)^{Alt}\to Z^q(X,*)^c.$$

For a homological complex (C_*, d), let $Z_p(C_*)$ denote the kernel of $d\colon C_p \to C_{p-1}$. The action of F_p on $\square^p$ induces actions of F_p on $Z_p(Z^q(X,*)^c)$ which descends to an action on $\mathrm{CH}^q(X,p)$. Although a single element $\sigma \in F_p$ does not canonically give rise to an automorphism of the complex $Z^q(X,*)^c$, a compatible family of automorphisms does. For future use we consider on some special examples of such a compatible family.

For a homological complex C_*, let $C_*^{\tau \geq p}$ be the subcomplex

$$C_n^{\tau \geq p} = \begin{cases} 0 & \text{for } n < p \\ \ker(d\colon C_p \to C_{p-1}) & \text{for } n = p \\ C_n & \text{for } n > p, \end{cases}$$

and let $C_*^{* \geq p}$ be the subcomplex

$$C_n^{* \geq p} = \begin{cases} 0 & \text{for } n < p \\ C_n & \text{for } n \geq p. \end{cases}$$

For $0 < i \leq p$, let $\sigma_p^i \in \Sigma_p$ be the permutation (i,p), and let $\sigma_p = \sigma_p^1 \cdot \sigma_p^2 \cdot \ldots \cdot \sigma_p^{p-1}$. We have the inclusion $\Sigma_p \to \Sigma_n$ for $n > p$ where $\sigma \in \Sigma_p$ acts by the identity on $\{p+1, \ldots, n\}$, and by σ on $\{1, \ldots, p\}$. The automorphism

$$(-1)^{p-i}\sigma_{p*}^i\colon Z^q(X,n)^{Alt} \to Z^q(X,n)^{Alt}; \quad n \geq p,$$

extends to the automorphism

$$\sigma^{i,p}\colon Z^q(X,*)^{Alt} \to Z^q(X,*)^{Alt}$$

of the complex $Z^q(X,*)^{Alt}$ by operating by $(-1)^{p-i}\sigma_{p*}^i$ on $Z^q(X,n)^{Alt}$ for $n \geq p$, by $(-1)^{n-i}\sigma_{n*}^i$ on $Z^q(X,n)^{Alt}$ for $i < n < p$ and by the identity on $Z^q(X,n)^{Alt}$ for $n \leq i$. This in turn gives us the endomorphism

$$s^{i,p}\colon Z^q(X,*)^c \to Z^q(X,*)^c$$

by

$$s^{i,p}(Z) = \pi(\sigma^{i,p}(i(Z))).$$

Finally, since $s_n^{i,p}(Z) = Z$ for $Z \in Z^q(X,n)^c$, $n \leq i$, the compositions

$$s_*^{1,p} \circ s_*^{2,p} \circ \ldots \circ s_*^{p-1,p} \tag{4.3}$$

$p \geq n$, all have the same action on $Z^q(X,n)^c$. Letting

$$s_n\colon Z^q(X,n)^c \to Z^q(X,n)^c$$

be the composition (4.3) for $p \geq n$, the s_n define the map of complexes

$$s : Z^q(X,*)^c \to Z^q(X,*)^c.$$

Clearly, $s_p(Z) = (-1)^{\frac{p(p+1)}{2}}\sigma_p(Z)$ for $Z \in Z_p(Z^q(X,*)^c)$.

We have a similar construction for the map $\tau(x_1, x_2, \ldots, x_p) = (1 - x_1, x_2, \ldots, x_p)$. The automorphism

$$-\tau_*\colon Z^q(X,n)^{Alt} \to Z^q(X,n)^{Alt}; \quad n \geq 1$$

extends to automorphism

$$-\tau\colon Z^q(X,*)^{Alt} \to Z^q(X,*)^{Alt}$$

by acting by the identity on $Z^q(X,0)^{Alt}$. We let

$$t\colon Z^q(X,*)^c \to Z^q(X,*)^c$$

be the composition $\pi_* \circ -\tau \circ i$.

Lemma 4.10.

(a) *The maps*

$$\sigma^{i,p} \circ i \colon Z^q(X,*)^c \to Z^q(X,*)^{Alt}$$

$$-\tau \circ i \colon Z^q(X,*)^c \to Z^q(X,*)^{Alt}$$

are homotopic to the inclusion i.

(b) *The map*

$$s_* : Z^q(X,p)^c \to Z^q(X,p)^c.$$

is homotopic to the identity.

(c) *For* $\rho \in F_p$, *the map*

$$(-1)^{sgn(\rho)}\rho_* \colon Z_p(Z^q(X,*)^c) \to Z_p(Z^q(X,*)^c)$$

acts by the identity on $CH^q(X,p)$.

Proof. We begin with the first assertion. We first consider the case of $\sigma = \sigma_p^{p-1} \in \Sigma_p$. Let

$$t_j = \begin{cases} x_j & \text{for } j \neq p-1, p \\ x_{p-1}x_p - x_{p-1} - x_p + 1 & \text{for } j = p-1 \\ x_{p-1}x_p & \text{for } j = p \end{cases}$$

Define the map $q_n \colon \square^n \to \square^n$ by $q_n(x_1, \ldots, x_n) = (t_1, \ldots, t_n)$.

We form the complex $B(X,*)$ by setting

$$B(X,n) = Z^q(X \times \square^n; X \times \partial\square^n - (x_{p-1} = 1) - (x_p = 1) - (x_n = 0))_{(x_n=0)}$$

and defining $d \colon B^q(X,n) \to B^q(X,n-1)$ by $d(Z) = Z \cdot (x_n = 0)$.

The maps $q_{n*} \colon Z^q(X \times \square^n) \to Z^q(X \times \square^n)$ and $q_n^* \colon \mathcal{Z}^q(X \times \square^n) \to \mathcal{Z}^q(X \times \square^n)$ induce maps

$$q_* \colon Z^q(X,*)^{c\ *\geq p} \to B^q(X,*)^{*\geq p}$$

$$q^* \colon B^q(X,*)^{*\geq p} \to Z^q(X,*)^{Alt\ *\geq p}$$

with $q^*(q_*(Z)) = i(Z) + \sigma_*(i(Z))$ for $Z \in Z^q(X,p)^{c*\geq p}$.

Since the map $i - \sigma^{p-1,p} \circ i$ is the zero map on $Z^q(X,n)^c$ for $n < p$, we have the factorization

$$\begin{array}{ccc} Z^q(X,*)^c & \xrightarrow{i-\sigma^{p-1,p}\circ i} & Z^q(X,*)^{Alt} \\ q_* \searrow & & \nearrow q^* \\ & B^q(X,*)^{*\geq p}, & \end{array}$$

where we extend q_* and q^* by zero to give the above maps.

Arguing as in the proof of of Lemma 4.6, the homotopy property Theorem 4.5, together with Proposition 4.4, shows that the complex $B^q(X,*)^{*\geq p}$ is acyclic. Since $Z^q(X,*)^c$ is a complex of free $\mathbb{Z}$-modules, the map

$$q_* \colon Z^q(X,*)^c \to B^q(X,*)^{*\geq p}$$

is homotopic to zero. Thus $i - \sigma^{p-1,p} \circ i$ is homotopic to zero, proving (a) in this case. To prove (a) for the map $\sigma^{i,p}$, we use the identity $\sigma^{i,p} = \sigma^{i+1,p} \circ \sigma^{i,i+1}$ to give

$$i - \sigma^{i,p} = \sigma^{i+1,p} \circ (i - \sigma^{i,i+1} \circ i) + i - \sigma^{i+1,p} \circ i.$$

By induction, $i - \sigma^{i+1,p} \circ i$ is homotopic to zero; we have already shown that $i - \sigma^{i,i+1} \circ i$ is homotopic to zero, proving (a) for $\sigma^{i,p}$. We note that we may take the homotopy $h_*(\sigma^{i,p})\colon Z^q(X,*)^c \to Z^q(X,*+1)^{Alt}$ of $i - \sigma^{i+1,p} \circ i$ to zero to be zero for $* < i$.

The argument for the map $-\tau$ is similar, after replacing the maps q_n with the map $r_n(x_1, x_2, \ldots, x_n) = (x_1(1-x_1), x_2, \ldots, x_n)$, and replacing $B^q(k,*)$ with the complex $A^q(k,*)$

$$A^q(k,n) = Z^q(X \times \square^n; X \times \partial\square^n - (x_1 = 1) - (x_n = 0))_{(x_n=0)}.$$

For (b), following the homotopy $h_*(\sigma^{i,p})$ with π gives the homotopy $h_*(s^{i,p})$ of $s^{i,p}$ with the identity, with $h_n(s^{i,p}) = 0$ for $n < i$. These in turn gives the homotopy $h_*(j,p)$ of $s^{1,p} \circ s^{2,p} \circ \ldots \circ s^{j,p}$ with the identity. Since $h_n(s^{i,p}) = 0$ for $n < i$ and $s^{i,p}_n$ is the identity for $n < i$, we have $h_n(j,p) = h_n(j+l, p+m)$ for $n < j < p$ and for $l, m > 0$. Thus, we may define the homotopy h_* from s_* to the identity by taking $h_n = h_n(n+1, n+2)$, proving (b).

The assertion (c) follows directly from (a), the identities

$$\pi \circ \sigma^{i,j} \circ i = (-1)^{sgn(\sigma^i_j)} \sigma^i_j \text{ on } Z_p(\mathcal{Z}^q(X,*)^c), \text{ for } i < j \le p$$

$$\pi \circ -\tau \circ i = -\tau \text{ on } Z_p(\mathcal{Z}^q(X,*)^c),$$

and the fact that F_p is generated by the σ^i_j and τ. This completes the proof. □

Theorem 4.11. *The map*

$$Alt^q\colon Z^q(X,*) \otimes \mathbb{Q} \to N^q(X)_*.$$

is a quasi-isomorphism.

Proof. For each n, and for each cycle Z on $X \times \square^n$, the cycle $W^X_n(Z)$ on $X \times \square^{n+1}$ is symmetric with respect to the automorphism $(x_1, \ldots, x_n, x_{n+1}) \mapsto (x_1, \ldots, x_{n+1}, x_n)$. Similarly, the cycle $Z \times \mathbb{A}^1$ is symmetric with respect to the automorphism $(x_1, \ldots, x_n, x_{n+1}) \mapsto (x_1, \ldots, x_n, 1 - x_{n+1})$. From these facts, together with a simple direct computation, we have

$$Alt^q(\pi(j(Z))) = Z$$

for $Z \in \mathcal{N}^q(k)_*$. On the other hand, by Lemma 4.10, the composition $\pi \circ j \circ Alt^q$ induces the identity map on the homology of $Z^q(X,*) \otimes \mathbb{Q}$, hence is a quasi-isomorphism. This proves the theorem. □

§5 Products and the projective bundle formula

Denote by $(-)_{\mathbb{Q}}$ the functor $(-)\otimes\mathbb{Q}$. In this section, we define, for X smooth and quasi-projective, a product

$$Z^a(X,*)_{\mathbb{Q}} \otimes Z^b(X,*)_{\mathbb{Q}} \to Z^{a+b}(X,*)_{\mathbb{Q}}$$

in the derived category, giving $\oplus_{q,p}\mathrm{CH}^q(X,p)^c_{\mathbb{Q}}$ the structure of a bi-graded ring, commutative with respect to the q-grading and graded commutative with respect to the p-grading. We also prove the projective bundle formula for $\mathrm{CH}^q(X,p)^c$.

Let Y be a k-scheme, s a finite set of closed subsets of Y, and let

$$Z^q(Y,m,n)^c_s \subset Z^q(Y\times\square^m\times\square^n)_{s(Y\times(\partial\square^m\times\square^n+\square^m\times\partial\square^n))}$$

be the subgroup consisting of cycles Z such that

$$\begin{gathered} Z\cdot(Y\times(x_i=0)\times\square^n)=0 \quad \text{for } i=1,\ldots,m-1\\ Z\cdot(Y\times(x_i=1)\times\square^n)=0 \quad \text{for } i=1,\ldots,m\\ Z\cdot(Y\times\square^m\times(x_i=0))=0 \quad \text{for } i=1,\ldots,n-1\\ Z\cdot(Y\times\square^m\times(x_i=1))=0 \quad \text{for } i=1,\ldots,n. \end{gathered}$$

We also assume that Z intersects $S\times D_I\times D_J$ properly for each $S\in s$, and each face D_I of $\square^m$ and face D_J of $\square^n$. Let $d'\colon \mathcal{Z}^q(Y,m,n)^c_s \to \mathcal{Z}^q(Y,m-1,n)^c_s$ be the map $Z\mapsto Z\cdot(Y\times(x_m=0)\times\square^n)$, and let $d''\colon \mathcal{Z}^q(Y,m,n)^c\to\mathcal{Z}^q(Y,m,n-1)^c$ be the map $Z\mapsto Z\cdot(Y\square^m\times(x_n=0))$. This gives us a double complex $(\mathcal{Z}^q(Y,m,n)^c_s,d',d'')$; we let $Tot(Y)^c_s$ be the associated total complex with differential $d=d'+(-1)^m d''$ on $\mathcal{Z}^q(Y,m,n)^c$. We have the map $\epsilon\colon\mathcal{Z}^q(Y,*)^c_s\to Tot(Y)^c_s$ gotten by identifying $\mathcal{Z}^q(Y,*)^c_s$ with $\mathcal{Z}^q(Y,0,*)^c_s$ and the map $\epsilon'\colon\mathcal{Z}^q(Y,*)^c_s\to Tot(Y)^c_s$ gotten by identifying $\mathcal{Z}^q(Y,*)^c_s$ with $\mathcal{Z}^q(Y,*,0)^c_s$.

Lemma 5.1. *The maps*

$$\epsilon\colon\mathcal{Z}^q(Y,*)^c_s\to Tot(Y)^c_s$$

and

$$\epsilon'\colon\mathcal{Z}^q(Y,*)^c_s\to Tot(Y)^c_s$$

are quasi-isomorphisms. The composition $\epsilon'\circ\epsilon^{-1}$ is the identity (in $D_+(\mathbf{Ab})$).

Proof. The proof of the first assertion is essentially the same as the argument used in the proof of Theorem 4.7. We have the spectral sequence

$$E^1_{a,b}=H_b(Z^q(Y,a,*)^c_s)\Rightarrow H_{a+b}(Tot(Y)_s).$$

As in the proof of Lemma 4.6, the homotopy property Theorem 4.5, together with Proposition 4.4, shows that $E^1_{a,b}=0$ for $a>0$, hence the spectral sequence degenerates at E^1 and ϵ is a quasi-isomorphism. The proof for ϵ' is the same.

For the second assertion, let $Z_p(Z^q(Y,*)^c_s)$ denote the kernel of d on $Z^q(Y,p)^c_s$, let $Z'_{p,q}(Z^q(Y,*,*)^c_s)$ and $Z''_{p,q}(Z^q(Y,*,*)^c_s)$ denote the kernel of d' and d'', respectively, on $Z^q(Y,p,q)^c_s$. Take $\eta\in Z_p(Z^q(Y,*)^c_s)$, and let $\eta_{0,p}\in Z^q_s(Y,0,p)^c$, $\eta_{p,0}\in Z^q_s(Y,p,0)^c$ be the elements

$$\eta_{0,p}=\epsilon(\eta),\quad \eta_{p,0}=\epsilon'(\eta).$$

Identify $\square^a \times \square^{b+1}$ with $\square^{a+b+1}$ by

$$((x_1, \ldots, x_a), (y_1, \ldots, y_{b+1})) \mapsto (x_1, \ldots, x_{a-1}, y_1, \ldots, y_b, x_a, y_{b+1}),$$

and let $W^Y_{a,b} \subset Y \times \square^a \times \square^{b+1}$ be the image of $W^Y_{a+b} \subset Y \times \square^{a+b+1}$ under this identification. Using the obvious modification of the construction of the map

$$W^Y_n : Z^q(Y \times \square^n) \to Z^q(Y \times \square^{n+1})$$

we construct the map

$$W^Y_{a,b}: Z^q(Y \times \square^a \times \square^b) \to Z^q(Y \times \square^a \times \square^{b+1})$$

satisfying the analog of Lemma 4.1. In particular, $W^Y_{a,b}$ defines the map

$$W^Y_{a,b}: Z''_{a,b}(Z^q(Y, *, *)^c_s) \to Z^q(Y, a, b+1)^c_s$$

with

$$d''(W^Y_{a,b}(Z)) = (-1)^a Z \quad \text{ for } Z''_{a,b}(Z^q(Y, a, b)^c_s)$$
$$d'(W^Y_{a,b}(Z)) = \tau_{a,b*}(Z) \quad \text{ for } Z \in Z''_{a,b}(Z^q(Y, a, b)^c_s), \tag{5.1}$$

where $\tau_{a,b}((x_1, \ldots, x_a), (y_1, \ldots, y_b)) = (((x_1, \ldots, x_{a-1}), (x_a, y_1, y_2 \ldots, y_b))$.

This gives us the elements

$$\begin{aligned} W_{p,0}(\eta_{p,0}) &\in \mathcal{Z}^q(Y, p, 1)^c_s \\ W_{p-1,1}(d'(W_{p,0}(\eta_{p,0}))) &\in \mathcal{Z}^q(Y, p-1, 2)^c_s \\ W_{p-2,2}(d'(W_{p-1,1}(d'(W_{p,0}(\eta_{p,0})))) &\in \mathcal{Z}^q(Y, p-2, 3)^c_s \\ &. \\ &. \\ &. \\ W_{1,p-1}(\ldots(d'(W_{p,0}(\eta_{p,0})))\ldots) &\in \mathcal{Z}^q(Y, 1, p)^c_s \end{aligned}$$

Then it follows from (5.1) that

$$(d' + d'')((-1)^p W_{p,0}(\eta_{p,0}) - (-1)^{2p-1} W_{p-1,1}(d'(W_{p,0}(\eta_{p,0}))+$$
$$\ldots - (-1)^{\frac{p(p+1)}{2}} W_{1,p-1}(\ldots(d'(W_{p,0}(\eta_{p,0})))\ldots))$$

$$= \eta_{p,0} - (-1)^{\frac{p(p+1)}{2}} \sigma_p(\eta_{0,p}).$$

Define $h^{p-a,a+1}_p(\eta)$ inductively by $h^{p,1}_p(\eta) = (-1)^p W_{p,0}(\eta_{p,0})$, and

$$h^{p-a,a+1}_p(\eta) = (-1)^{p-a+1} W_{p-a,a}(d' h^{p-a+1,a}_p(\eta))$$

for $a = 1, \ldots, p-1$. Letting $h_p(\eta) = \sum_{a=0}^{p-1} h_p^{p-a,a+1}(\eta)$, we have

$$(d' + d'')(h_p(\eta)) = \eta_{p,0} - (-1)^{\frac{p(p+1)}{2}} \sigma_p(\eta_{0,p})$$

for $\eta \in Z_p(\mathcal{Z}^q(Y,*)_s^c)$. We now proceed to extend h_p to all of $\mathcal{Z}^q(Y,p)_s^c$.

For $Z \in Z^q(Y,p)_s^c$, let $h_p^{p,1}(Z) = (-1)^p W_{p,0}(Z)$. Then $h_p^{p,1}(Z)$ is in $Z^q(Y,p,1)_s^c$, and

$$d''(h_p^{p,1}(Z)) = Z$$

$$d'(h_p^{p,1}(Z)) = -h_{p-1}^{p-1,1}(dZ).$$

Define $h_p^{p-a,a+1}(Z)$ inductively, satisfying

$$d''(h_p^{p-a,a+1}(Z)) = -d' h_p^{p-a+1,a}(Z) - h_{p-1}^{p-a,a}(dZ).$$

Then

$$\begin{aligned} d'' \circ d'(h_p^{p-a,a+1}(Z)) =& d' h_{p-1}^{p-a,a}(dZ) \\ =& -d'' h_{p-1}^{p-a-1,a+1}(dZ), \end{aligned}$$

so $d''(d' h_p^{p-a,a+1}(Z) + h_{p-1}^{p-a-1,a+1}(dZ)) = 0$. Thus, if we define

$$h_p^{p-a-1,a+2}(Z) = (-1)^{p-a} W_{p-a-1,a+1}(d' h_p^{p-a,a+1}(Z) + h_{p-1}^{p-a-1,a+1}(dZ)),$$

we have

$$d''(h_p^{p-a-1,a+2}(Z)) = -d' h_p^{p-a,a+1}(Z) - h_{p-1}^{p-a-1,a+1}(dZ)$$

and the induction goes through.

Let $h_p(Z) = \sum_{a=1}^{p-1} h_p^{p-a,a}(Z)$, for $Z \in \mathcal{Z}^q(Y,p)_s^c$. Then this extends our earlier definition of h_p on $Z_p(\mathcal{Z}^q(Y,*)_s^c)$. Let σ_p^i be the permutation $(i,p) \in \Sigma_p$. Then $\sigma_p = \sigma_p^1 \ldots \sigma_p^{p-2} \sigma_p^{p-1}$; let

$$Z' = (-1)^{\frac{p(p+1)}{2}} \pi \circ \sigma_{p*}^1 \circ i(\ldots(\pi \circ \sigma_{p*}^{p-2} \circ i(\pi \circ \sigma_{p*}^{p-1} \circ i(Z) \ldots) = s(Z),$$

where s is the map defined just before Lemma 4.10. Then a direct computation gives

$$(d' + d'')h_p(Z) + h_{p-1}(dZ) = \epsilon(Z) - \epsilon'(Z') = \epsilon(Z) - \epsilon'(s(Z)).$$

By Lemma 4.10(b), the map $Z \mapsto s(Z)$ is homotopic to the identity. Thus ϵ' and ϵ are homotopic, completing the proof. □

The complex $Tot(Y)^c$ is covariantly functorial for proper maps, and $Tot(Y)_s^c$ contravariantly functorial for appropriate maps (depending on s).

Suppose we have non-negative integers q, q' and q'' with $q' + q'' = q$. Let

$$\times_{m,n}: \mathcal{Z}^{q'}(X,m)^c \otimes \mathcal{Z}^{q''}(X,n)^c \to \mathcal{Z}^q(X \times X, m, n)^c$$

be the map $\times_{m,n}(Z \otimes W) = \sigma_{23*}(Z \times W)$, where

$$\sigma_{23}\colon (X \times \square^m) \times (X \times \square^n) \to (X \times X) \times (\square^m \times \square^n)$$

is the obvious isomorphism. Then the maps $\times_{m,n}$ give rise to a map of total complexes

$$Tot(\times)^{q',q''}\colon \mathcal{Z}^{q'}(X,*)^c \otimes \mathcal{Z}^{q''}(X,*)^c) \to Tot_s(X \times X)^c;$$

composing with the inverse of the quasi-isomorphism ϵ defines the map in $D_+(\mathbf{Ab})$

$$\times_X^{q',q''}\colon Z^{q'}(X,*)^c \otimes^L Z^{q''}(X,*)^c \to \mathcal{Z}^q(X \times X,*)^c.$$

Let $\Delta_X\colon X \to X \times X$ be the diagonal. If X is smooth and quasi-projective over k, we have the pull-back map $\Delta_X^*\colon Z^q(X \times X,*)^c_{\mathbb{Q}} \to \mathcal{Z}^q(X,*)^c_{\mathbb{Q}}$ in $D_+(\mathbf{Ab})$; define

$$\cup^{q',q''}\colon Z^{q'}(X,*)^c_{\mathbb{Q}} \otimes Z^{q''}(X,*)^c_{\mathbb{Q}} \to Z^q(X,*)^c_{\mathbb{Q}}$$

as the composition $\Delta_X^* \circ \times_X$. This gives product maps

$$\cup_{p',p''}^{q',q''}\colon \mathrm{CH}^{q'}(X,p')_{\mathbb{Q}} \otimes \mathrm{CH}^{q''}(X,p'')_{\mathbb{Q}} \to \mathrm{CH}^{q'+q''}(X,p'+p'')_{\mathbb{Q}}$$

Theorem 5.2. *Let X be smooth and quasi-projective over k. The maps $\cup_{p',p''}^{q',q''}$ define the structure of a bi-graded ring (graded commutative with respect to p and commutative with respect to q) on the bi-graded group $\oplus_{p,q}CH^q(X,p)_{\mathbb{Q}}$ such that*

(a) for each morphism $f\colon X \to Y$ of smooth quasi-projective varieties, the map f^* is a ring homomorphism.

(b) if $f\colon X \to Y$ is a proper morphism of smooth quasi-projective varieties, we have the projection formula

$$f_*(\alpha \cup f^*(\beta)) = f_*(\alpha) \cup \beta$$

for $\alpha \in CH^*(X,*)_{\mathbb{Q}}$, $\beta \in CH^*(Y,*)_{\mathbb{Q}}$.

(c) the restriction of $\cup$ to $\oplus_q CH^q(X,0)_{\mathbb{Q}}$ is the usual product structure on the rational Chow ring of X.

(d) Suppose $Z \in Z^q(X,p)^c$, $W \in \mathcal{Z}^{q'}(X,p')^c$ represent classes in $CH^q(X,p)$, $CH^{q'}(X,')$, resp., then

$$Z \cup W = (-1)^{pp'}\Delta_X^*(Z \times W) = \Delta_X^*(W \times Z)$$

Proof. We first verify that $\cup$ is graded commutative with respect p and commutative with respect to q. Let $tr_{a,b}\colon Z^q(X \times X \times \square^a \times \square^b) \to \mathcal{Z}^q(X \times X \times \square^b \times \square^a)$ be the automorphism induced by exchanging the factors X and X, and the factors $\square^a$ and $\square^b$. The maps $tr_{a,b}$ give rise to the automorphism tr of $Tot_s(X \times X)^c$ by $tr(Z) = (-1)^{ab}tr_{a,b}(Z)$ for $Z \in \mathcal{Z}^q(X \times X \times \square^a \times \square^b)$. Let τ be the canonical isoomorphism

$$\tau\colon Tot(Z^{q'}(X,*) \otimes Z^{q''}(X,*)) \to Tot(Z^{q''}(X,*) \otimes Z^{q'}(X,*))$$

induced by the exchange of factors in the tensor product. Then we have

$$Tot(\times) \circ \tau = tr \circ Tot(\times)$$

and

$$\epsilon' = \epsilon \circ tr$$

From Lemma 5.1, it follows that $\cup \circ \tau = \cup$ on homology; as $\tau(A \otimes B) = (-1)^{ab}(B \otimes A)$ for $A \in Z^{q'}(X, a)$, $B \in \mathcal{Z}^{q''}(X, b)$, we have

$$A \cup B = (-1)^{ab}(B \cup A)$$

for $A \in \mathrm{CH}^{q'}(X, a)_{\mathbb{Q}}$, $B \in \mathrm{CH}^{q''}(X, b)_{\mathbb{Q}}$.

Associativity of the product $\cup$ follows by considering the triple complex analogue of the double complex considered in Lemma 5.1; we leave the details to the reader.

To prove (a), note that the exterior product $Tot(\times)$ clearly satisfies

$$f^*(Tot(\times)(Z \otimes W)) = Tot(\times)(f^*(Z) \otimes f^*(W))$$

The result then follows from the naturality of the quasi-isomorphism ϵ and the relation

$$\Delta_X^* \circ (f \times f)^* = f^* \circ \Delta_Y^*.$$

We now prove the projection formula (b). Let Z be in $Z^q(Y \times X, p)$ such that $((f \times id) \circ \Delta_X)^*(Z)$ is defined. Then $\Delta_Y^*((id \times f)_*(Z))$ is also defined, and we have the identity of cycles

$$\Delta_Y^*((id \times f)_*(Z)) = ((f \times id) \circ \Delta_X)^*(Z). \tag{5.1}$$

The maps $(id \times f)_*$ and $(f \times id)^*$ induce maps

$$(f \times id)^*: Tot_f(Y \times X)^c \to Tot_s(X \times X)^c$$

and

$$(id \times f)_*: Tot_f(Y \times X)^c \to Tot_s(Y \times Y)^c.$$

By the naturality of the quasi-isomorphisms ϵ, we have the commutative diagram

$$\begin{array}{ccccc} Tot(X \times X)^c & \overset{(f\times id)^*}{\leftarrow} & Tot_f(Y \times X)^c & \overset{(id\times f)_*}{\rightarrow} & Tot(Y \times Y)^c \\ \epsilon_{X\times X} \downarrow & & \epsilon_{Y\times X} \downarrow & & \epsilon_{Y\times Y} \downarrow \\ \mathcal{Z}^q(X \times X, *)^c & \overset{(f\times id)^*}{\leftarrow} & \mathcal{Z}^q_{f\times id}(Y \times X, *)^c & \overset{(id\times f)_*}{\rightarrow} & \mathcal{Z}^q(Y \times Y, *)^c \end{array} \tag{5.2}$$

We have as well the commutative diagram

$$\begin{array}{ccccc} \mathcal{Z}^q(X, *)^c \otimes \mathcal{Z}^q(X, *)^c & \overset{f^*\otimes id^*}{\leftarrow} & \mathcal{Z}^q_f(Y, *)^c \otimes \mathcal{Z}^q(X, *)^c & \overset{(id\times f)_*}{\rightarrow} & \mathcal{Z}^q(Y, *)^c \otimes \mathcal{Z}^q(Y, *)^c \\ Tot(\times) \downarrow & & Tot(\times) \downarrow & & Tot(\times) \\ Tot(X \times X)^c & \overset{(id\times f)^*}{\leftarrow} & Tot_{f\times id}(Y \times X)^c & \overset{(id\times f)_*}{\rightarrow} & Tot(Y \times Y)^c \end{array} \tag{5.3}$$

Putting (5.1), (5.2) and (5.3) together proves (b).

For (d) we retain the notation of the proof of Lemma 5.1. Let $\tau: \square^{p+p'} \to \square^{p+p'}$ be the automorphism

$$\tau_{p,p'}(x_1, \ldots, x_p, y_1, \ldots, y_{p'}) = (y_1, \ldots, y_{p'}, x_p, \ldots, x_1).$$

We have

$$(d'+d'')(W_{p,p'}(\times_{p,p'}(Z\otimes W)-W_{p-1,p'+1}(d'(W_{p,p'}(\times_{p,p'}(Z\otimes W)))+\ldots$$
$$+(-1)^{p-1}W_{1,p+p'-1}(\ldots(d'(W_{p,0}(\times_{p,p'}(Z\otimes W))))\ldots))$$

$$=(-1)^p(\times_{p,p'}(Z,W)-(-1)^{\frac{p(p+1)}{2}}\times_{0,p+p'}(\tau_{p,p'*}(Z\times W)).$$

Since $sgn(\tau_{p,p'})=(-1)^{pp'+\frac{p(p+1)}{2}}$, we have

$$\epsilon^{-1}(\times_{p,p'}(Z,W))=(-1)^{pp'}(Z\times W).$$

By Lemma 4.10, $(-1)^{pp'}(Z\times W)=W\times Z$ in homology. The formula (d) then follows from the definition of the product $\cup$. The assertion (c) follows from (d). □

Let X and Y be smooth quasi-projective varieties, with X projective. Let $d_{X/Y}=\dim(X)-\dim(Y)$. For a codimension d cycle W on $Y\times X$, form the homomorphism

$$W_*\colon\oplus_{q,p}\mathrm{CH}^q(X,p)_{\mathbb{Q}}\to\oplus_{q,p}\mathrm{CH}^{q+d-d_{X/Y}}(Y,p)_{\mathbb{Q}}$$

by $W_*(\eta)=p_{1*}(W\cup p_2^*(\eta))$. We recall the pairing

$$\circ\colon\mathrm{CH}^a(Z\times Y)_{\mathbb{Q}}\times\mathrm{CH}^b(Y\times X)_{\mathbb{Q}}\to\mathrm{CH}^{a+b}(Z\times X)_{\mathbb{Q}}$$

defined by

$$W_2\circ W_1:=pr_{Z\times X*}(pr^*_{Z\times Y}(W_2)\cup pr^*_{Y\times X}(W_1))$$

This is defined if Y is projective and X, Y and Z are smooth and quasi-projective over k, and gives $\mathrm{CH}^*(X\times X)_{\mathbb{Q}}$ the structure of a graded ring, if X is smooth and projective over k. In addition, we have $(W_2\circ W_1)_*=W_{2*}\circ W_{1*}$. Finally, if W is the graph of a morphism $f\colon Y\to X$, then $W_*(\eta)=f^*(\eta)$.

Corollary 5.3. *Suppose X and Y are smooth and quasi-projective over k, and X is projective. Sending Z to γ_Z descends to a homomorphism*

$$\gamma\colon\oplus_d CH^d(X\times Y)_{\mathbb{Q}}\to\oplus_{q,p}Hom(CH^q(X,p)_{\mathbb{Q}}CH^{q+d-d_{X/Y}}(Y,p)_{\mathbb{Q}}).$$

This makes $\oplus_{p,q}CH^q(X,p)_{\mathbb{Q}}$ into a graded $CH^*(X\times X)$-module.

Proof. This follows directly from Theorem 5.2. □

Corollary 5.4. *Let $E\to X$ be a vector bundle of rank $n+1$ over a smooth, quasi-projective variety X, and let $\pi\colon P\to X$ be the associated projective space bundle. Let ζ be the class of $\mathcal{O}(1)$ in $CH^1(P)$. Then the maps*

$$\alpha_i\colon CH^{q-i}(X,*)\to CH^q(P,*)$$

$$\alpha_i(\eta)=\pi^*(\eta)\cup\zeta^i$$
$$i=0,\ldots,n$$

define an isomorphism for each p:

$$\sum_{i=0}^{n} \alpha_i \colon \oplus_{i=0}^{n} CH^{q-i}(X,p) \to CH^q(P,p).$$

Proof. That $\sum_{i=0}^{n} \alpha_i$ gives an isomorphism for $p=0$ is well-known. In particular, the $CH^n(P \times_X P)$-class of the diagonal $\Delta \subset P \times_X P$ can be written as

$$[\Delta] = \sum_{i=0}^{n} p_1^*(a_i) \cup p_2^*(\zeta^{n-i}).$$

Let η be in $\mathrm{CH}^q(P,p)$. Then

$$\begin{aligned}
\eta &= [\Delta]_*(\eta) \\
&= p_{2*}(p_1^*(\eta) \cup \Delta) \\
&= p_{2*}(p_1^*(\eta) \cup \sum_{i=0}^{n} p_1^*(a_i) \cup p_2^*(\zeta^{n-i})) \\
&= \sum_{i=0}^{n} \zeta^{n-i} \cup (p_{2*}(p_1^*(\eta \cup a_i)),
\end{aligned}$$

so $\sum_{i=0}^{n} \alpha_i$ is in general surjective. Suppose $\sum_{i=0}^{n-j} \alpha_i(\tau_i) = 0$ for $\tau_i \in \mathrm{CH}^{q-i}(X,p)$, $i = 0, \ldots, n-j$ with $\tau_{n-j} \neq 0$. Then $\zeta^j \cup \sum_{i=0}^{n-j} \pi^*(\tau_i) \cup \zeta^i = 0$, so

$$\begin{aligned}
0 &= \pi_*(\sum_{i=j}^{n} \pi^*(\tau_{i-j}) \cup \zeta^i)) \\
&= \sum_{i=j}^{n} \tau_{i-j}) \cup \pi_*(\zeta^i) \\
&= \tau_{n-j},
\end{aligned}$$

since

$$\pi_*(\zeta^i) = \begin{cases} 0 & \text{if } 0 \le i < n \\ [X] & \text{if } i = n. \end{cases}$$

Thus all the τ_i were zero, and $\sum_{i=0}^{n} \alpha_i$ is injective. □

We recall from §4 the product

$$\cup \colon \mathcal{N}^q(k)_* \otimes \mathcal{N}^{q'}(k)_* \to \mathcal{N}^{q+q'}(k)_*$$

defined by

$$Z \cup W = Alt^{q+q'}(Z \times W).$$

Corollary 5.5. *Let*

$$t \colon \mathcal{Z}^q(X,*)^c \otimes^L \mathcal{Z}^{q'}(X,*)^c \to \mathcal{Z}^{q'}(X,*)^c \otimes^L \mathcal{Z}^q(X,*)^c$$

be the canonical isomorphism induced by the exchange of factors in $\otimes$*. Then the diagram*

$$
(5.4)\qquad \begin{array}{ccc}
\mathcal{Z}^q(X,*)^c \otimes^L \mathcal{Z}^{q'}(X,*)^c & \xrightarrow{\cup \circ t} & \mathcal{Z}^{q+q'}(X,*)^c \\
Alt^q \otimes Alt^{q'} \downarrow & & \downarrow Alt^{q+q'} \\
\mathcal{N}^q(k)_* \otimes^L \mathcal{N}^{q'}(k)_* & \xrightarrow{\cup} & \mathcal{N}^{q+q'}(k)_*
\end{array}
$$

commutes in $D_+(\mathbf{Ab})$*.*

Proof. By Theorem 5.2(d), we have

$$Z \cup W = (-1)^{pp'}(Z \times W) = W \times Z$$

for $Z \in \mathrm{CH}^q(k,p) \otimes \mathbb{Q}$, $W \in \mathrm{CH}^{q'}(k,p') \otimes \mathbb{Q}$. From this and the definition of the product on $\mathcal{N}^*(k)_*$, the diagram (5.4) induces a commutative diagram after taking homology. Since the complexes in (5.4) are complexes of $\mathbb{Q}$-vector spaces, this implies that (5.4) commutes in $D_+(\mathbf{Ab})$. □

§6 Chern character, relative cycles and K

In ([B2], §7) Bloch gives an argument for the construction of Chern classes with values in $\mathrm{CH}^q(-,p)$. This construction, however, relies on a Mayer-Vietoris property for the complexes $\mathcal{Z}^q(X,*)$; as the gap in the proof of localization for the complexes $\mathcal{Z}^q(X,*)$ leaves unproved this Mayer-Vietoris property, the construction of the integral Chern classes in [B2] is incomplete. However, as we have proved the relevant Mayer-Vietoris property for the $\mathbb{Q}$-complexes $\mathcal{Z}^q(X,*)^c\otimes\mathbb{Q}$ (Theorem 3.3), and as the complexes $\mathcal{Z}^q(X,*)$ and $\mathcal{Z}^q(X,*)^c$ are naturally quasi-isomorphic (Theorem 4.7), the Mayer-Vietoris property holds for $\mathcal{Z}^q(X,*)\otimes\mathbb{Q}$ as well. Bloch's argument then goes through to construct natural Chern classes with values in $\mathrm{CH}^q(-,p)\otimes\mathbb{Q}$, as well as the Chern character

$$ch^p\colon K(X)\to\oplus_q\mathrm{CH}^q(X,p)\otimes\mathbb{Q}.$$

In this section, we recall some salient points from Bloch's argument for the construction of the Chern classes

$$c_{q,p}\colon K_{2q-p}(X)\otimes\mathbb{Q}\to\mathrm{CH}^q(X,2q-p)\otimes\mathbb{Q}.$$

and give a slight refinement of his construction which shows that the Chern character gives a natural decomposition of the localized space $K(X)\otimes\mathbb{Q}$ as $\prod_q K(\mathcal{Z}^q(X,*)^c\otimes\mathbb{Q},0)$ via the weak homotopy equivalence

$$ch\colon K(X)\otimes\mathbb{Q}\to\prod_q K(\mathcal{Z}^q(X,*)^c\otimes\mathbb{Q},0),$$

where $K(C_*,0)$ denotes the $0th$ space in the Eilenberg-Maclane spectrum $EM(C_*)$ associated to a complex of abelian groups C_*. This raises a natural question. Let $K(X)_*$ be the K-theory spectrum of X, i.e. $K(X)_n$ is the geometric realization of the category $Q^n(\mathcal{P}_X)$, and the map $K(X)_n\to\Omega K_{n+1}(X)$ is the homotopy equivalence defined by Waldhausen. Is there a natural homotopy equivalence of spectra

$$ch\colon K(X)_*\otimes\mathbb{Q}\to\prod_q EM(\mathcal{Z}^q(X,*)^c\otimes\mathbb{Q})?$$

Presumably, Schechtman's delooping of the Chern character suffices to give such a decomposition of $K(X)_*\otimes\mathbb{Q}$, but we have not checked this. We also have not checked that our map ch defines a decomposition of $K(X)\otimes\mathbb{Q}$ into "eigenspaces for the Adams operations", i.e., if the diagram

$$\begin{array}{ccc} K(X)\otimes\mathbb{Q} & \xrightarrow{ch} & \prod_q K(\mathcal{Z}^q(X,*)^c\otimes\mathbb{Q},0)\\ \psi^k\downarrow & & \downarrow\prod_q\times k^q\\ K(X)\otimes\mathbb{Q} & \xrightarrow{ch} & \prod_q K(\mathcal{Z}^q(X,*)^c\otimes\mathbb{Q},0)\end{array}$$

commutes up to homotopy, where $\times k^q$ is the map on $K(\mathcal{Z}^q(X,*)^c\otimes\mathbb{Q},0)$ induced by multiplication by k^q on the complex $\mathcal{Z}^q(X,*)^c\otimes\mathbb{Q}$. The induced map on homotopy groups does however commute.

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