Lambda-operations, K-theory and motivic cohomology

Marc Levine

Introduction

This paper is in two parts: in part one, our main object is to give a construction of natural λ-operations for relative K-theory with supports, satisfying the special λ-ring identities. In the second part, we give an application to the relation of motivic cohomology and algebraic K-theory of a smooth quasi-projective variety over a field.

The main idea in the construction of the λ-operations is to re-do the constructions of Quillen (Hiller gives an exposition of Quillen's construction in [H]) and Kratzer [Kr] for the K-theory of a commutative ring, in the setting of the K-theory of an *I-diagram of commutative rings*, i.e., a functor

$$A: I \to \mathbf{Rings}$$

with I a small category. This approach can contrasted with the construction of Soulé [So], where use is made of the closed model structure on the category of sheaves of simplicial sets. Our point of view is to look at the "discrete" setting, viewing for example a scheme as a finite diagram of affine schemes, and replace the use of topology and sheaf theory with the Mayer-Vietoris property of K-theory provided by the Thomason-Trobaugh theorem [T-T], together with the remark that the operations we construct are natural with respect to refinements.

We rely on the general machinery of closed simplicial model categories, especially that of I-diagrams of simplicial sets: the category of functors

$$X: I \to \mathcal{S},$$

where I is a fixed small category, and $\mathcal{S}$ is the category of simplicial sets. Much of the general theory, as well as the fundamental properties, of Quillen's $+$-construction carry over without change to the setting of I-diagram of commutative rings. However, at some point we need to work with an analog of finite CW complexes, which restricts our theory to *finite* categories, i.e., a category I whose nerve has only finitely many non-degenerate simplices. Fortunately, the diagrams of affine schemes needed to express the relative K-theory of a noetherian scheme, with supports in a closed subset, are all finite.

Besides the works mentioned above, there is a recent construction of Lecomte [Le], building on previous work of Schechtman [Sc], giving Adams operations for the K-groups of schemes, which in turn gives a special λ-ring structure for the rational K-groups. It seems that her construction would also give natural Adams operations for relative K-theory with supports, and for other groups expressable as finite diagrams of schemes, or perhaps even for arbitrary small diagrams. There is also the unpublished work of Gillet and Soulé [G-S], which uses the sheaf-theoretic point of view. Together with the paper [J] of Jardine, it seems that the arguments of [G-S] would give a special λ-ring structure for the K-groups of an arbitrary small diagram of schemes satisfying a certain cohomological finiteness (see

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e.g. [So2]). Grayson ([Gr], [Gr2]) has constructed λ-operations and Adams operations on the simplicial level, but it is not clear that these operations satisfy special λ-ring identities. Our justification for producing this work is that there is currently nothing in the literature which constructs λ-operations, satisfying the special λ-ring identities, in the generality required for relative K-theory with supports.

The argument given here consists of reworking the basic ingredients of the construction of Quillen in the setting of a diagram of rings. These ingredients are:

i) the special λ-ring structure on the representation ring of a group

ii) the universal property of the plus construction, and the structure of an H-space on BGL^+

iii) the construction of a "universal" natural transformation from the representation ring of $\pi_1(X)$ to the set of homotopy classes of maps $[X, \mathrm{BGL}^+]$,

The representation theory is quite straightforward and is discussed in §1. The generalization of (ii) is done in §2, and §3, and requires the machinery of closed simplicial model categories. For (iii), we require the finiteness of the parameter category; this is discussed in §4. We give the applications to the construction of the special λ-ring structure for relative K-theory with supports in §5. For the reader's convenience, we have included two appendices, the first recalling the basic notions from the theory of closed simplicial model categories, and the second reviewing basic properties of homotopy limits, and relations with Hom-complexes in the model category of functors to simplicial sets.

The application in the second part is a refinement of our earlier rational computation of motivic cohomology in [L]: for X a smooth quasi-projective variety over a field k, there is a natural isomorphism of the motivic cohomology, defined as Bloch's higher Chow groups [Bl], with the weight-graded pieces of K-theory, after tensoring with $\mathbb{Q}$

$$\mathrm{CH}^q(X,p)_{\mathbb{Q}} \cong K_p(X)^{(q)}_Q. \tag{1}$$

As the γ-filtration on the K-group $K_p(X)$ splits into eigenspaces for the Adams operations ψ^k after inverting $(d+p-1)!$ (at least for $p \geq 1$; for $p = 0$ one needs to restrict to the subgroup $F^2_\gamma K_0(X)$), where d is the dimension of X over k, it is natural to ask if there is an isomorphism as in (1), after inverting only $(d+p-1)!$. This turns out to be the case: there are natural isomorphisms for all p and q (see corollary 8.2)

$$\mathrm{CH}^q(X,p)[\frac{1}{(d+p-1)!}] \cong \mathrm{gr}^q_\gamma K_p(X)[\frac{1}{(d+p-1)!}]. \tag{2}$$

There is in fact a somewhat finer result; we refer the reader to theorem 8.1 for details. Soulé [So2] has proved a similar result for $X = \mathrm{Spec}(F)$, where F is a field of characteristic zero, using the "Atiyah-Hirzebruch" spectral sequence of Bloch and Lichtenbaum [B-L].

The main idea of the proof is to note that, although the γ-filtration does not in general behave well with respect to exact sequences, a decomposition into characteristic subspaces for the Adams operations does. This remark is systematized in §6. In §7, we give the applications of the Adams decomposition to the case of relative K-theory with supports, and we prove the main result in §8.

The construction of λ-operations given here is a somewhat refined version of an earlier construction I gave in a course on algebraic K-theory and motivic cohomology at MIT in the winter of 1993. I would like to thank all the participants in that course for their comments and suggestions, especially Thomas Geisser and Lorenzo Ramero; I would like to thank Thomas Geisser doubly for preparing an excellent set of notes. The systematic use of techniques of closed model categories that appears here is largely due to a comment of Rick Jardine, who pointed out to me the use of closed model categories in solving coherence problems. Finally, I would like to thank the University of Essen, especially Hélène Esnault, Eckart Viehweg, and the DFG Forschergruppe "Arithmetik und Geometrie", for their hospitality and support.

Part I: Lambda-operations

§1. Representation theory over a category

For a small category I, and a category $\mathcal{C}$, we let $\mathcal{C}^I$ denote the category of functors $F: I \to \mathcal{C}$. Let **Rings** be the category of commutative rings (with unit); for a commutative ring S, we let $\mathbf{Rings}_S$ be the subcategory of **Rings** consisting of S-algebras and S-algebra maps.

Let I be a small category, and let

$$A: I \to \mathbf{Rings}_S; \qquad G: I \to \mathbf{Groups}$$

be functors. We have the category $\mathbf{Mod}_A$ of A-modules, defined as the category of pairs (M, μ), with M in $\mathbf{Ab}^I$, and $\mu: A \times M \to M$ a map in $\mathbf{Ab}^I$ such that $\mu(i)$ makes $M(i)$ into an $A(i)$-module for each $i \in I$. Maps of A-modules are defined similarly. We have the full subcategory $\mathcal{P}_{A/S}$ of $\mathbf{Mod}_A$ with objects M isomorphic to an A-module of the form $P \otimes_S A$:

$$i \mapsto P \otimes_S A(i)$$

for some finitely generated projective S-module P.

If M is in $\mathbf{Mod}_A$, a *representation* ρ *of* G *on* M is a collection of homomorphisms

$$\rho(i): G(i) \to \mathrm{Aut}_{A(i)}(M(i)); \qquad i \in I,$$

such that, for each morphism $s: i \to j$ in I, we have

$$M(s)(\rho(i)(g)(m)) = \rho(j)(G(s)(g))(M(s)(m))$$

for each $m \in M(i)$ and $g \in G(i)$. We let $\mathrm{Rep}_S(G; A)$ denote the category of pairs (M, ρ), with M in $\mathcal{P}_{A/S}$, and ρ a representation of G on M, with the obvious notion of morphism. If we choose an isomorphism of M with $P \otimes_S A$, then a representation ρ of G on M is the same as a homomorphism

$$\rho: G \to \mathrm{Aut}_A(P \otimes_S A)$$

where $\mathrm{Aut}_A(P \otimes_S A)$ is the group over I given by

$$i \mapsto \mathrm{Aut}_{A(i)}(P \otimes_S A(i)).$$

We make $\mathrm{Rep}_S(G; A)$ into an exact category by taking the sequences in $\mathrm{Rep}_S(G; A)$

$$0 \to (M', \rho') \to (M, \rho) \to (M'', \rho'') \to 0$$

such that

$$0 \to M' \to M \to M'' \to 0$$

is *split* exact in $\mathbf{Mod}_A$. We let $R_S(G; A)$ denote the Grothendieck group of the exact category $\mathrm{Rep}_S(G; A)$, and define $K_0(A/S)$ by

$$K_0(A/S) := R_S(\{\mathrm{id}\}; A).$$

The inclusion $\mathrm{id} \to G$ gives the augmentation

$$\epsilon_{G,A}: R_S(G; A) \to K_0(A/S). \tag{1.1}$$

Sending a finitely generated projective S-module P to the A-module $P \otimes_S A$ determines the surjection

$$K_0(S) \to K_0(A/S). \tag{1.2}$$

Taking the point-wise tensor product of representations gives $R_S(G; A)$ the natural structure of a commutative ring; point-wise exterior product gives the natural operations

$$\lambda^k: R_S(G; A) \to R_S(G; A) \tag{1.3}$$

We recall from [A-T], [G] or [B] the notions of a λ-ring and a special λ-ring.

Proposition 1.1. *The ring $R_S(G; A)$ with the operations (1.3) is a special λ-ring; the augmentation (1.1) is a map of λ-rings.*

Proof. The second assertion follows directly from the first, and the naturality of the operations (1.3).

For a finitely generated projective S-module P, let $Aut_{P/S}$ denote the group scheme over S representing the functor

$$R \mapsto \mathrm{Aut}_R(P \otimes_S R)$$

For $P = S^n$, we write $\mathcal{GL}_{n/S}$ for $Aut_{P/S}$. Let $\mathcal{G}$ be a group scheme over S. We have the exact category $\mathrm{Rep}_S(\mathcal{G})$ with objects consisting of pairs (P, ρ), P a finitely generated projective S-module, and

$$\rho: \mathcal{G} \to Aut_{P/S} \tag{1}$$

a homomorphism of group schemes over S.

Let $R_S(\mathcal{G})$ denote the Grothendieck group of $\mathrm{Rep}_S(\mathcal{G})$. The operation of tensor product of representations gives $R_S(\mathcal{G})$ the structure of a ring, and the operation sending ρ to $\Lambda^k\rho$ extends to give $R_S(\mathcal{G})$ the structure of a λ-ring. Using the fact that the K_0-class of a representation of $\mathcal{GL}_{n/S}$ on a free S-module is determined by its character (see e.g. [S]), it is easy to show that $R_S(\mathcal{GL}_{n/S})$ is a special λ-ring. This easily implies that $R_S(\mathcal{G})$ is a special λ-ring for $\mathcal{G}$ of the form

$$\mathcal{G} = \prod_{i=1}^{N} \mathcal{GL}_{n_i/S},$$

as

$$R_S(\prod_{i=1}^{N} \mathcal{GL}_{n_i/S}) \cong \bigotimes_{i=1}^{N} R_S(\mathcal{GL}_{n_i/S}).$$

As an arbitrary homomorphism (1) is stably equivalent to a homomorphism with $P = S^n$ for some n, this implies that $R_S(\mathcal{G})$ has the natural structure of a special λ-ring for all algebraic group schemes over S.

If we now have a finite collection of representations ρ_i of some $G \in \mathbf{Groups}^I$ on A-modules $P_i \otimes A$, these representations factor through the identity representation of

$$\prod_i \mathrm{Aut}_S(P_i \otimes A)$$

As this latter object of $\mathbf{Groups}^I$ is the functor gotten by composing A with

$$\mathcal{G}(-) := \prod_i Aut_{P_i/S}(-),$$

the special λ-ring identities in $R_S(\mathcal{G})$ imply the special λ-ring identities in the sub-λ-ring of $R_S(G; A)$ generated by the ρ_i. As $R_S(G; A)$ is a direct limit of its finitely generated sub-λ-rings, it follows that $R_S(G; A)$ is a special λ-ring. $\square$

§2. The classifying space, and the classifying map

For a group G, we have the simplicial set $\mathrm{E}G_*$ with free G-action, and the simplicial set $\mathrm{B}G$. The geometric realization $|\mathrm{B}G|$ is a functorial model for the classifying space of G, i.e., $|\mathrm{B}G|$ is pointed and connected, $\pi_1(|\mathrm{B}G|, *) = G$ and $|\mathrm{B}G|$ has no higher homotopy groups. In particular, if $(X, *)$ is a pointed simplicial set with $|X|$ connected, sending a map $f: (X, *) \to (\mathrm{B}G, *)$ to the induced map on π_1 gives an isomorphism

$$\phi_X: [(X, *), (\mathrm{B}G, *)] \to \mathrm{Hom}_{\mathbf{Groups}}(\pi_1(|X|, *), G).$$

(here $[-,-]_*$ is the set of pointed homotopy classes of pointed maps). Indeed, since $\mathrm{B}G$ is a simplicial group, $\mathrm{B}G$ is fibrant, hence

$$[(X, *), (\mathrm{B}G, *)] \cong [(|X|, *), (|\mathrm{B}G|, *)],$$

and the result follows from classical obstruction theory.

Now let I be a small category, and $G: I \to \mathbf{Groups}$ a functor. We let $\tilde{\mathrm{B}}G: I \to \mathcal{S}$ denote a bifibrant model of the functor

$$i \mapsto \mathrm{B}G(i).$$

Proposition 2.1. *Let $(X, *)$ be a cofibrant object of $\mathcal{S}^{*I}$, and let $G: I \to$ **Groups** be a functor. Suppose $|X(i)|$ is connected for each $i \in I$. Then sending a map $f: (X, *) \to (\tilde{\mathrm{B}}G, *)$ to the induced map on π_1 defines an isomorphism*

$$\phi_{X,G}: [(X, *), (\tilde{\mathrm{B}}G, *)] \to \mathrm{Hom}_{\mathbf{Groups}^I}(\pi_1(X, *), G).$$

Proof. For each i and j in I, the space

$$\mathcal{H}om_{\mathcal{S}^*}((X(i), *), (\tilde{\mathrm{B}}G(j), *))$$

has

$$\pi_n = \begin{cases} \mathrm{Hom}_{\mathbf{Groups}}(\pi_1(X(i), *), G(j)) & \text{for } n = 0 \\ 0 & \text{for } n \geq 1. \end{cases}$$

Thus, sending $\mathcal{H}om_{\mathcal{S}^*}((X(i), *), (\tilde{\mathrm{B}}G(j), *))$ to the constant simplicial set

$$\pi_0 \mathcal{H}om_{\mathcal{S}^*}((X(i), *), (\tilde{\mathrm{B}}G(j), *))$$

defines a weak equivalence in $\mathcal{S}^{*aI}$ (see Appendix B, following Theorem B.1, for the definition of the *twisted arrow category* aI):

$$\Pi_0: \mathcal{H}om((X, *), (\tilde{\mathrm{B}}G, *)) \to \pi_0 \mathcal{H}om((X, *), (\tilde{\mathrm{B}}G, *))$$

Since $(X,*)$ is cofibrant and $(\tilde{\mathrm{B}}G,*)$ is fibrant, this is a trivial fibration of fibrant objects in $\mathcal{S}^{*aI}$. Thus, by ([B-K], XI, §5, lemma 5.6) Π_0 induces a weak equivalence

$$\operatorname*{holim}_{aI} \mathcal{H}om((X,*),(\tilde{\mathrm{B}}G,*)) \to \operatorname*{holim}_{aI} \pi_0\mathcal{H}om((X,*),(\tilde{\mathrm{B}}G,*)). \tag{1}$$

It is easy to see that the natural map

$$\begin{aligned}\mathrm{Hom}_{\mathbf{Groups}^I}(\pi_1(X,*),G) &= \lim_{\substack{\leftarrow \\ aI}} \pi_0\mathcal{H}om((X,*),(\tilde{\mathrm{B}}G,*)) \\ &\to \pi_0(\operatorname*{holim}_{aI} \pi_0\mathcal{H}om((X,*),(\tilde{\mathrm{B}}G,*)))\end{aligned}$$

is an isomorphism; by theorem B.2, the weak equivalence (1) induces the isomorphism

$$[(X,*),(\tilde{\mathrm{B}}G,*)] \cong \pi_0 \operatorname*{holim}_{aI} \mathcal{H}om((X,*),(\tilde{\mathrm{B}}G,*)) \cong \mathrm{Hom}_{\mathbf{Groups}^I}(\pi_1(X,*),G).$$

$\square$

Let G and H be in $\mathbf{Groups}^I$. Taking $X = \tilde{\mathrm{B}}H$, proposition 2.1 gives the isomorphism

$$\Phi_{H,G}\colon \mathrm{Hom}_{\mathbf{Groups}^I}(H,G) \to [(\tilde{\mathrm{B}}H,*),(\tilde{\mathrm{B}}G,*)] \tag{2.1}$$

In particular, if $g\colon I \to G$ is a section, the automorphism of G given by conjugation by g determines the pointed map

$$C_g\colon \tilde{\mathrm{B}}G \to \tilde{\mathrm{B}}G, \tag{2.2}$$

uniquely up to pointed homotopy.

For a cofibrant $(Y,*)$ in $(\mathcal{S}^*)^I$, let $\pi_1(Y,*)$ denote the functor

$$i \mapsto \pi_1(|Y(i)|,*).$$

and let

$$q_Y\colon (Y,*) \to \tilde{\mathrm{B}}\pi_1(Y,*) \tag{2.3}$$

be the map $\phi_{Y,\pi_1(Y,*)}(\mathrm{id}_{\pi_1(Y,*)})$.

§3. BGL and BGL$^+$

For $R \in$ **Rings**, n a positive integer, we have the functorial map of the permutation group S_n into $\mathrm{GL}_n(R)$ as the group of permutation matrices; this gives the functorial map $A_5 \to \mathrm{GL}_n(R)$. We let

$$\mathrm{B}A_5 \to \mathrm{B}A_5^+$$

be a cofibration of simplicial sets whose geometric realization is Quillen's $+$-construction applied to the classifying space $|\mathrm{B}A_5|$. Define the simplicial set $\mathrm{BGL}_n(R)^+$, $n \geq 5$, by

$$\mathrm{BGL}_n(R)^+ = \mathrm{BGL}_n(R) \cup_{\mathrm{B}A_5} \mathrm{B}A_5^+. \tag{3.1}$$

This gives the natural transformation of functors

$$\mathrm{BGL}_n(-) \to \mathrm{BGL}_n(-)^+$$

from **Rings** to $\mathcal{S}^*$.

It is shown in [H] that $|\mathrm{BGL}_n(R)^+|$ is a model for the $+$-construction on $|\mathrm{BGL}_n(R)|$; i.e., the cofibration $i\colon |\mathrm{BGL}_n(R)| \to |\mathrm{BGL}_n(R)^+|$ satisfies

i) $i_*\colon \pi_1(|\mathrm{BGL}_n(R)|, *) = \mathrm{GL}_n(R) \to \pi_1(|\mathrm{BGL}_n(R)^+|, *)$ is surjective, with kernel $\mathrm{E}_n(R)$

ii) i is a homology isomorphism for all local systems on $|\mathrm{BGL}_n(R)^+|$.

Let I be a small category, and $A\colon I \to$ **Rings** a functor. We may apply the functors $\mathrm{BGL}_n(-)$ and $\mathrm{BGL}_n(-)^+$, giving the objects $\mathrm{BGL}_n(A)$ and $\mathrm{BGL}_n(A)^+$ of $\mathcal{S}^{*I}$, and the map

$$i_n\colon \mathrm{BGL}_n(A) \to \mathrm{BGL}_n(A)^+ \tag{3.2}$$

in $\mathcal{S}^{*I}$. We have as well the stabilization maps

$$\begin{aligned} \iota_{n,n+1}\colon \mathrm{BGL}_n(A) &\to \mathrm{BGL}_{n+1}(A) \\ \iota^+_{n,n+1}\colon \mathrm{BGL}_n(A)^+ &\to \mathrm{BGL}_{n+1}(A)^+ \end{aligned} \tag{3.3}$$

induced by the stabilization maps $\mathrm{GL}_n \to \mathrm{GL}_{n+1}$. Passing to the direct limit over n gives the functors

$$\begin{aligned} \mathrm{BGL}(A)\colon I &\to \mathcal{S}^* \\ \mathrm{BGL}(A)^+\colon I &\to \mathcal{S}^*. \end{aligned} \tag{3.4}$$

and the maps

$$\begin{aligned} \iota_n\colon \mathrm{BGL}_n(A) &\to \mathrm{BGL}(A) \\ \iota_n^+\colon \mathrm{BGL}_n(A)^+ &\to \mathrm{BGL}(A)^+ \\ i\colon \mathrm{BGL}(A) &\to \mathrm{BGL}(A)^+. \end{aligned} \tag{3.5}$$

We sometimes denote $\mathrm{BGL}(A)$ and $\mathrm{BGL}(A)^+$ by $\mathrm{BGL}_\infty(A)$ and $\mathrm{BGL}_\infty(A)^+$.

If we apply the functor $\mathrm{Sin}|-|$ (singular complex of the geometric realization), we have the canonical weak equivalences with fibrant objects

$$\mathrm{BGL}_n(A) \to \mathrm{Sin}|\mathrm{BGL}_n(A)|$$
$$\mathrm{BGL}_n(A)^+ \to \mathrm{Sin}|\mathrm{BGL}_n(A)^+|$$

as well as for $n = \infty$.

We may find commutative diagrams in $\mathcal{S}^I$:

$$\begin{array}{ccc} \mathrm{B\tilde{G}L}_n(A) & \xrightarrow{\tilde{i}_n} & \mathrm{B\tilde{G}L}_n(A)^+ \\ \downarrow & & \downarrow \\ \mathrm{Sin}|\mathrm{BGL}_n(A)| & \xrightarrow{\mathrm{Sin}|i_n|} & \mathrm{Sin}|\mathrm{BGL}_n(A)^+| \end{array}$$

$n = 5, 6, \ldots$, and $n = \infty$, with the vertical arrows trivial fibrations, and $\tilde{i}_n$ a cofibration of bifibrant objects of $\mathcal{S}^I$. We may assume that there are stabilization maps

$$\tilde{\iota}_{n,n+1} \colon \mathrm{B\tilde{G}L}_n(A) \to \mathrm{B\tilde{G}L}_{n+1}(A)$$
$$\tilde{\iota}^+_{n,n+1} \colon \mathrm{B\tilde{G}L}_n(A)^+ \to \mathrm{B\tilde{G}L}_{n+1}(A)^+ \tag{3.6}$$

extending the stabilization maps $\mathrm{Sin}|\iota_{n,n+1}|$ and $\mathrm{Sin}|\iota^+_{n,n+1}|$, that the maps (3.6) are cofibrations, and that there are cofibrations

$$\tilde{\iota}_n \colon \mathrm{B\tilde{G}L}_n(A) \to \mathrm{B\tilde{G}L}(A)$$
$$\tilde{\iota}^+_n \colon \mathrm{B\tilde{G}L}_n(A)^+ \to \mathrm{B\tilde{G}L}(A)^+ \tag{3.7}$$

extending the maps $\mathrm{Sin}|\iota_n|$ and $\mathrm{Sin}|\iota^+_n|$. In addition, we may assume that this fits together so that all the obvious diagrams commute: the maps $\tilde{i}_*$, $\tilde{\iota}_{*,*+1}$, $\tilde{\iota}_*$ $\tilde{\iota}^+_{*,*+1}$ and $\tilde{\iota}^+_*$ all commute, $* = 5, 6, \ldots, \infty$, and this data commutes with the corresponding data for $\mathrm{Sin}|\mathrm{BGL}_n(A)|$ and $\mathrm{Sin}|\mathrm{BGL}_n(A)^+|$. The construction of this data satisfying the required commutativities is an easy exercise using elementary properties of closed model categories (see Appendix A, axioms CM1-CM5, and properties following the axioms).

Lemma 3.1. *i) Let $(Z, *)$ be a fibrant object of $\mathcal{S}^{*I}$, such that $\pi_1(|Z(i)|, *)$ has no nontrivial perfect subgroup for each $i \in I$. Let*

$$f \colon X \to X^+$$

be a pointed map of cofibrant objects of $\mathcal{S}^{*I}$, such that, for each $i \in I$,

a) $|X(i)|$ and $|X^+(i)|$ are connected

b) the map $f(i)$ is acyclic ($f(i)$ induces an isomorphism in homology for all local systems on $|X(i)^+|$).

Then the map

$$f^*\colon \mathcal{H}om_{\mathcal{S}^{*I}}(X^+, Z) \to \mathcal{H}om_{\mathcal{S}^{*I}}(X, Z)$$

is a weak equivalence. In particular, for each $n \geq 5$, including $n = \infty$, the map

$$\tilde{i}_n\colon \mathrm{B}\tilde{\mathrm{G}}\mathrm{L}_n(A) \to \mathrm{B}\tilde{\mathrm{G}}\mathrm{L}_n(A)^+$$

induces an isomorphism

$$\tilde{i}_n^*\colon [\mathrm{B}\tilde{\mathrm{G}}\mathrm{L}_n(A)^+, Z]_* \to [\mathrm{B}\tilde{\mathrm{G}}\mathrm{L}_n(A), Z]_*.$$

Proof. It follows from classical obstruction theory that the map

$$f^*(i)\colon \mathcal{H}om_{\mathcal{S}^*}((X^+(i), *), (Z(j), *)) \to \mathcal{H}om_{\mathcal{S}^*}((X(i), *), (Z(j), *))$$

is a weak equivalence for each $i, j \in I$. The result them follows from corollary B.4. $\square$

Let $g \in \Gamma(\mathcal{C}, \mathrm{GL}_n(A))$ be a section. By lemma 3.1, the map (2.2)

$$C_g\colon \mathrm{B}\tilde{\mathrm{G}}\mathrm{L}_n(A) \to \mathrm{B}\tilde{\mathrm{G}}\mathrm{L}_n(A)$$

extends to the pointed map

$$C_g^+\colon \mathrm{B}\tilde{\mathrm{G}}\mathrm{L}_n(A)^+ \to \mathrm{B}\tilde{\mathrm{G}}\mathrm{L}_n(A)^+$$

uniquely up to (pointed) homotopy.

Lemma 3.2. *Let*

$$\tilde{\iota}_{n,2n}^+\colon \mathrm{B}\tilde{\mathrm{G}}\mathrm{L}_n(A)^+ \to \mathrm{B}\tilde{\mathrm{G}}\mathrm{L}_{2n}(A)^+$$

be the stabilization map. Then $\tilde{\iota}_{n,2n}^+$ and $\tilde{\iota}_{n,2n}^+ \circ C_g^+$ induce homotopic maps

$$\mathrm{B}\tilde{\mathrm{G}}\mathrm{L}_n(A)^+ \to \mathrm{B}\tilde{\mathrm{G}}\mathrm{L}_{2n}(A)^+,$$

for $n \geq 5$.

Proof. By the usual trick of writing the matrix

$$\begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix}$$

as an element in $\mathrm{E}_{2n}(R)$:

$$\begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix} = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -m^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

we need only show that C_g^+ is homotopically trivial for each section g of the form $g = e_{ij}^s$, where s is a section of A over I.

Let $(X, *)$ be a pointed simplicial set. We have the canonical homomorphism

$$\nu_{|X|}\colon \pi_1(|X|, *) \to [(|X|, *), (|X|, *)]$$

giving the action of $\pi_1(|X|, *)$ on X. Applying Sin, we have the map

$$\nu_X\colon \pi(|X|, *) \to [(\mathrm{Sin}|X|, *), (\mathrm{Sin}|X|, *)]$$

For $(X, *)$ in $\mathcal{S}^{*I}$, define $\pi_1^I(|X|, *)$ as the homotopy classes of pointed maps in $\mathbf{Top}^I$

$$\sigma\colon (S^1, *) \times I \to (|X|, *),$$

where a homotopy between two pointed maps $f, g\colon (Y, *) \to (Z, *)$ is given in the obvious way by a map

$$h\colon (Y \times [0,1], * \times [0,1]) \to (Z, *)$$

It is then easy to see that there is an action of $\pi_1^I(|X|, *)$ on $|X|$, given as a homomorphism

$$\nu_{|X|}\colon \pi_1^I(|X|, *) \to [(|X|, *), (|X|, *)].$$

The main point to note is that, taking the barycentric subdivision of each $X(i)$, the resulting closed star neighborhood of $*$ in each $|X(i)|$ gives a functorial system of closed neighborhoods of $*$, functorially contractible to $*$. Taking the singular complex gives the homomorphism

$$\nu_X\colon \pi_1^I(|X|, *) \to [(\widetilde{\mathrm{Sin}|X|}, *), (\widetilde{\mathrm{Sin}|X|}, *)]$$

where $\tilde{}$ denotes bifibrant model. When evaluated at a point $i \in I$, these give the maps $\nu_{|X(i)|}$ and $\nu_{X(i)}$, respectively.

If we apply this to the case $X = \mathrm{B}G$, and $\sigma := \sigma_g \in \pi_1$ is given by a section g of G over I, it follows from proposition 2.1 that $\nu_X(\sigma_g)$ is homotopic to the map

$$\widetilde{\mathrm{Sin}|\mathrm{B}G|} \to \widetilde{\mathrm{Sin}|\mathrm{B}G|}$$

given by conjugation by g. Thus, to show that $C_{e_{ij}^s}$ is homotopically trivial, it suffices to show that the map

$$|i_n| \circ \sigma_{e_{ij}^s}\colon (S^1 \times I, * \times I) \to (|\mathrm{BGL}_n(A)^+|, *)$$

is homotopically trivial. If we take A to be the constant ring $\mathbb{Z}[t]$, and s the constant section $s(i) = t$, the homotopy of $\sigma_{e_{ij}^t}$ to $*$ in $|\mathrm{BGL}_n(\mathbb{Z}[t])^+|$ gives the desired homotopy over I; in general, a choice of a section s of A over I defines the map

$$\tilde{s}\colon \mathbb{Z}[t] \times I \to A$$
$$\tilde{s}(i)(t) = s(i)$$

and the homotopy for e^s_{ij} is gotten by applying $\tilde{s}$ to the homotopy for e^t_{ij}. $\square$

For a ring R, the shuffle sum of matricies:

$$((g) \oplus (h))_{ij} = \begin{cases} g_{i'j'} & \text{if } i = 2i' - 1,\ j = 2j' - 1, \\ h_{i'j'} & \text{if } i = 2i',\ j = 2j', \\ 0 & \text{otherwise} \end{cases}$$

defines the map

$$\oplus \colon \mathrm{BGL}(R) \times \mathrm{BGL}(R) \to \mathrm{BGL}(R)$$

and extends, uniquely up to homotopy, to the map

$$\oplus \colon \mathrm{BGL}(R)^+ \times \mathrm{BGL}(R)^+ \to \mathrm{BGL}(R)^+$$

This gives $\mathrm{BGL}(R)^+$ the structure of a commutative H-group, i.e., an H-space which is homotopy associative and homotopy commutative, and with homotopy inverse.

Let $\mathcal{A}$ be a closed model category. An *H-space in $\mathcal{A}$*, (X, e, μ), is a monoid object in the homotopy category $\mathrm{Ho}\mathcal{A}$ (see Appendix A); an *H-group in $\mathcal{A}$* is an H-space in $\mathcal{A}$ which is a group object in $\mathrm{Ho}\mathcal{A}$.

Theorem 3.3. *The point-wise shuffle sum gives the bifibrant model $\mathrm{B\tilde{G}L}(A)^+$ the structure of a commutative H-group in $\mathcal{S}^I$.*

Proof. An injection

$$\alpha \colon \mathbb{N} \to \mathbb{N}$$

induces the map

$$\tilde{\mathrm{B}}\alpha^+ \colon \mathrm{B\tilde{G}L}(A)^+ \to \mathrm{B\tilde{G}L}(A)^+$$

by rearranging the matrix entries. To show that that $(\mathrm{B\tilde{G}L}(A)^+, \oplus, *)$ is a commutative H-space, it suffices to show that each $\tilde{\mathrm{B}}\alpha^+$ is homotopically trivial.

It is well-known that, for each i, the map

$$\tilde{\mathrm{B}}\alpha^+(i) \colon \mathrm{B\tilde{G}L}(A(i))^+ \to \mathrm{B\tilde{G}L}(A(i))^+$$

is homotopically trivial; in particular, a weak equivalence. Thus each $\tilde{\mathrm{B}}\alpha^+$ is a weak equivalence. Since $\mathrm{B\tilde{G}L}(A)^+$ is bifibrant, each $\tilde{\mathrm{B}}\alpha^+$ is a homotopy equivalence. Sending α to $\tilde{\mathrm{B}}\alpha^+$ defines a homomorphism of the monoid $\mathcal{M}$ of injective self-maps of $\mathbb{N}$ (under composition) to the monoid of homotopy classes of self-maps of $\mathrm{B\tilde{G}L}(A)^+$, which thus factors through the group completion of $\mathcal{M}$. As this group completion is the trivial group, all the $\tilde{\mathrm{B}}\alpha^+$ are homotopic to the identity. We now show that $\mathrm{B\tilde{G}L}(A)^+$ is an H-group. It suffices to prove

Claim. *Let (X, e, μ) be an H-space in $\mathcal{S}^I$, with X bifibrant. Suppose that $|X(i)|$ is connected for each $i \in I$. Then there is a map*

$$\iota \colon X \to X$$

which is a homotopy inverse for μ.

Proof of claim. $X \times X$ is fibrant. Let

$$p: X \times X \to X \times X$$

be the map

$$p = (\mu, p_2).,$$

Applying the classical result for connected H-spaces (in **Top**), the map

$$|p(i)|: |X(i)| \times |X(i)| \to |X(i)| \times |X(i)|$$

is a weak equivalence for each $i \in I$, hence p is a weak equivalence in $\mathcal{S}^I$.

Since X is cofibrant and $X \times X$ is fibrant there is an up-to-homotopy lifting

$$\phi: X \to X \times X$$

in the diagram

$$\begin{array}{ccc} & & X \times X \\ & & \downarrow p \\ X = * \times X & \overset{e \times \mathrm{id}_X}{\to} & X \times X \end{array}$$

Then $\pi \circ \phi$ is homotopic to (ι, id_X), for some map $\iota: X \to X$, and ι is thus a left homotopy inverse for μ. As μ is homotopy associative, ι is the desired homotopy inverse. □

For a ring R, let $\mathrm{GL}_{n,m}(R)$ be the subgroup of $\mathrm{GL}_{n+m}(R)$ consisting of matrices of the form

$$\begin{pmatrix} g_n & a_{n,m} \\ 0 & g_m \end{pmatrix}$$

with $g_n \in \mathrm{GL}_n(R)$, $g_m \in \mathrm{GL}_m(R)$, and $a_{n,m}$ an $n \times m$ matrix. The shuffle sum

$$\mathrm{GL}_n(R) \times \mathrm{GL}_m(R) \to \mathrm{GL}(R)$$

extends in the obvious way to the inclusion

$$s_{n,m}: \mathrm{GL}_{n,m}(R) \to \mathrm{GL}(R).$$

Classifying and taking $+$ gives us the map

$$Bs^+_{n,m}: \mathrm{BGL}_{n,m}(R) \to \mathrm{BGL}(R)^+.$$

Similarly, the stabilization maps

$$\mathrm{GL}_n(R) \times \mathrm{GL}_m(R) \to \mathrm{GL}_{n+1}(R) \times \mathrm{GL}_{m+1}(R)$$

extends to

$$\mathrm{BGL}_{n,m}(R) \to \mathrm{BGL}_{n+1,m+1}(R);$$

let $\mathrm{GL}_{*,*}(R)$ the direct limit of the $\mathrm{GL}_{n,m}(R)$. We let $j_{n,m}\colon \mathrm{GL}_n(R) \times \mathrm{GL}_m(R) \to \mathrm{GL}_{n,m}(R)$ denote the inclusion,

$$r_{n,m}\colon \mathrm{GL}_{n,m}(R) \to \mathrm{GL}_n(R) \times \mathrm{GL}_m(R)$$
$$\begin{pmatrix} g_n & a_{n,m} \\ 0 & g_m \end{pmatrix} \mapsto (g_n, g_m)$$

the projection.

If $A\colon I \to \mathbf{Rings}$ is a functor, the above constructions generalize to define the groups $\mathrm{GL}_{n,m}(A)$, maps

$$s_{n,m}\colon \mathrm{GL}_{n,m}(A) \to \mathrm{GL}(A),$$

etc. We may take as in (3.3)-(3.7) a system of bifibrant models $\mathrm{B\tilde{G}L}_{n,m}(A)$ of $\mathrm{BGL}_{n,m}(A)$, together with a compatible system of cofibrations ($n' \geq n$, $m' \geq m$)

$$\mathrm{B\tilde{G}L}_{n,m}(A) \to \mathrm{B\tilde{G}L}_{n',m'}(A) \to \ldots \to \mathrm{B\tilde{G}L}_{*,*}(A)$$

$$\mathrm{BGL}_n(A) \tilde{\times} \mathrm{BGL}_m(A) \to \mathrm{BGL}_{n'}(A) \tilde{\times} \mathrm{BGL}_{m'}(A) \to \ldots$$

and compatible maps

$$\mathrm{BGL}_n(A) \tilde{\times} \mathrm{BGL}_m(A) \xrightarrow{\tilde{\mathrm{B}}j_{n,m}} \mathrm{B\tilde{G}L}_{n,m}(A)$$

$$\mathrm{BGL}(A) \tilde{\times} \mathrm{BGL}(A) \xrightarrow{\tilde{\mathrm{B}}j_{*,*}} \mathrm{B\tilde{G}L}_{*,*}(A)$$

and

$$\mathrm{B\tilde{G}L}_{n,m}(A) \xrightarrow{\tilde{\mathrm{B}}r_{n,m}} \mathrm{BGL}_n(A) \tilde{\times} \mathrm{BGL}_m(A);$$

as well as a commutative diagram

$$\begin{array}{ccc} \mathrm{B\tilde{G}L}_{n,m}(A) & \xrightarrow{\tilde{\mathrm{B}}s^+_{n,m}} & \mathrm{B\tilde{G}L}(A)^+ \\ \downarrow & & \| \\ \mathrm{B\tilde{G}L}_{*,*}(A) & \xrightarrow{\tilde{\mathrm{B}}s^+_{*,*}} & \mathrm{B\tilde{G}L}(A)^+ \end{array}$$

Lemma 3.4. *The maps*

$$\tilde{\mathrm{B}}s^+_{n,m}\colon \mathrm{B\tilde{G}L}_{n,m}(A) \to \mathrm{B\tilde{G}L}(A)^+$$

and

$$\tilde{\mathrm{B}}s^+_{n,m} \circ \tilde{\mathrm{B}}j_{n,m} \circ \tilde{\mathrm{B}}r_{n,m}\colon \mathrm{B\tilde{G}L}_{n,m}(A) \to \mathrm{B\tilde{G}L}(A)^+$$

are homotopic for all m, n.

Proof. From Quillen [Q4], we know that the map

$$\mathrm{B}j_{*,*}\colon \mathrm{BGL}(R) \times \mathrm{BGL}(R) \to \mathrm{BGL}_{*,*}(R)$$

induces a weak equivalence

$$\mathrm{B}j^*_{*,*}\colon \mathcal{H}om_{\mathcal{S}^*}(\mathrm{BGL}_{*,*}(R), \mathrm{B\tilde{G}L}(R')^+) \to \mathcal{H}om_{\mathcal{S}^*}(\mathrm{BGL}(R) \times \mathrm{BGL}(R), \mathrm{B\tilde{G}L}(R')^+)$$

for all rings R and R', where $\mathrm{B\tilde{G}L}(R')^+$ is a fibrant model. We may apply corollary B.4, which implies that

$$\tilde{\mathrm{B}}j^*_{*,*}\colon \mathcal{H}om_{\mathcal{S}^{*I}}(\mathrm{B\tilde{G}L}_{*,*}(A), \mathrm{B\tilde{G}L}(A)^+) \to \mathcal{H}om_{\mathcal{S}^{*I}}(\mathrm{BGL}(A) \tilde{\times} \mathrm{BGL}(A), \mathrm{B\tilde{G}L}(A)^+)$$

is a weak equivalence. As the maps $\tilde{\mathrm{B}}s_{n,m}$ factor through $\mathrm{B\tilde{G}L}_{*,*}(A)$ this, together with the commutative diagrams given above, completes the proof. $\square$

§4. λ-operations

A simplicial set X is called *finite* if X has only finitely many non-degenerate simplices. We have the full subcategories $\mathcal{S}_{\mathrm{fin}}$ and $\mathcal{S}_{\mathrm{c}}$ of $\mathcal{S}$, consisting respectively of the finite and the connected simplicial sets; we let $\mathcal{S}_{\mathrm{cfin}}$ denote the intersection $\mathcal{S}_{\mathrm{fin}} \cap \mathcal{S}_{\mathrm{c}}$ This gives us the functor category $\mathcal{S}_{\mathrm{c}}^I$, which is a full subcategory of $\mathcal{S}^I$.

Call an object X of $\mathcal{S}^I$ *finite* if $X(i)$ is finite for each $i \in I$. We let $\mathcal{S}^I_{\mathrm{fin}}$ denote the full subcategory of $\mathcal{S}^I$ consisting of finite cofibrant objects, and let $\mathcal{S}^I_{\mathrm{bfin}}$ denote the full subcategory of $\mathcal{S}^I$ with objects X such that

i) X is bifibrant

ii) there is a weak equivalence $f\colon X_0 \to X$, with X_0 in $\mathcal{S}^I_{\mathrm{fin}}$.

We note that (ii) implies the map

$$f^*\colon \mathcal{H}om(X, Z) \to \mathcal{H}om(X_0, Z)$$

is a weak equivalence for all fibrant Z. Indeed, factor f as uv with

$$u\colon X_0 \to X'$$

a trivial cofibration, and

$$v\colon X' \to X$$

a trivial fibration. Then X' is also bifibrant, hence v is a homotopy equivalence. Thus, we may assume that f is a trivial cofibration; as Z is fibrant, f^* is a trivial fibration by axiom SM7.

We let $\mathcal{S}^I_c$, $\mathcal{S}^I_{\mathrm{cfin}}$ and $\mathcal{S}^I_{\mathrm{cbfin}}$ be the full subcategories of $\mathcal{S}^I$, $\mathcal{S}^I_{\mathrm{fin}}$ and $\mathcal{S}^I_{\mathrm{bfin}}$ of X such that $|X(i)|$ is connected for each $i \in I$. We have the pointed versions defined similarly.

We now suppose I is a *finite* category, i.e., that the nerve of I is a finite simplicial set. This implies that I is *direct* (in the sense of [D-K]), i.e., that the nerve $\mathcal{N}(I/i)$ of the over category I/i is finite dimensional for each $i \in I$. Concretely, the condition that I is finite is simply that I has finitely many objects, finitely many morphisms, and there is an N such that each sequence of morphisms

$$i_0 \xrightarrow{f_0} i_1 \xrightarrow{f_1} \ldots \xrightarrow{f_N} i_{N+1}$$

contains an identity morphism.

For a direct category J, we let J_n be the full subcategory of objects i such that

$$\dim \mathcal{N}(J/i) \leq n.$$

and J^n the set of objects with

$$\dim \mathcal{N}(J/i) = n.$$

If X is in $\mathcal{S}^J$, and i is in J^n, have the canonical map

$$\lim_{\substack{\rightarrow \\ (J/i)_{n-1}}} X/i \to X(i),$$

where

$$X/i \colon J/i \to \mathcal{S}$$

is the functor

$$X(s \colon j \to i) = X(j).$$

We write the inductive limit as

$$X(J//i) := \lim_{\substack{\rightarrow \\ (J/i)_{n-1}}} X/i.$$

Lemma 4.1. *Each cofibrant object X in $\mathcal{S}^I$ is an inductive limit:*

$$X = \lim_{\substack{\rightarrow \\ \alpha \in A}} Z_\alpha; \qquad Z_\alpha \in \mathcal{S}^I_{\text{fin}},$$

with A a totally ordered set, and

$$\begin{array}{ccc} Z_\alpha & \rightarrow & Z_\beta \\ i_\alpha & \searrow \quad \swarrow & i_\beta \\ & X & \end{array}$$

a commutative diagram of cofibrations in $\mathcal{S}^I$ for $\alpha \leq \beta \in A$. If $(X,*)$ is a cofibrant object of $\mathcal{S}^{*I}$, and each $X(i)$ is connected, then the above holds for $(X,*)$, where we may take the inductive limit in $\mathcal{S}^{*I}_{\text{cfin}}$.

Proof. We proceed by induction on the number of objects of I. If $I = I_0$, then there are no non-identity maps in I, hence the result follows from the analogous statement in

$\mathcal{S}$. In general $I = I_N$ for some minimal N; take some $i \in I^N$, and consider the category $J := I\backslash\{i\}$.

If j is in J, there are no morphisms $i \to j$ in I; if there were, this would put j in I^M for some $M > N$. The restriction functor

$$\mathrm{res}^I_J \colon \mathcal{S}^I \to \mathcal{S}^J$$

has the left adjoint

$$\mathrm{cor}^J_I \colon \mathcal{S}^J \to \mathcal{S}^I$$

defined by

$$\begin{aligned} &\mathrm{res}^I_J \mathrm{cor}^J_I = \mathrm{id} \\ &\mathrm{cor}^J_I(X)(i) = X(I//i). \end{aligned}$$

As res^I_J preserves fibrations and weak equivalences, cor^J_I sends cofibrations to cofibrations. It is easy to see that, if $f \colon B \to C$ is a map in $\mathcal{S}^I$, then f is a cofibration if and only if

i) $\mathrm{res}^I_J f$ is a cofibration in $\mathcal{S}^J$, and

ii) the natural map

$$C(I//i) \cup_{B(I//i)} B(i) \to C(i)$$

is a cofibration in $\mathcal{S}$.

In addition, the natural map

$$f(I//i) \colon B(I//i) \to C(I//i)$$

is a cofibration in $\mathcal{S}$.

If now X is a cofibrant object of $\mathcal{S}^I$, we may write $\mathrm{res}^I_J X$ as the nested inductive limit of cofibrations

$$i_\alpha \colon Z^J_\alpha \to \mathrm{res}^I_J X; \alpha \in A_J$$

with Z^J_α in $\mathcal{S}^J_{\mathrm{fin}}$, which satisfy the various conditions of the lemma. Since I is finite, the objects $\mathrm{cor}^J_I Z^J_\alpha$ are in $\mathcal{S}^I_{\mathrm{fin}}$. Thus $\mathrm{cor}^J_I \mathrm{res}^I_J X$ is the inductive limit of the cofibrations

$$\mathrm{cor}^J_I Z^J_\alpha \to \mathrm{cor}^J_I \mathrm{res}^I_J X.$$

Evaluating at i, this writes the subcomplex $X(I//i)$ of $X(i)$ as the nested inductive limit of subcomplexes

$$X(I//i) = \varinjlim_\alpha Z^J_\alpha(I//i). \tag{1}$$

We may write $X(i)$ as a nested inductive limit of finite subcomplexes

$$X(i) = \varinjlim_\beta Z^i_\beta$$

indexed by a totally ordered set B. For each $\beta \in B$, we may find an $\alpha_\beta \in A_J$ such that the map

$$X(I//i) \cup_{Z^J_\alpha(I//i)} [Z^J_\alpha(I//i) \cup Z^i_\beta] \to X(i)$$

is a cofibration for all $\alpha \geq \alpha_\beta$. Let $A_J B$ be the partially ordered set consisting of (α, β) with $\alpha \geq \alpha_\beta$; for $(\alpha, \beta) \in A_J B$, let $Z_{\alpha\beta}$ be the object of $\mathcal{S}^I$ given by

$$Z_{\alpha\beta}(j) = \begin{cases} Z^J_\alpha(j) & \text{for } j \neq i, \\ Z^J_\alpha(I//i) \cup Z^i_\beta & \text{for } j = i, \end{cases}$$

with the obvious maps. We thus have X written as the inductive limit

$$X = \varinjlim_{(\alpha,\beta)\in A_J B} Z_{\alpha\beta},$$

via commutative diagrams of cofibrations in $\mathcal{S}^I$

$$\begin{tikzcd} Z_{\alpha\beta} \arrow[rr] \arrow[dr] & & Z_{\alpha'\beta'} \arrow[dl] \\ & X & \end{tikzcd}$$

for $\alpha\beta \leq \alpha'\beta'$ in $A_J B$. By an obvious diagonalization, we may find a totally ordered subset A of $A_J B$, which gives a cofinal subfamily of the family $\{Z_{\alpha\beta}\}$.

The proof for $(X, *)$ in $\mathcal{S}^{*I}_{\mathrm{c}}$ is similar, and is left to the reader. $\square$

We now proceed to make minor modifications in the argument of Quillen [H] to extend the universality statement of ([H], Corollary 2.4) to our setting. Let F denote, as in ([H], pg. 245), the two-skeleton of the simplicial set BA_5, and $F \to F^+$ a simplicial version of the $+$-construction on F: $F \to F^+$ is a cofibration of finite pointed connected simplicial sets. Applying the functor (B.3)(ii)

$$(-) \wedge \mathcal{N}(I/-) \colon \mathcal{S}^* \to \mathcal{S}^{*I}$$

to $F \to F^+$, we have the cofibration

$$i \colon \mathcal{F} \to \mathcal{F}^+$$

of cofibrant objects of $\mathcal{S}^{*I}_{\mathrm{c}}$, which is weakly equivalent to the map

$$F \times I \to F^+ \times I$$

As I is finite dimensional, i is a map in $\mathcal{S}^{*I}_{\mathrm{cfin}}$. The natural maps

$$F \to \mathrm{BGL}(R)$$

gives rise to map

$$j\colon \mathcal{F} \to \mathrm{BGL}(A).$$

Replacing $\mathrm{BGL}(A)$ with a weakly equivalent model, and changing notation, we may assume that j is a cofibration; in particular, that $\mathrm{BGL}(A)$ is cofibrant. By ([H], pg. 245), the push-out

$$\mathrm{BGL}(A) \to \mathrm{BGL}(A) \cup_{\mathcal{F}} \mathcal{F}^+ \tag{4.1}$$

is (point-wise) weakly equivalent to the map

$$\mathrm{BGL}(A) \to \mathrm{BGL}(A)^+$$

By lemma 3.1, a bifibrant model of (4.1) is weakly equivalent to

$$\tilde{i}\colon \mathrm{B\tilde{G}L}(A) \to \mathrm{B\tilde{G}L}(A)^+$$

We now apply lemma 4.1, writing $\mathrm{BGL}(A)$ as the nested inductive limit of cofibrations

$$j_\alpha\colon \mathcal{F}_\alpha \to \mathrm{BGL}(A);$$

with $\mathcal{F}_\alpha$ in $\mathcal{S}^{*I}_{\mathrm{cfin}}$; we may assume that j factors (uniquely) through each j_α.

The geometric realization functor

$$|-|\colon \mathcal{S}^I \to \mathbf{Top}^I$$

has the singular complex functor

$$\mathrm{Sin}\colon \mathbf{Top}^I \to \mathcal{S}^I$$

as right adjoint. For each $T \in \mathbf{Top}^I$, the object $\mathrm{Sin}T$ is fibrant in $\mathcal{S}^I$. The functor $|-|$ commutes with direct limits. This is not in general the case for the functor Sin, however, if a simplicial complex X is written as an nested inductive limit of subcomplexes

$$\begin{aligned}
&X = \cup_{\alpha\in A} Z_\alpha \\
&Z_\alpha \subset \ldots \subset Z_\beta \subset \ldots \subset X; \quad \alpha \le \beta \in A,
\end{aligned}$$

indexed by a totally ordered set A, then we have

$$\mathrm{Sin}|X| = \varinjlim_{\alpha\in A} \mathrm{Sin}|Z_\alpha|.$$

Take B in $\mathcal{S}^I_{\mathrm{fin}}$, and let $\tilde{B}$ be a bifibrant model. Let

$$\begin{aligned}
&\tilde{\mathcal{F}}_\alpha \to \mathrm{Sin}|\mathcal{F}_\alpha| \\
&\mathcal{F}_\alpha \tilde{\cup}_{\mathcal{F}} \mathcal{F}^+ \to \mathrm{Sin}|\mathcal{F}_\alpha \cup_{\mathcal{F}} \mathcal{F}^+|
\end{aligned}$$

be trivial fibrations, with $\tilde{\mathcal{F}}_\alpha$ and $\mathcal{F}_\alpha \mathbin{\tilde{\cup}_{\mathcal{F}}} \mathcal{F}^+$ bifibrant. Then we have the isomorphisms (for both the pointed and unpointed homotopy classes)

$$\begin{aligned}
[\tilde{B}, \mathrm{B}\tilde{\mathrm{G}}\mathrm{L}(A)^+] &\cong [B, \mathrm{B}\tilde{\mathrm{G}}\mathrm{L}(A)^+] \\
&\cong [B, \mathrm{Sin}|\mathrm{BGL}(A) \cup_{\mathcal{F}} \mathcal{F}^+|] \\
&\cong \varinjlim_{\alpha} [B, \mathrm{Sin}|\mathcal{F}_\alpha \cup_{\mathcal{F}} \mathcal{F}^+|] \\
&\cong \varinjlim_{\alpha} [B, \mathcal{F}_\alpha \mathbin{\tilde{\cup}_{\mathcal{F}}} \mathcal{F}^+] \\
&\cong \varinjlim_{\alpha} [\tilde{B}, \mathcal{F}_\alpha \mathbin{\tilde{\cup}_{\mathcal{F}}} \mathcal{F}^+].
\end{aligned} \tag{4.2}$$

Similarly, we have the isomorphism

$$[\tilde{B}, \mathrm{B}\tilde{\mathrm{G}}\mathrm{L}(A)] \cong \varinjlim_{\alpha} [\tilde{B}, \tilde{\mathcal{F}}_\alpha]. \tag{4.3}$$

Let G be in $\mathbf{Groups}^I$, P a finitely generated projective S-module, and

$$\rho: G \to \mathrm{Aut}(P \otimes_S A)$$

be a representation of G on the A-module $P \otimes_S A$. Let Q be a finitely generated projective S-module with $P \oplus Q$ isomorphic to S^N, and choose an isomorphism $P \oplus Q \to S^N$. The representation $\rho \oplus \mathrm{id}_Q$ then determines the homomorphism

$$\tilde{\rho}(A): G \to \mathrm{GL}_N(A).$$

Classifying, taking $+$, and using lemma 3.1, we have the map in $\mathcal{S}^{*I}$

$$\tilde{\mathrm{B}}\tilde{\rho}^+: \tilde{\mathrm{B}}G \to \mathrm{B}\tilde{\mathrm{G}}\mathrm{L}_N(A)^+, \tag{4.4}$$

uniquely determined up to homotopy by $\tilde{\rho}$. We may then compose with

$$\tilde{\iota}_N: \mathrm{B}\tilde{\mathrm{G}}\mathrm{L}_N(A)^+ \to \mathrm{B}\tilde{\mathrm{G}}\mathrm{L}(A)^+.$$

Lemma 4.2. *The (pointed) homotopy class of $\tilde{\iota}_N \circ \tilde{\mathrm{B}}\tilde{\rho}^+$ is uniquely determined by ρ.*

Proof. If Q' is another choice of inverse to P, we have

$$Q \oplus S^m \cong Q' \oplus S^{m'}$$

for certain m and m'. If ρ_1 and ρ_2 are isomorphic representations

$$\rho_i: G \to \mathrm{GL}_{M/S},$$

then they differ by conjugation by a matrix in $\mathrm{GL}_M(S)$. Thus, the choices made alter $\tilde{\mathrm{B}}\tilde{\rho}^+$ by stabilizing, and by conjugation by some section g of $\mathrm{GL}_M(A)$ over I. Applying lemma 3.2 completes the proof. $\square$

Let $\mathcal{C}$ be the category of pairs (Z, B_Z), with Z a fibrant object in $\mathcal{S}^I$, B_Z a set of base points

$$s: * \to Z; s \in B_Z,$$

such that $\pi_1(|Z(i)|, s(i))$ has no non-trivial perfect subgroup for each i in I. Maps $(Z, B_Z) \to (W, B_W)$ are given by pairs of maps $h: B_Z \to B_W$, $g: Z \to W$ such that g defines a pointed map $(Z, s) \to (W, h(s))$ for each $s \in B_Z$.

For $(X, *)$ in $\mathcal{S}^{*I}$, set

$$[X, (Z, B_Z)]_* = \coprod_{s \in B_Z} [(X, *), (Z, s)]$$

We usually omit the B_Z from the notation. The set of base-points $K_0(A/S) \times *$ for $K_0(A/S) \times \mathrm{B}\tilde{\mathrm{GL}}(A)^+$ makes $K_0(A/S) \times \mathrm{B}\tilde{\mathrm{GL}}(A)^+$ an object of $\mathcal{C}$.

Let $(X, *)$ be in $\mathcal{S}^{*I}_{\mathrm{cfin}}$. Define the map

$$\mathrm{Rep}_S(\pi_1(X, *); A) \to [X, K_0(A/S) \times \mathrm{B}\tilde{\mathrm{GL}}(A)^+]_*$$

by sending

$$\rho: \pi_1(X, *) \to \mathrm{Aut}(P \otimes_S A)$$

to the map (see (2.3), (3.7) and (4.4))

$$([P], \tilde{\iota}_N \circ \tilde{\mathrm{B}}\tilde{\rho}^+ \circ q_X).$$

This determines the natural transformation of functors

$$q^s: \mathrm{Rep}_S(\pi_1(-); A) \to [-, K_0(A/S) \times \mathrm{B}\tilde{\mathrm{GL}}(A)^+]_* \tag{4.5}$$

from $[\mathcal{S}^{*I}_{\mathrm{cfin}}]^{\mathrm{op}}$ to **Sets**.

We let $\mathrm{Ho}\mathcal{S}^{*I}_{\mathrm{cbfin}}$ denote the full subcategory of $\mathrm{Ho}\mathcal{S}^{*I}$ with objects the objects of $\mathcal{S}^{*I}_{\mathrm{cbfin}}$.

Theorem 4.3. *The natural transformation (4.5) extends to the natural transformation*

$$q^s: R_S(\pi_1(-); A) \to [-, K_0(A/S) \times \mathrm{B}\tilde{\mathrm{GL}}(A)^+]_* \tag{4.6}$$

of contravariant functors from $\mathrm{Ho}\mathcal{S}^{*I}_{\mathrm{cbfin}}$ to **Sets**, *which is universal relative to $\mathcal{C}$: for each natural transformation*

$$g: R_S(\pi_1(-); A) \to [-, Z]_*$$

of contravariant functors from the pointed homotopy category $\mathrm{Ho}\mathcal{S}^{*I}_{\mathrm{cbfin}}$ to **Sets**, *with Z in $\mathcal{C}$, there is a unique natural transformation*

$$h: [-, K_0(A/S) \times \mathrm{B}\tilde{\mathrm{GL}}(A)^+]_* \to [-, Z]_*$$

with $g = h \circ q^s$.

Proof. The arguments of ([H], §1, especially proposition 1.1), combined with theorem 3.3 and lemma 3.4, give the extension of the functor (4.5) to the functor (4.6). The arguments of [H] (theorem 2.2, corollary 2.3 and corollory 2.4), together with the results of §3 mentioned above, as well as lemma 4.1 and lemma 4.2, prove the universality: we replace Quillen's categories $\mathcal{C}_1$ and $\mathcal{C}_2$ with

$$\begin{aligned}\mathcal{C}_1 := \{(Z, *) \text{ in } \mathcal{S}^{*I} \mid Z \text{ is fibrant and } &\pi_1(|Z(i)|, *(i)) \text{ has no} \\ &\text{non-trivial perfect subgroup for each } i \in I\} \\ \mathcal{C}_2 := \mathcal{C}_1,&\end{aligned}$$

and replace $\mathcal{C}_3$ with our $\mathcal{C}$. In Quillen's proof of theorem 2.2 of [H], we replace his F_α and $F_\alpha \cup_F F^+$ with our $\tilde{\mathcal{F}}_\alpha$ and $\mathcal{F}_\alpha \tilde{\cup}_{\mathcal{F}} \mathcal{F}^+$, use the isomorphisms (4.2) and (4.3), and replace his use of obstruction theory with lemma 3.1. We use pointed maps and pointed homotopy classes of maps throughout.

The same argument shows that the natural transformation

$$[-, (\mathrm{B}\tilde{\mathrm{G}}\mathrm{L}(A))^n]_* \to [-, (\mathrm{B}\tilde{\mathrm{G}}\mathrm{L}(A)^+)^n]_* \tag{1}$$

is universal relative to $\mathcal{C}_1$, for all $n \geq 1$.

One notes that, for X in $\mathcal{S}^{*I}_{\mathrm{cbfin}}$, we have

$$\begin{aligned}[][X, \mathrm{B}\tilde{\mathrm{G}}\mathrm{L}(A)]_* &\cong \mathrm{Hom}_{\mathbf{Groups}^I}(\pi_1(X), \mathrm{GL}) \\ &= \varinjlim_n \mathrm{Hom}_{\mathbf{Groups}^I}(\pi_1(X), \mathrm{GL}_n) \\ &= \varinjlim_n [X, \mathrm{B}\tilde{\mathrm{G}}\mathrm{L}_n(A)]_*,\end{aligned}$$

since $\pi_1(X(i))$ is finitely generated for each i, and I is finite.

In Quillen's proof of ([H], corollary 2.3), we replace his $\tilde{K}(-, A)$ with $[-, \mathrm{B}\tilde{\mathrm{G}}\mathrm{L}(A)^+]_*$, and replace his $\tilde{R}(-, A)$ with the subgroup $\tilde{R}_S(-; A)$ of $R_S(-; A)$ generated by representations into A-modules of the form A^n. We also replace his statement that "products of universal transformations relative to $\mathcal{C}_2$ are universal relative to $\mathcal{C}_2$" with the universality of (1) for all n. The argument of ([H], corollary 2.3), together with lemma 3.2, then shows that the natural transformation

$$\tilde{R}_S(\pi_1(-); A) \to [-, \mathrm{B}\tilde{\mathrm{G}}\mathrm{L}(A)^+]_* \tag{2}$$

defined as the composition

$$\tilde{R}_S(\pi_1(-); A) \to R_S(\pi_1(-); A) \to [-, K_0(A/S) \times \mathrm{B}\tilde{\mathrm{G}}\mathrm{L}(A)^+]_* \xrightarrow{p_{2*}} [-, \mathrm{B}\tilde{\mathrm{G}}\mathrm{L}(A)^+]_*,$$

is universal relative to $\mathcal{C}_2$; the change in the argument mentioned above accounts for the difference in our $\mathcal{C}_2$ from his.

Now suppose we have a natural transformation

$$\eta\colon R_S(\pi_1(-);A) \to [-,(Z,B_Z)]_*$$

with (Z,B_Z) in $\mathcal{C}$. Applying η to $(\mathcal{N}(I/-),*)$ (which is point-wise contractible) gives the map

$$\eta(*)\colon K_0(A/S) = R_S(\mathrm{id};A) \to B_Z.$$

This, together with the universality of (2), shows that (4.5) is universal relative to $\mathcal{C}$. $\square$

Theorem 4.4. *The functor*

$$[-,K_0(A/S)\times \mathrm{B\tilde{G}L}(A)^+]_*\colon \mathcal{S}^{*I}_{\mathrm{cbfin}} \to \mathbf{Sets} \tag{4.7}$$

admits a canonical structure of a functor from $\mathcal{S}^{*I}_{\mathrm{cbfin}}$ to the category of special λ-rings such that the natural transformation (4.6) is a natural transformation of functors to the category of special λ-rings. In addition, this functor is natural in the pair (I,A).

Proof. The first assertion follows directly from theorem 4.3 and the arguments of *loc. cit.*. The second follows similarly from theorem 4.3, and the naturality of the functor

$$R_S(-;A)\colon \mathbf{Groups}^I \to \mathbf{Rings}$$

in (I,A). $\square$

We now pass from the pointed category to the unpointed category.

Theorem 4.5. *The functor*

$$[-,K_0(A/S)\times \mathrm{B\tilde{G}L}(A)^+]\colon \mathcal{S}^{I}_{\mathrm{fin}} \to \mathbf{Sets} \tag{4.8}$$

admits a canonical structure of a functor from $\mathcal{S}^{I}_{\mathrm{fin}}$ to the category of special λ-rings. In addition, this functor is natural in the pair (I,A).

Proof. We extend the functor (4.7) to finite disjoint unions of objects in $\mathcal{S}^{*I}_{\mathrm{cbfin}}$ by

$$[\coprod_{i=1}^{N}(X_i,*),-]_* = \prod_{i=1}^{N}[(X_i,*),-]_*$$

The functor (4.7) then extends to a functor to special λ-rings in the obvious way.

Refering to (4.2), let

$$\tilde{W}_\alpha = \mathcal{F}_\alpha \tilde{\cup}_{\mathcal{F}} \mathcal{F}^+$$

From (4.2) we have the canonical isomorphisms of functors from $\mathrm{Ho}\mathcal{S}^{*I}_{\mathrm{cbfin}}$ to pointed sets (resp. from $\mathcal{S}_{\mathrm{fin}}$ to **Sets**):

$$\begin{aligned} [-, \mathrm{B}\tilde{\mathrm{G}}\mathrm{L}(A)^+]_* &\cong \varinjlim_{\alpha} [-, \tilde{W}_\alpha]_* \\ [-, \mathrm{B}\tilde{\mathrm{G}}\mathrm{L}(A)^+] &\cong \varinjlim_{\alpha} [-, \tilde{W}_\alpha] \end{aligned} \tag{1}$$

Write the set $K_0(A/S)$ as a nested union of finite subsets S_α. The pointed version in (1) gives the isomorphisms

$$\begin{aligned} &\mathrm{Nat}([-, K_0(A/S) \times \mathrm{B}\tilde{\mathrm{G}}\mathrm{L}(A)^+]_*, [-, K_0(A/S) \times \mathrm{B}\tilde{\mathrm{G}}\mathrm{L}(A)_+]^*) \\ &\quad \cong \varprojlim_{\alpha} \mathrm{Nat}([-, S_\alpha \times \tilde{W}_\alpha]_*, [-, K_0(A/S) \times \mathrm{B}\tilde{\mathrm{G}}\mathrm{L}(A)^+]_*) \\ &\quad \cong \varprojlim_{\alpha} [S_\alpha \times \tilde{W}_\alpha, K_0(A/S) \times \mathrm{B}\tilde{\mathrm{G}}\mathrm{L}(A)^+]_* \end{aligned}$$

Thus, by theorem 4.4, the set

$$\varprojlim_{\alpha} [S_\alpha \times \tilde{W}_\alpha, K_0(A/S) \times \mathrm{B}\tilde{\mathrm{G}}\mathrm{L}(A)^+]_* \tag{2}$$

has the natural structure of a special λ-ring. Now, if B is in $\mathcal{S}^I_{\mathrm{fin}}$, the isomorphism (1) (in the unpointed case), the canonical pairing

$$\varinjlim_{\alpha} [B, S_\alpha \times \tilde{W}_\alpha] \times \varprojlim_{\alpha} [S_\alpha \times \tilde{W}_\alpha, K_0(A/S) \times \mathrm{B}\tilde{\mathrm{G}}\mathrm{L}(A)^+]_* \to [B, K_0(A/S) \times \mathrm{B}\tilde{\mathrm{G}}\mathrm{L}(A)^+],$$

together with the natural λ-ring structure on (2), gives the natural special λ-ring structure on the unpointed homotopy set

$$[B, K_0(A/S) \times \mathrm{B}\tilde{\mathrm{G}}\mathrm{L}(A)^+].$$

□

Remark 4.6. i) We will usually use take $S = \mathbb{Z}$, in which case $K_0(A/\mathbb{Z})$ is $\mathbb{Z}$ by rank.

ii) Using the fact that the functor

$$\operatorname*{holim}_{I} \colon \mathcal{S}^I \to \mathcal{S}$$

is right adjoint to the functor (B.1)

$$(-) \times \mathcal{N}(I/-) \colon \mathcal{S} \to \mathcal{S}^I,$$

and noting that this latter functor sends $\mathcal{S}_{\mathrm{fin}}$ to $\mathcal{S}_{\mathrm{fin}}^I$ in case I is finite, theorem 4.5 gives a natural special λ-ring structure to the functor

$$[-, \operatorname*{holim}_I K_0(A/S) \times \mathrm{B}\tilde{\mathrm{G}}\mathrm{L}(A)^+] \colon \mathcal{S}_{\mathrm{fin}} \to \mathbf{Sets}.$$

iii) One may replace GL_n with SL_n, and get analogous results, with minor changes. Indeed, for a group G and a commutative ring R, we let $\mathrm{Rep}^0(G; R)$ denote the set of conjugacy classes (by $\mathrm{GL}_n(R)$) of homomorphisms

$$\rho \colon G \to \mathrm{SL}_n(R)$$

and $R^0(G; R)$ the Grothendieck group of the $\mathrm{Rep}^0(G; R)$. The same argument as in this and the preceeding section gives $\mathrm{B}\tilde{\mathrm{S}}\mathrm{L}(A)^+$ the structure of an H-group, and we have the universal natural transformation

$$q_0^s : R^0(\pi_1(-); A) \to [-, \mathbb{Z} \times \mathrm{B}\tilde{\mathrm{S}}\mathrm{L}(A)^+] \tag{4.6}$$

which gives a natural special λ-ring structure to the functor $[-, \mathbb{Z} \times \mathrm{B}\tilde{\mathrm{S}}\mathrm{L}(A)^+]$. The map

$$[-, \mathbb{Z} \times \mathrm{B}\tilde{\mathrm{S}}\mathrm{L}(A)^+] \to [-, \mathbb{Z} \times \mathrm{B}\tilde{\mathrm{G}}\mathrm{L}(A)^+]$$

is a map of functors from $\mathcal{S}_{\mathrm{fin}}$ to special λ-rings. $\square$

§5. λ-operations on the K-theory of schemes and variants

Let X be a (noetherian, separated) scheme over a commutative noetherian ring S. We let $K^Q(X)$ denote the Quillen K-theory space of X:

$$K^Q(X) = \Omega \mathrm{BQ}\mathcal{P}_X$$

where $\mathcal{P}_X$ is the category of locally free coherent sheaves on X. If we have a small category I, and a functor

$$X: I \to \mathbf{Sch}$$

one can make $X \mapsto \mathcal{P}_X$ into a functor, giving the functor

$$\mathrm{BQ}\mathcal{P}_X: I \to \mathcal{S}.$$

We let

$$K^Q(X): I \to \mathcal{S}$$

be a bifibrant model of the loop space functor applied to a fibrant model of $\mathrm{BQ}\mathcal{P}_X$, and let $K_0(A)$ be the functor

$$i \mapsto K_0(A(i)).$$

Lemma 5.1. *Let I be a small category, $A: I \to$ **Rings** a functor. Let $\mathrm{B\tilde{GL}}_n(A)^+$, $\mathrm{B\tilde{GL}}(A)^+$ be bifibrant models of $\mathrm{BGL}_n(A)^+$, $\mathrm{BGL}(A)^+$, resp., with the sequence of cofibrations*

$$\begin{array}{ccccc} \mathrm{B\tilde{GL}}_n(A)^+ & \to & \mathrm{B\tilde{GL}}_{n+1}(A)^+ & \to \ldots \\ & \searrow & \downarrow & \\ & & \mathrm{B\tilde{GL}}(A)^+ & \end{array} \tag{5.1}$$

as in (3.3)-(3.7). Then there is a weak equivalence

$$\psi_A: K_0(A) \times \mathrm{B\tilde{GL}}(A)^+ \to K^Q(\mathrm{Spec}(A))$$

ψ_A is not necessarily natural in A, but the composition

$$K_0(A) \times \mathrm{B\tilde{GL}}_n(A)^+ \to K_0(A) \times \mathrm{B\tilde{GL}}(A)^+ \to K^Q(\mathrm{Spec}(A))$$

is natural in A, up to homotopy, for each n.

Proof. The definition of the map ψ_A follows essentially from Grayson's article [Gr] relating $\Omega K^Q(\mathrm{Spec} R)$ and $\mathrm{BGL}(R)^+$ for a commutative ring R, together with some techniques from the theory of the simplicial closed model categories applied to $\mathcal{S}^I$. Indeed, for a small category I, and a functor

$$A: I \to \mathbf{Rings}$$

we have the functors

$$P(A): I \to \mathbf{cat}; \qquad S(A): I \to \mathbf{cat}$$
$$QP(A): I \to \mathbf{cat}; \qquad E(A): I \to \mathbf{cat}$$

where, for a ring R, $P(R)$ is the category of finitely generated projective R-modules, S is the category $\mathrm{Iso}(P(R))$, with the same objects as $P(R)$, with only isomorphisms for morphisms, $QP(R)$ is Quillen's Q-construction on $P(R)$, and $E(R)$ is the extension construction on $S(R)$ ([Gr], pg. 226). One needs to fiddle a bit to make P into a functor, but this is easily done. We have the commutative diagram of functors from I to **cat** ([Gr], pg. 228):

$$\begin{array}{ccc} S^{-1}S(A) & \to & S^{-1}E(A) \\ \downarrow & & \downarrow \\ * & \to & QP(A); \end{array}$$

taking the nerves gives the resulting diagram in $\mathcal{S}^I$, which we write the same way. If we pass to a commuting diagram of bifibrant models in $\mathcal{S}^{*I}$:

$$\begin{array}{ccc} \tilde{S^{-1}}S(A) & \to & \tilde{S^{-1}}E(A) \\ \downarrow & & \downarrow \\ * & \to & \tilde{Q}P(A); \end{array} \tag{1}$$

the first two theorems of ([Gr], pg. 228) imply that the diagram (1) is homotopy cartesian. Applying the results of ([B-K], XI, 5.6), taking homotopy limits results in a weak equivalence

$$\beta(A)\colon \operatorname*{holim}_I \tilde{S^{-1}}S(A) \to \operatorname*{holim}_I \Omega\tilde{Q}P(A) = \operatorname*{holim}_I K^Q(A)$$

On the other hand, for a ring R, there is, for each n, a canonical map

$$\mathrm{BGL}_n(R) \to S^{-1}S(R)_0 \tag{2}$$

(see [Gr], pg. 224), which is natural, up to homotopy, with respect to the stabilization maps

$$\mathrm{BGL}_n(R) \to \mathrm{BGL}_{n+1}(R).$$

Taking a functor $A\colon I \to$ **Rings** as above, the maps (2) define maps

$$\alpha_n\colon K_0(A) \times \mathrm{BGL}_n(A) \to S^{-1}S(A);$$

if we pass to the sequence (5.1) of bifibrant models, the maps α_n induce (uniquely up to homotopy for each n) a sequence of maps

$$\tilde{\alpha}_n\colon K_0(A) \times \mathrm{B\tilde{G}L}_n(A) \to \tilde{S^{-1}}S(A),$$

which strictly commute with the stabilization maps

$$\mathrm{B\tilde{G}L}_n(A) \to \mathrm{B\tilde{G}L}_{n+1}(A).$$

This gives a well-defined map on the inductive limit

$$\alpha_\infty\colon K_0(A) \times \varinjlim_n \mathrm{B\tilde{G}L}_n(A) \to \tilde{S^{-1}}S(A).$$

In addition, it is easy to see that $\varinjlim_n \mathrm{B}\tilde{\mathrm{GL}}_n(A)$ is cofibrant, and that the canonical map

$$\varinjlim_n \mathrm{B}\tilde{\mathrm{GL}}_n(A) \to \mathrm{B}\tilde{\mathrm{GL}}(A)$$

is a weak equivalence. This gives the map

$$\alpha\colon K_0(A) \times \mathrm{B}\tilde{\mathrm{GL}}(A) \to \widetilde{S^{-1}S}(A),$$

which restricted to $K_0(A) \times \mathrm{B}\tilde{\mathrm{GL}}_n(A)$ is homotopic to $\tilde{\alpha}_n$.

As $\widetilde{S^{-1}S}(A)$ is an H-space, the map α extends canonically (up to homotopy) to

$$\alpha^+(A)\colon K_0(A) \times \mathrm{B}\tilde{\mathrm{GL}}(A)^+ \to \widetilde{S^{-1}S}(A),$$

by lemma 3.1. Applying the theorem of ([Gr], pg. 224), α^+ is a weak equivalence. Composing $\alpha^+(A)$ with $\beta(A)$ gives the desired weak equivalence.

For the naturality, the map β is canonical. It follows from the various uniqueness statements of §5, that the restriction of α^+ to $K_0(A) \times \mathrm{B}\tilde{\mathrm{GL}}_n(A)$ is canonical, up to homotopy, for each n. Suppose we have a functor

$$f\colon J \to I$$

of small categories, a functor

$$B\colon J \to \mathbf{Rings}$$

and a natural transformation

$$\omega\colon B \to A \circ f$$

Let $(f\colon J \to I)$ denote the category with objects $\mathrm{Obj}I \coprod \mathrm{Obj}J$, such that I and J are embedded as full subcategories, there are no maps $i \to j$ for $i \in I$ and $j \in J$, and where

$$\mathrm{Hom}(j, i) = \mathrm{Hom}_I(f(j), i).$$

The data (A, B, ω) determines the functor

$$(A, B, \omega)\colon (f\colon J \to I) \to \mathbf{Rings}$$

The weak equivalence $\beta(A, B, \omega)\alpha^+(A, B, \omega)$ then gives the naturality of $\beta(-)\alpha^+(-)$. $\square$

Lemma 5.2. *Let I be a finite category, $A\colon I \to$ **Rings** a functor. For each B in $\mathcal{S}^I_{\mathrm{fin}}$, there is an isomorphism*

$$\psi_{A,B}\colon [B, K_0(A) \times \tilde{\mathrm{B}}\mathrm{GL}(A)^+] \to [B, K^Q(A)]$$

*which is natural in B, and in functors $A\colon I \to$ **Rings** for finite categories I. In particular, for each finite simplicial complex B, there is an isomorphism*

$$\Psi_{A,B}\colon [B, \operatorname{holim}_I K_0(A) \times \mathrm{BGL}(A)^+] \to [B, \operatorname{holim}_I K^Q(A)]$$

*which is natural in in B, and in functors $A\colon I \to$ **Rings** for finite categories I.*

Proof. The first assertion follows from lemma 5.1, noting that the natural map

$$\lim_n [X, K_0(A) \times \mathrm{BGL}_n(A)^+] \to [X, K_0(A) \times \mathrm{BGL}(A)^+]$$

is an isomorphism. For the second, we note that the functor holim_I is right adjoint to the functor (see Appendix B)

$$(-) \times \mathcal{N}(I/-)\colon \mathcal{S} \to \mathcal{S}^I$$

and that this functor sends $\mathcal{S}_{\mathrm{fin}}$ to $\mathcal{S}^I_{\mathrm{fin}}$. $\square$

For a set T, we let $\mathcal{M}(T)$ denote the category with objects the elements $a_* := a_1 + \ldots + a_n$, $n = 0, 1, \ldots$, of the free abelian monoid $M(T)$ on T, with a unique morphism $a_* \to b_*$ if and only if there is a $c_* \in M(T)$ with

$$b_* = a_* + c_*.$$

We let $<T>$ denote the full subcategory of $\mathcal{M}(T)$ with objects the sums $a_1 + \ldots + a_n$ with *distinct* a_i in T (and the empty sum).

Let $\mathfrak{U} = \{U_\alpha \mid \alpha \in T\}$ be an affine cover of X; $U_\alpha = \mathrm{Spec}(R_\alpha)$. Let $<\mathfrak{U}>$ be the full subcategory of $<T>$ consisting of the non-empty sums $\alpha_1 + \ldots + \alpha_n$. Let $R_{\mathfrak{U}}$ be the S-algebra over $<\mathfrak{U}>$ defined by

$$R_{\mathfrak{U}}(\alpha_1 + \ldots + \alpha_n) = \Gamma(\cap_j U_{\alpha_j}, \mathcal{O}_X)$$

(we make the convention that $\Gamma(\emptyset, \mathcal{O}_X)$ is the zero ring, and that $K_0(S) \times \mathrm{BGL}(0)^+$ is the one-point space).

Theorem 5.3. *Let X be a noetherian scheme over S which admits an ample family of line bundles, and let $\mathfrak{U}$ be a finite affine open cover of X. Then*

i) There is a map

$$\psi_{\mathfrak{U}}\colon \operatorname*{holim}_{<\mathfrak{U}>} (K_0(A/S) \times \mathrm{BGL}(R_{\mathfrak{U}})^+) \to K^Q(X).$$

For each finite simplicial complex B, the induced map on $[B, -]$ is natural in the triple $(B, \mathfrak{U}, X)$.

ii) $\psi_{\mathfrak{U}}$ induces an isomorphism on homotopy groups π_n for $n > 0$ and an injection for $n = 0$; the image of $\psi_{\mathfrak{U}}$ on π_0 is the subgroup of $K_0(X)$ generated by locally free sheaves P on X such that $[P \otimes_{\mathcal{O}_X} \mathcal{O}_{U_\alpha}]$ is in the image of $K_0(S) \to K_0(U_\alpha)$ for each $\alpha \in A$. Furthermore, $\psi_{\mathfrak{U}}$ induces an isomorphism

$$\lim_{\to} \pi_0(\operatorname*{holim}_{<\mathfrak{U}>} (K_0(A/S) \times \mathrm{BGL}(R_{\mathfrak{U}})^+)) \to K_0(X)$$

where the limit is over finite affine covers $\mathfrak{U}$ of X.

Proof. We note that the category $<\mathfrak{U}>$ is finite. By the two previous lemmas, we need only show that the natural map

$$K^Q(X) \to \operatorname*{holim}_I K^Q(R_{\mathfrak{U}}) \tag{1}$$

is a weak equivalence, and compute the difference between

$$\pi_0 \operatorname*{holim}_{<\mathfrak{U}>} K_0(R_{\mathfrak{U}}/S) \times \mathrm{BGL}(R_{\mathfrak{U}})^+$$

and

$$\pi_0 \operatorname*{holim}_{<\mathfrak{U}>} K_0(R_{\mathfrak{U}}) \times \mathrm{BGL}(R_{\mathfrak{U}})^+.$$

In the case of regular X, (1) being a weak equivalence is a direct consequence of the Mayer-Vietoris spectral sequence arising from Quillen's localization theorem [Q3], which one compares with the Bousfield-Kan spectral sequence (theorem B.1) for $\operatorname{holim}^*_{<\mathfrak{U}>}$. In fact, for a pre-sheaf $\mathcal{T}$ of simplicial spaces on X, we have the space $\mathcal{T}(\mathfrak{U})$ over $<\mathfrak{U}>$, gotten by taking global sections over the various intersections of elements of $\mathfrak{U}$. The Bousfield-Kan spectral sequence for

$$\operatorname*{holim}_{<\mathfrak{U}>} \mathcal{T}(\mathfrak{U})$$

is exactly the Mayer-Vietoris spectral sequence for $\mathcal{T}$ with respect to the cover $\mathfrak{U}$.

In general, a similar argument using the Thomason-Trobaugh extension [T-T] of Quillen's localization theorem to the case of arbitrary schemes gives the proof; although the theorem of ThomasonTrobaugh uses spectra with (possibly) negative homotopy groups, the homotopy groups in non-negative degree agree with the Quillen K-groups, which implies that the natural map

$$K^Q(X) \to \operatorname*{holim}_{<\mathfrak{U}>}(K^Q(-))$$

is a weak equivalence.

The difference in two π_0's is also readily computable from the Mayer-Vietoris spectral sequence, with the result as claimed. □

Corollary 5.4. *Let X be a noetherian scheme over a noetherian ring S, admitting an ample family of line bundles. Then there is a special $K_0(S)$-λ-algebra structure on the K-groups of X over S, which is natural in X.*

Proof. By Remark 4.6(ii), there is a natural special $K_0(S)$-λ-algebra structure for the functor

$$[-, \operatorname*{holim}_{<\mathfrak{U}>} (K_0(R_{\mathfrak{U}}/S) \tilde{\times} \mathrm{BGL}(R)^+)] \colon \mathcal{S}_{\mathrm{fin}} \to \mathbf{Sets}.$$

In particular, this gives a natural special $K_0(S)$-λ-algebra structure on the homotopy groups of the H-space

$$\operatorname*{holim}_{<\mathfrak{U}>} (K_0(R_{\mathfrak{U}}/S) \tilde{\times} \mathrm{BGL}(R)^+).$$

The result then follows from theorem 5.3. □

If U is an open subset of a scheme X, we have the K-theory space with supports in $W := X\backslash U$, $K^W(X)$, defined as the homotopy fiber of the natural map

$$i_U^*: K(X) \to K(U).$$

If we have an affine open cover of X as above, we have the ring $R_{\mathfrak{U}}$ over $<\mathfrak{U}>$, and similarly the ring $R_{U,\mathfrak{U}}$ over $<\mathfrak{U}>$ gotten by restricting the open cover to U.

We have the category $[0,1]$, associated to the ordered set $0 < 1$, and for each $n = 1, 2, \ldots$ the n-cube $[0,1]^n$; we may identify $[0,1]^n$ with the category associated to the partially ordered set of subsets of $\{1, \ldots, n\}$, ordered under inclusion.

The map $i_U^*: R_{\mathfrak{U}} \to R_{U,\mathfrak{U}}$ defines the ring $R_{\mathfrak{U}}^U$ over $<\mathfrak{U}> \times [0,1]$, where

$$\begin{aligned}(R \to R_U)_{|<\mathfrak{U}>\times 0} &= R_{\mathfrak{U}}\\ (R \to R_U)_{|<\mathfrak{U}>\times 1} &= R_{U,\mathfrak{U}}\end{aligned}$$

As above, we may compute the component of the identity of $K^W(X)$ is given as the component of the identity of

$$\overset{*}{\operatorname*{holim}_{<\mathfrak{U}>\times[0,1]}} K_0(R_{\mathfrak{U}}^U/S) \tilde{\times} \mathrm{BGL}(R_{\mathfrak{U}}^U)^+,$$

and $K_0^W(X)$ may be computed as a limit over finite open covers. Here $\overset{*}{\mathrm{holim}}$ is the pointed version of holim which gives the desired homotopy fiber. This gives

Corollary 5.5. *Let X be a noetherian scheme over a noetherian ring S, admitting an ample family of line bundles, W a closed subset. There is a special $K_0(S)$-λ-algebra structure on the $K_p^W(X)$, which is natural in the pair (X, W). In addition, the long exact sequence*

$$\ldots \to K_p^W(X) \to K_p(X) \to K_p(X\backslash W) \to K_{p-1}^W(X) \to \ldots \tag{5.1}$$

associated to the fiber sequence

$$K^W(X) \to K(X) \to K(U)$$

is a sequence of $K_0(S)$-λ-algebras.

Proof. We may apply the special $K_0(S)$-λ-algebra structure on the functor

$$[-, \overset{*}{\operatorname*{holim}_{<\mathfrak{U}>\times[0,1]}} K_0(R_{\mathfrak{U}}^U/S) \tilde{\times} \mathrm{BGL}(R_{\mathfrak{U}}^U)^+)],$$

given by theorem 4.5 and Remark 4.6(ii), to give a natural special $K_0(S)$- λ-algebra structure on the homotopy groups of

$$\operatorname*{\overset{*}{holim}}_{<\mathfrak{U}>\times[0,1]} K_0(R^U_{\mathfrak{U}}/S)\,\tilde{\times}\,\mathrm{BGL}(R^U_{\mathfrak{U}})^+.$$

The first assertion then follows from theorem 5.3.

For the second, the naturality portion of theorem 4.5 implies that the sequence of functors on finite simplicial sets

$$[-,K^W(X)]\to[-,K(X)]\to[-,K(U)]$$

is a sequence of functors from finite simplicial sets to $K_0(S)$- λ-algebras. This shows that the long exact sequence (5.1) is a sequence of $K_0(S)$-λ- algebras. $\square$

Finally, if $Y_1,\ldots,Y_n$ are closed subschemes of X, we may form the n-cube of spaces $K(X;Y_1,\ldots,Y_n)_*$, with

$$K(X;Y_1,\ldots,Y_n)_I=K(\cap_{i\in I}Y_i);\qquad I\subset\{1,\ldots,n\}$$

The relative K-theory space $K(X;Y_1,\ldots,Y_n)$ defined as

$$K(X;Y_1,\ldots,Y_n)=\operatorname*{\overset{*}{holim}}_{[0,1]^n}(K(X;Y_1,\ldots,Y_n)_*).$$

If U is an open subset of X, with complement W, we have the relative K-theory space with supports, $K^W(X;Y_1,\ldots,Y_n)$, defined as the homotopy fiber of

$$K(X;Y_1,\ldots,Y_n)\to K(U;Y_1\cap U,\ldots,Y_n\cap U);$$

we may also consider $K^W(X;Y_1,\ldots,Y_n)$ as a (pointed) homotopy limit over the obvious $n+1$-cube. If we have an affine open cover of X as above, we may restrict this cover to each intersection $\cap_{i\in I}Y_i$ or $\cap_{i\in I}Y_i\cap U$, forming the ring

$$R^U_{\mathfrak{U}}(Y_1,\ldots,Y_n)$$

over $<\mathfrak{U}>\times[0,1]^n\times[0,1]=<\mathfrak{U}>\times[0,1]^{n+1}$. As above, the component of the identity of $K^W(X;Y_1,\ldots,Y_n)$ is given as the component of the identity of

$$\operatorname*{\overset{*}{holim}}_{<\mathfrak{U}>\times[0,1]^{n+1}} K_0(R^U_{\mathfrak{U}}(Y_1,\ldots,Y_n)/S)\,\tilde{\times}\,\mathrm{BGL}(R^U_{\mathfrak{U}}(Y_1,\ldots,Y_n))^+,$$

and $K_0^W(X;Y_1,\ldots,Y_n)$ may be computed as a limit over finite open covers. This gives as before

Corollary 5.6. *Let X be a noetherian scheme over a noetherian ring S, admitting an ample family of line bundles. There is a special $K_0(S)$-λ-algebra structure for relative K-theory with supports:*

$$K_p^W(X; Y_1, \ldots, Y_n),$$

which is natural in the tuple $(X, Y_1, \ldots, Y_n, W)$. The long exact relativization sequence

$$\ldots \to K_p^W(X; Y_1, \ldots, Y_n) \to K_p^W(X; Y_1, \ldots, Y_{n-1}) \\ \to K_p^{W\cap Y_n}(X \cap Y_n; Y_1 \cap Y_n, \ldots, Y_{n-1} \cap Y_n) \to \ldots$$

and the long exact localization sequence

$$\ldots \to K_p^W(X; Y_1, \ldots, Y_n) \to K_p(X; Y_1, \ldots, Y_n) \to K_p(X \backslash W; Y_1 \backslash W, \ldots, Y_n \backslash W) \to \ldots$$

are sequences of special $K_0(S)$-λ-algebras. □

Part II: Motivic cohomology

§6. Adams decompositions

For a set S of primes in $\mathbb{Z}$, let $\mathbb{Z}_S$ be the subring of $\mathbb{Q}$

$$\mathbb{Z}_S = \mathbb{Z}[\{\frac{1}{l} \mid l \in S\}].$$

Let $\mathcal{A}_S$ denote the category of $\mathbb{Z}_S$-modules M, together with a collection of endomorphisms

$$\psi_M^k : M \to M; \qquad k = 1, 2, \ldots$$

satisfying

$$\psi_M^k \circ \psi_M^l = \psi_M^{kl}$$

for all k and l. Morphisms $(M, \psi_M^*) \to (N, \psi_N^*)$ are $\mathbb{Z}_S$-module maps $\phi: M \to N$ with

$$\phi \circ \psi_M^k = \psi_N^k \circ \phi$$

for all k. Clearly, $\mathcal{A}_S$ is an abelian category, with forgetful functor to the abelian category of $\mathbb{Z}_S$-modules.

If N is an integer, we let $S(N)$ denote the set of primes p which divide N, and write $\mathcal{A}_N$ for $\mathcal{A}_{S(N)}$. Similarly, if T is a set of primes, we write (T) for the set of primes not in T, giving the category $\mathcal{A}_{(T)}$, and the category of $\mathbb{Z}_{(T)}$-modules. We write $\mathbb{Z}_{(l)}$ for $\mathbb{Z}_{(\{l\})}$ when l is a prime. We let ∞ denote the set of all primes.

Let M be a $\mathbb{Z}_S$-module, $\phi: M \to M$ a locally nilpotent endomorphism (for each $x \in M$, there is an N such that $\phi^N(x) = 0$). We have the increasing filtration on M

$$N_s(M, \phi) = \{m \in M \mid \phi^s(m) = 0\}$$

which is a finite filtration if and only if ϕ is nilpotent.

Let (M, ψ_M^*) be in $\mathcal{A}_S$. An *Adams decomposition of* (M, ψ_M^*) is a finite direct sum decomposition of (M, ψ_M^*) into subobjects in $\mathcal{A}_S$:

$$(M, \psi_M^*) = \oplus_{i=a}^b (M_i, \psi_{M_i}^*); \qquad 0 \leq a \leq b,$$

such that

i) $\psi_{M_i}^k - k^i \mathrm{id}_{M_i}$ is nilpotent.

ii) There is a finite, exhaustive, increasing filtration N_*M ($* \geq 0$) on M such that $N_sM \cap M_i \subset N_s(M_i, \phi_{M_i}^k)$ for all k.

We call $b - a + 1$ the *length* of the Adams decomposition; we call the summand $(M_i, \psi_{M_i}^*)$ the *summand of weight* i in the Adams decomposition, and say a decomposition of the form above has weights $a, \ldots, b$.

Let (M, ψ_M^*) be an object of $\mathcal{A}_S$. If $S' \supset S$, then an Adams decomposition of (M, ψ_M^*) induces an Adams decomposition of $(M \otimes \mathbb{Z}_{S'}, \psi_M^* \otimes \mathrm{id})$ by extension of scalars.

If (M, ψ_M^*) in $\mathcal{A}_S$ has an Adams decomposition as above, then the endomorphisms

$$\Psi_M^k(a, b) := \prod_{i=a}^b \psi_M^k - k^i \mathrm{id}_M$$

are all nilpotent, and $N_sM \subset N_s(M, \Psi_M^k(a, b))$ for all k. If we set $N_s^{\mathrm{can}}M$ equal to the intersection

$$N_s^{\mathrm{can}}M := \cap_k N_s(M, \Psi_M^k(a, b)),$$

then we may take $N_*M = N_*^{\mathrm{can}}M$ in (ii) above. We call $N_*^{\mathrm{can}}M$ the *canonical filtration* of (M, ψ_M^*), associated to the Adams decomposition $\oplus_{i=a}^b (M_i, \psi_{M_i}^*)$.

Lemma 6.1. *i) Let (M, ψ_M^*) be in $\mathcal{A}_\infty$. An Adams decomposition of (M, ψ_M^*) is unique.*

ii) Let l be a prime, and let (M, ψ_M^*) be in $\mathcal{A}_{(l)}$. Fix $0 \leq a \leq b$ with $b - a + 1 < l$. An Adams decomposition of (M, ψ_M^*) of the form

$$(M, \psi_M^*) = \oplus_{i=a}^b (M_i, \psi_{M_i}^*)$$

is unique

Proof. (i). Take any $k \geq 2$. If (M, ψ_M^*) has an Adams decomposition

$$(M, \psi_M^*) = \oplus_{i=a}^b (M_i, \psi_{M_i}^*)$$

then $\psi_M^k - k^i \mathrm{id}_M$ is nilpotent on M_i and invertible on M_j for $j \neq i$. Thus

$$M_i = \cup_{n=1}^\infty \ker[(\psi_M^k - k^i \mathrm{id}_M)^n]$$

This characterizes M_i.

(ii). The group of units in $\mathbb{Z}/l\mathbb{Z}$ is cyclic of order $l-1$, hence there is an integer k prime to l such that the powers k^i, $i = a, a+1, \ldots, b$ are distinct modulo l. Thus, if we have an Adams decomposition

$$(M, \psi_M^*) = \oplus_{i=a}^b (M_i, \psi_{M_i}^*)$$

then $\psi_M^k - k^i\mathrm{id}$ acts nilpotently on M_i and invertibly on M_j for $j \neq i$. We may then characterize M_i as above.

□

Lemma 6.2. *Let (M, ψ_M^*) be in $\mathcal{A}_S$, and suppose*

i) $(M \otimes \mathbb{Z}_{(l)}, \psi_M^* \otimes \mathrm{id})$ *has an Adams decomposition*

$$(M \otimes \mathbb{Z}_{(l)}, \psi_M^* \otimes \mathrm{id}) = \oplus_{i=a}^b (M_i^l, \psi_{M_i^l}^*)$$

for each prime l outside S, where a and b are independent of l.

ii) There is a finite increasing filtration N_*M $(* = 1, 2 \ldots)$ on M such that

$$N_s M \otimes \mathbb{Z}_{(l)} \subset N_s(M_i^l, \phi_{M_i^l}^k) \text{ for all } l \text{ and } k.$$
$$M = \cup_{s=1}^\infty N_s M.$$

Then there is a unique Adams decomposition of M:

$$(M, \psi_M^*) = \oplus_{i=a}^b (M_i, \psi_{M_i}^*)$$

such that

$$M_i \otimes \mathbb{Z}_{(l)} = M_i^l.$$

Proof. If $S = (l)$ for some prime l, or if $S = \infty$, there is nothing to prove, so we may assume there are at least two primes not in S.

For $a \leq i \leq b$, let M_i be the subset of M defined by

$$M_i = \{m \in M \mid m \otimes 1 \in M_i^l \text{ for all } l \notin S\}.$$

Then M_i is a $\mathbb{Z}_S$-submodule of M, stable under ψ_M^k for all k. Let p and q be distinct primes outside of S. By lemma 6.1, the $\mathbb{Q}$-subspaces $M_i^p \otimes \mathbb{Q}$ and $M_i^q \otimes \mathbb{Q}$ of $M \otimes \mathbb{Q}$ agree. Thus, by an elementary argument, we have

$$M_i \otimes \mathbb{Z}_{(l)} = M_i^l \tag{1}$$

for all $l \notin S$, and

$$M = \oplus_i M_i. \tag{2}$$

This gives the direct sum decomposition in $\mathcal{A}_S$

$$(M, \psi_M^*) = \oplus_{i=a}^{b} (M_i, \psi_{M_i}^*). \tag{3}$$

A similar argument shows that the direct sum decomposition (3) is the unique such which induces the given Adams decomposition of $M \otimes \mathbb{Z}_{(l)}$ for each l not in S. The assumption (ii) readily implies that $\psi_M^k - k^i \mathrm{id}$ is nilpotent on M_i for each k, and that (3) defines an Adams decomposition of (M, ψ^*). The uniqueness follows from the uniqueness of the direct sum decomposition (3). $\square$

Lemma 6.3. *Let N be a positive integer, let S be a set of primes containing all primes $p \leq N$, and let (M, ψ_M^*) be in $\mathcal{A}_S$. Let $0 \leq a \leq b$ be integers with $b - a < N$. Then an Adams decomposition of (M, ψ_M^*) as*

$$(M, \psi_M^*) = \oplus_{i=a}^{b} (M_i, \psi_{M_i}^*)$$

is unique.

Proof. This follows from lemma 6.1(ii) and lemma 6.2. $\square$

In particular, for an Adams decomposition as in lemma 6.3, the canonical filtration $N_*^{\mathrm{can}} M$ is an invariant of M.

Proposition 6.4. *Let $0 \leq a \leq b$ be integers, let S be a set of primes containing all primes $p \leq b - a + 1$, and let*

$$(A, \psi_A^*) \xrightarrow{\iota} (M, \psi_M^*) \xrightarrow{\pi} (B, \psi_B^*) \tag{1}$$

be an exact sequence in $\mathcal{A}_S$. Suppose we have Adams decompositions

$$(A, \psi_A^*) = \oplus_{i=a}^{b} (A_i, \psi_{A_i}^*); \qquad (B, \psi_B^*) = \oplus_{i=a}^{b} (B_i, \psi_{B_i}^*).$$

Then there is a unique Adams decomposition of (M, ψ_M^*) of the form

$$(M, \psi_M^*) = \oplus_{i=a}^{b} (M_i, \psi_{M_i}^*).$$

In addition, the sequence (1) induces exact sequences in $\mathcal{A}_S$

$$(A_i, \psi_{A_i}^*) \to (M_i, \psi_{M_i}^*) \to (B_i, \psi_{B_i}^*) \tag{1}$$

for each i.

Proof. The uniqueness follows from lemma 6.3. For existence, suppose $N_r A = A$. Set

$$N_s M = \begin{cases} \iota(N_s A) & \text{for } s = 1, \ldots, r \\ \pi^{-1}(N_{s-r} B) & \text{for } s = r+1, \ldots \end{cases}$$

Take a prime $l > N$. Write M^l for $M \otimes \mathbb{Z}_{(l)}$, $\psi^k_{M^l}$ for the endomorphism induced by ψ^k_M, and similarly for A and B. By lemma 6.2, it suffices to construct an Adams decomposition for $M \otimes \mathbb{Z}_{(l)}$ of the form

$$(M^l, \psi^*_{M^l}) = \oplus_{i=a}^b (M^l_i, \psi^*_{M_i}).$$

with the filtration $N_*M^l := N_*M \otimes \mathbb{Z}_{(l)}$ being finer than the canonical filtration.

For this, choose an integer k prime to l which gives a generator for $(\mathbb{Z}/l)^\times$, as in the proof of lemma 6.1(ii). Let M^l_i be given by

$$M^l_i = \cup_{n=1}^\infty \ker[(\psi^k_{M^l} - k^i\mathrm{id})^n].$$

The endomorphism $\psi^k_{A^l} - k^i\mathrm{id}$ is nilpotent on A^l_i, invertible on on A^l_j for $i \neq j$, and similarly for B^l. Thus, we have the exact sequence

$$A^l_i \to M^l_i \to B^l_i \tag{1}$$

and inclusion

$$N_sM^l_i \subset N_s(M^l_i, \psi^k_{M^l} - k^i) \tag{2}$$

for each $i = a, \ldots, b$.

We claim that

$$B^l_i \cap \pi(M^l) = \pi(M^l_i). \tag{3}$$

The inclusion $\supset$ is clear. To prove the inclusion $\subset$, take m in M^l, and suppose $\pi(m)$ is in B^l_i. There is an integer $N > 0$ such that $(\psi^k_{A^l} - k^j)^N$ is zero on A^l_j and $(\psi^k_{B^l} - k^j)^N$ is zero on B^l_j for all j. Let $P_N(T)$ be the polynomial

$$P_N(T) = T\sum_{i=0}^{N-1}(-1)^i(T-1)^i.$$

Letting

$$p^B_i := \prod_{j\neq i,\ a\leq j\leq b} P_N(\frac{(\psi^k_{B^l} - k^j)^N}{(k^i - k^j)^N})\colon B^l \to B^l,$$

p^B_i is the projection of B^l onto B^l_i. We have a similar formula for the projection p^A_i of A^l onto A^l_i. Let

$$p^M_i := \prod_{j\neq i,\ a\leq j\leq b} P_N(\frac{(\psi^k_{M^l} - k^j)^N}{(k^i - k^j)^N})\colon M^l \to M^l.$$

Then p^M_i commutes with $(\psi^k_{M^l} - k^i)$,

$$\pi(p^M_i(m)) = p^B_i(\pi(m)) = \pi(m),$$

and

$$\pi((\psi^k_{M^l} - k^i)^N(p^M_i(m))) = (\psi^k_{M^l} - k^i)^N(p^B_i(\pi(m))) = 0.$$

Thus $(\psi^k_{M^l} - k^i)^N(p_i^M(m))$ is in the image of A^l:

$$(\psi^k_{M^l} - k^i)^N(p_i^M(m)) = \iota(x); \quad x \in A^l.$$

We have the same identities with p_i^M replaced by $[p_i^M]^2$; in addition

$$\begin{aligned}(\psi^k_{M^l} - k^i)^{2N}([p_i^M]^2(m)) &= [(\psi^k_{M^l} - k^i)^N \circ p_i^M] \circ [(\psi^k_{M^l} - k^i)^N \circ p_i^M](m) \\ &= \iota((\psi^k_{A^l} - k^i)^N(p_i^A(x))) \\ &= 0.\end{aligned}$$

Thus $[p_i^M]^2(m)$ is an element of M_i^l lifting $\pi(m)$, proving (3).

By (3) and the exactness of (1), it follows that we have the direct sum decomposition of M^l as a $\mathbb{Z}_{(l)}[\psi^k_{M^l}]$-module

$$M^l = \oplus_{i=a}^b M_i^i. \tag{4}$$

Since $\psi^k_{M^l}$ and $\psi^t_{M^l}$ commute for all t and k, the decomposition (4) is a direct sum decompostion of $(M^l, \psi^*_{M^l})$ in $\mathcal{A}_{(l)}$, and we have the exact sequences in $\mathcal{A}_{(l)}$,

$$A_i^l \to M_i^l \to B_i^l,$$

for all $i = a, \ldots, b$. Since $\psi^t_{A^l} - t^i$ is nilpotent on A_i^l, and $\psi^t_{B^l} - t^i$ is nilpotent on B_i^l, it follows that $\psi^t_{M^l} - t^i$ is nilpotent on M_i^l; similarly, we have

$$N_s M_i^l \subset N_s(M_i^l, \psi^t_{M^l} - t^i)$$

for all s, t and i. Thus (4) gives the desired Adams decomposition of M^l. $\square$

We let $\mathcal{A}_S(a,b)$ denote the full subcategory of $\mathcal{A}_S$ consisting of (M, ψ^*_M) which admit an Adams decomposition with weights $a, \ldots, b$.

Theorem 6.5. *Let $0 \leq a \leq b$ be integers, and let S be a set of primes, containing all primes $p \leq b - a + 1$. Then*

1) The category $\mathcal{A}_S(a,b)$ is a strictly full abelian subcategory of $\mathcal{A}_S$.

2) The category $\mathcal{A}_S(a,b)$ is closed under extensions in $\mathcal{A}_S$.

3) Each object of $\mathcal{A}_S(a,b)$ has a unique Adams decomposition with weights $a, \ldots, b$.

4) Each morphism $f\colon (M, \psi^*_M) \to (N, \psi^*_N)$ in $\mathcal{A}_S(a,b)$ is strict, i.e.,

$$f(M_i) \subset N_i$$

for all i, where M_i and N_i are the weight i summands of M and N, respectively.

Proof. (3) follows from lemma 6.1 and lemma 6.2, and (2) follows from proposition 6.4. For (4), it suffices to show that

$$f(M_i \otimes \mathbb{Z}_{(l)}) \subset N_i \otimes \mathbb{Z}_{(l)}$$

for all primes l not in S, i.e., we may assume that $S = (l)$. In this case, for each object (A, ψ_A^*) of $\mathcal{A}_S(a, b)$, we have the projector p_i^A of A onto the weight i summand A_i, defined in the proof of proposition 6.4:

$$p_i^A := \prod_{j \neq i,\ a \leq j \leq b} P_N(\frac{(\psi_{A^l}^k - k^j)^N}{(k^i - k^j)^N}),$$

where N is any sufficiently large integer. Clearly, we have

$$p_i^N \circ f = f \circ p_i^M$$

for each morphism $f\colon (M, \psi_M^*) \to (N, \psi_N^*)$ in $\mathcal{A}_S(a, b)$, whence (4).

(1) follows easily from (4): If we have a morphism $f\colon (M, \psi_M^*) \to (N, \psi_N^*)$ in $\mathcal{A}_S(a, b)$, with kernel (A, ψ_A^*) and cokernel (B, ψ_B^*) in $\mathcal{A}_S$, from (3) we have

$$\begin{aligned} (A, \psi_A^*) &= \oplus_{i=a}^b (A \cap M_i, \psi_{A \cap M_i}) \\ (B, \psi_B^*) &= \oplus_{i=a}^b (\bar{N}_i, \psi_{\bar{N}_i}) \end{aligned}$$

where M_i and N_i are the weight i summands of M and N, and $\bar{N}_i$ is the image of N_i in B.

§7. Adams operations on relative K-theory with support

Let Y be a scheme, W, $D_1, \ldots, D_s$ closed subschemes. By corollary 5.6, we have natural λ-operations on the relative K-groups with support $K_*^W(Y; D_1, \ldots, D_s)$, which satisfy the special λ-ring identities. The resulting Adams operations ψ^k thus define the object

$$(K_p^W(Y; D_1, \ldots, D_s), \psi^*)$$

of $\mathcal{A}$. All the maps of groups of the form $K_*^W(Y; D_1, \ldots, D_s)$, arising from pull-back by morphisms of schemes, or from maps in the various long exact localization sequences or relativization sequences, define maps in $\mathcal{A}$.

Let Y be a scheme of finite type over a field k. We say that subschemes $D_1, \ldots, D_s$ define a *reduced normal crossing subscheme of* Y if each D_i has pure codimension, say $d_i > 0$ in Y, and each scheme-theoretic intersection

$$D_{i_1} \cap \ldots \cap D_{i_t}$$

(including the empty intersection) is smooth over k, and of pure codimension $\sum_j d_{i_j}$ on Y. For the multi-index $I = (i_1, \ldots, i_t)$, we let D_I denote the above intersection, and d_I the sum

$$d_I := \sum_j d_{i_j}.$$

We let $\mathcal{Z}^q(Y)$ denote the group of codimension q algebraic cycles on Y; for W a closed subset of Y, we let $\mathcal{Z}^q_W(Y)$ denote the subgroup of W consisting of cycles with support contained in W.

Lemma 7.1. *Let Y be a scheme finite type over a field k, W is a closed subset of Y of pure codimension q. Let $D_1, \ldots, D_s$ define a reduced normal crossing subscheme of Y, and suppose that*

$$W \cap D_I$$

has pure codimension q on D_I for each multi-index I. Suppose that W has pure dimension d over k. Then

i) the image of $K^W_p(Y; D_1, \ldots, D_s)$ in $\mathcal{A}_{(d+p+1-\epsilon)!}$ has an Adams decomposition with weights $q+\epsilon, \ldots, q+d+p$, where

$$\epsilon = \begin{cases} 0 & \text{for } p = 0 \\ 1 & \text{for } p = 1 \\ 2 & \text{for } p \geq 2 \end{cases}$$

ii) Letting

$$K^W_p(Y; D_1, \ldots, D_s)^{(a)}$$

denote the weight a-submodule of $K^W_p(Y; D_1, \ldots, D_s)[1/(d+p+1-\epsilon)!]$, we have the exact sequence of $\mathbb{Z}[1/(d+1)!]$-modules

$$0 \to K^W_0(Y; D_1, \ldots, D_s)^{(q)} \to K^W_0(Y)^{(q)} \to \oplus_{i=1}^s K_0^{W \cap D_i}(D_i)^{(q)}$$

iii) Let w denote the set of closed points in the semi-local scheme $\mathrm{Spec}(\mathcal{O}_{Y,W})$; sending an irreducible component W_i of W to the K_0-class of $\mathcal{O}_{W_i}$ gives the identification

$$K^w_0(\mathrm{Spec}(\mathcal{O}_{Y,W})) \cong \mathcal{Z}^q_W(Y).$$

Then the restriction map

$$K^W_0(Y) \to K^w_0(\mathrm{Spec}(\mathcal{O}_{Y,W}))$$

induces isomorphisms

$$K^W_0(Y)^{(q)} \cong K^w_0(\mathrm{Spec}(\mathcal{O}_{Y,W}))^{(q)} \cong \mathcal{Z}^q_W(Y)[1/(d+1)!].$$

of $\mathbb{Z}[1/(d+1)!]$-modules, and identifies the restriction maps in (2)

$$K^W_0(Y)^{(q)} \to K_0^{W \cap D_i}(D_i)^{(q)}$$

with the cycle intersection

$$Z \mapsto Z \cdot D_i.$$

Proof. (i)If $W = Y$, and $s = 0$, this is well-known (see e.g. [So]). More generally, suppose W is regular, and a codimension w complete intersection in Y, such that the inclusion

$$i: W \to Y$$

is split by a morphism

$$p: Y \to W$$

The structure sheaf $\mathcal{O}_W$ defines canonically a class $[\mathcal{O}_W]$ in $K_0^W(Y)$; as W is a complete intersection of codimension q, a direct computation shows that

$$\psi^k([\mathcal{O}_W]) = k^q[\mathcal{O}_W].$$

As the map

$$([\mathcal{O}_W] \cup (-)) \circ p^*: K_p(W) \to K_p^W(Y)$$

is an isomorphism, and the Adams operations are multiplicative, the result follows in this case as well. In general, W has a dense open subset W_0 such that there is an étale map

$$q: (Y', W_0) \to (Y, W)$$

which on W_0 is the inclusion $W_0 \to W$, with W_0 a closed codimension q split complete intersection in a smooth Y'. Let $\bar{W}$ be the complement $W \backslash W_0$. As q induces an isomorphism in $\mathcal{A}$

$$K_p^{W_0}(Y \backslash \bar{W}) \cong K_p^{W_0}(Y')$$

we have the result for $K_p^{W_0}(Y \backslash \bar{W})$. The result for general W and $s = 0$ follows by induction on dimension, using the exact sequence

$$K_p^{\bar{W}}(Y) \to K_p^W(Y) \to K_p^{W_0}(Y \backslash \bar{W})$$

and proposition 1.4. This completes the proof for $s = 0$.

For general s, we proceed by induction. We have the exact sequence

$$K_{p+1}^{W \cap D_s}(D_s; D_1 \cap D_s, \ldots, D_{s-1} \cap D_s) \to K_p^W(Y; D_1, \ldots, D_s) \to K_p^W(Y; D_1, \ldots, D_{s-1})$$

Since

$$\dim(W \cap D_s) = \dim W - d_s \leq d - 1$$

we may apply our induction hypothesis and proposition 1.4 to complete the proof of (i).

For (ii), we have the exact sequence

$$K_1^{W \cap D_s}(D_s; D_1 \cap D_s, \ldots, D_{s-1} \cap D_s) \to K_0^W(Y; D_1, \ldots, D_s) \to K_0^W(Y; D_1, \ldots, D_{s-1})$$

After inverting $(d+1)!$, this gives by (i) the exact sequence

$$\begin{aligned}0 = K_1^{W\cap D_s}(D_s; D_1\cap D_s, \ldots, D_{s-1}\cap D_s)^{(q)} &\to K_0^W(Y; D_1, \ldots, D_s)^{(q)}\\ &\to K_0^W(Y; D_1, \ldots, D_{s-1})^{(q)}.\end{aligned}$$

Thus, by induction, the map

$$K_0^W(Y; D_1, \ldots, D_s)^{(q)} \to K_0^W(Y)^{(q)}$$

is injective. This, together with the similar injectivity for the restrictions to D_I, proves (ii).

For (iii), we may apply the results of (i) to $K_0^w(\mathrm{Spec}(\mathcal{O}_{Y,W}))$ by taking a direct limit. As the kernel of the surjection

$$K_0^W(Y) \to K_0^w(\mathrm{Spec}(\mathcal{O}_{Y,W}))$$

comes from groups $K_0^{\bar{W}}(Y)$, with

$$\mathrm{codim}_Y(\bar{W}) > \mathrm{codim}_Y(W),$$

the map on the weight q submodule is thus an isomorphism. The final assertion in (iii) follows from Serre's intersection formula, and the fact that the appropriate higer Tor's vanish in this situation. □

Let A be a λ-ring, and let $a > 2$ be an integer. Form the operator

$$\phi^k_{\geq a} := \prod_{i=2}^{a-1}(\psi^k - k^i\mathrm{id}).$$

Lemma 7.2. *Let Y be a scheme, x an element of $F^2_\gamma K_0(Y)$. Then*

$$\phi^k_{\geq a}(x) \in F^a_\gamma K_0(Y).$$

Proof. This follows from the fact that $\psi^k - k^i\mathrm{id}$ maps $F^i_\gamma(K_0(Y))$ to $F^{i+1}_\gamma(K_0(Y))$. □

Lemma 7.3. *Let Y be a quasi-projective scheme over an infinite field k, $q \geq 1$ an integer, x an element of $F^q_\gamma K_0(Y)$, and $F_1, \ldots, F_s$ closed irreducible subsets of Y. Then there is a closed subset W of Y, of pure codimension q, such that*

i) $W \cap F_i$ has codimension q on F_i, for each $i = 1, \ldots, s$.

ii) There is an element $x_W \in K_0^W(Y)$ with image x in $K_0(Y)$.

Proof. Let E be a vector bundle. Since Y is quasi-projective, there is a map $f\colon Y \to H$, into a homogeneous space H for a linear reductive algebraic group G (G can be taken to be a product of general linear groups), and a vector bundle E_H of rank n on H such that

$$E \cong f^*(E_H)$$

Taking products of such maps, we see that the same holds true for each finite set of vector bundles on Y. This implies that there is a map $f\colon Y \to H$ as above, and an element y of $F^q_\gamma(H)$ with

$$f^*(y) = x.$$

As H is smooth over k, the result of ([G], Chap. II, §5, theorem 2.16) implies that y is in the image of $K_0^T(H)$ for some pure codimension q closed subset T of H:

$$y = i_{T*}(y_T); \quad y_T \in K_0^T(H).$$

As G acts trivially on $K_0(H)$, we may assume ([Kl], theorem 2) that $W := f^{-1}(T)$ has pure codimension q on Y, and intersects each F_i in codimension q. Taking

$$x_W = f^*(y_T) \in K_0^W(Y)$$

completes the proof. □

For a noetherian scheme Y, define the groups $\tilde{S}K_p(Y)$ as follows: For a ring R, we have the simplicial set $\mathrm{BSL}(R)^+$ (which we make into a functor as for $\mathrm{BGL}(R)^+$). Take a finite affine cover

$$\mathfrak{U} := \{\mathrm{Spec}(R_i) \mid i \in I\}$$

of Y, and set

$$\mathrm{B\tilde{S}L}(\mathfrak{U})^+ = \operatorname*{holim}_I \mathrm{B\tilde{G}L}(-)^+$$

A refinement $\mathfrak{V} \to \mathfrak{U}$ induces a map on the homotopy groups

$$\pi_*(\mathrm{B\tilde{S}L}(\mathfrak{U})^+) \to \pi_*(\mathrm{B\tilde{S}L}(\mathfrak{V})^+)$$

which is independent of the choice of refinement mapping. We then set

$$\tilde{S}K_p(Y) := \varinjlim_{\mathfrak{U}} \pi_*(\mathrm{B\tilde{S}L}(\mathfrak{U})^+)$$

The generalization to groups with support and the relative version are defined similarly, by taking compatible collections of open covers of the various schemes involved. The natural H-group structure on $\mathrm{BSL}(-)$ gives $\mathrm{B\tilde{S}L}(\mathfrak{U})^+$ the structure of an H-group, which in turn makes $\tilde{S}K_p(-)$ into an abelian group-valued functor, for all $p \geq 0$.

We define the functor $K^{(1)}(-)$ similarly, by using $\mathrm{B}\mathbb{G}_m = \mathrm{B}\mathbb{G}_m^+$ instead of $\mathrm{BSL}(-)^+$. The sequence of group schemes

$$\mathrm{SL}_n \to \mathrm{GL}_n \to \mathbb{G}_m$$

gives rise to the sequence of functors

$$\tilde{S}K_p(-) \to \tilde{K}_p(-) \to K_p^{(1)}(-).$$

where

$$\tilde{K}_p(-) = \begin{cases} K_p(-) & \text{for } p > 0 \\ \ker[\mathrm{rnk}\colon K_0(-) \to H^0(-,\mathbb{Z})] & \text{for } p = 0 \end{cases}$$

Theorem 7.4. *Let Y be a k scheme of finite type, W and $D_1, \dots, D_s$ closed subschemes.*

i) The sequence of abelian groups

$$0 \to \tilde{S}K_p^W(Y; D_1, \dots, D_s) \to \tilde{K}_p^W(Y; D_1, \dots, D_s) \to K_p^{(1)W}(Y; D_1, \dots, D_s) \to 0$$

is exact for all $p \geq 0$, and split for $p \geq 1$.
ii) Let

$$i\colon D := \cup_{i=1}^s D_i \to Y$$

be the inclusion, $j\colon U \to Y$ the inclusion of the complement

$$U := Y \backslash D,$$

and let $j_!\mathcal{O}_Y^*$ be the subsheaf of $\mathcal{O}_Y^*$ on Y given by the exactness of

$$0 \to j_!\mathcal{O}_Y^* \to \mathcal{O}_Y^* \to i_*\mathcal{O}_D^* \to 0.$$

Then $K_p^{(1)W}(Y; D_1, \dots, D_s)$ is 0 for $p \geq 2$; for $p \leq 1$, we have

$$K_1^{(1)}(Y; D_1, \dots, D_s) = \ker[\Gamma(Y, \mathcal{O}_Y^*) \to \prod_{i=1}^s \Gamma(D_i, \mathcal{O}_{D_i}^*)$$
$$K_0^{(1)}(Y; D_1, \dots, D_s) = H^1_{\mathrm{Zar}}(Y, j_!\mathcal{O}_Y^*)$$

Proof. Let $\mathrm{E}(R)$ be the group of elementary matrices. It follows from the definition of the $+$ contruction that the sequence

$$\mathrm{BE}(R)^+ \to \mathrm{BSL}(R)^+ \to \mathrm{BGL}(R)^+$$

is homotopy equivalent to a sequence of covering spaces of $\mathrm{BGL}(R)^+$, corresponding to the sequence of subgroups

$$\{0\} \subset \mathrm{SL}(R)/\mathrm{E}(R) \subset K_1(R) = \mathrm{GL}(R)/\mathrm{E}(R).$$

In particular, the sequence

$$\mathrm{BSL}(R)^+ \to \mathrm{BGL}(R)^+ \xrightarrow{\det^+} \mathrm{B}\mathbb{G}_m(R) \tag{1}$$

is a homotopy fiber sequence.

The stabilization map

$$\mathrm{B}\mathbb{G}_m(R) \to \mathrm{BGL}(R) \tag{2}$$

gives a natural section s to the sequence (1); since $\mathrm{BGL}(R)^+$ is an H-group, we have the homotopy equivalence

$$\mathrm{BGL}(R)^+) \sim \mathrm{BSL}(R)^+) \times \mathrm{B}\mathbb{G}_m(R)$$

Let

$$A: I \to \mathbf{Rings}$$

be a functor, giving the functors

$$\mathrm{BSL}(A)^+: I \to \mathcal{S}^*, \quad \mathrm{BGL}(A)^+: I \to \mathcal{S}^*, \quad \mathrm{B}\mathbb{G}_m(A): I \to \mathcal{S}^*,$$

and the sequence

$$\mathrm{BSL}(A)^+ \to \mathrm{BGL}(A)^+ \to \mathrm{B}\mathbb{G}_m(A) \tag{3}$$

of functors. The sequence (3) is weakly equivalent to a homotopy fiber sequence. The splitting (2) gives a splitting of (3); this and the H-group structure on $\mathrm{BGL}(A)^+$ defines a weak equivalence of bifibrant objects

$$\mathrm{B\tilde{G}L}(A)^+ \to \mathrm{B\tilde{S}L}(A)^+ \times \tilde{\mathrm{B}}\mathbb{G}_m(A). \tag{4}$$

Taking the homotopy limit over I gives ([B-K], XI, §5, lemma 5.6) the weak equivalence

$$\operatorname*{holim}_I \mathrm{B\tilde{S}L}(A)^+ \times \operatorname*{holim}_I \tilde{\mathrm{B}}\mathbb{G}_m(A) \sim \operatorname*{holim}_I \mathrm{B\tilde{G}L}(A)^+;$$

this proves (i).

For (ii), we note that the groups $K_p^{(1)}(-)$ are given by

$$\begin{aligned} K_1^{(1)}(Y) &= \Gamma(Y, \mathcal{O}_Y^*) \\ K_0^{(1)}(Y) &= H^1(Y, \mathcal{O}_Y^*) \end{aligned} \tag{5}$$

and $K_p^{(1)}(Y) = 0$ for $p > 1$. Indeed, the last equality follows from the spectral sequence of theorem B.1, which degenerates at E_2 (we avoid the fringe effects by taking deloopings of $\mathrm{B}\mathbb{G}_m$). The same spectral sequence gives

$$\pi_1 \operatorname*{holim}_I \mathrm{B}\mathbb{G}_m(A) = \varprojlim_I \mathbb{G}_m(A)$$

$$\pi_0 \operatorname*{holim}_I \mathrm{B}\mathbb{G}_m(A) = {\varprojlim_I}^1 \mathbb{G}_m(A)$$

For A the functor associated to an affine open cover $\mathfrak{U}$ of Y, the lim term is $H^0(\mathfrak{U}, \mathcal{O}_Y^*)$, and the $\lim^1$ term is $H^1(\mathfrak{U}, \mathcal{O}_Y^*)$, where $H^0(\mathfrak{U}, -)$ is the Čech cohomology with respect to the cover $\mathfrak{U}$. As the Zariski cohomology of a sheaf of abelian gorups is computable via Čech cohomology, for degrees ≤ 1, we arrive at the canonical identites (5). The computation of relative groups $K_p^{(1)}(Y; D_1, \ldots, D_s)$ follows from the case $s = 0$ by noting that the inclusion

$$j_! \mathcal{O}_Y^* \to \mathcal{O}_Y^*$$

induces a natural map

$$H^*_{\check{\mathrm{C}}\mathrm{ech}}(Y, j_! \mathcal{O}_Y^*) \to K^{(1)}_{1-*}(Y; D_1, \ldots, D_s)$$

and using the respective long exact relativization sequences. □

Corollary 7.5. *Let $D_1, \ldots, D_s$ define a reduced normal crossing subscheme of a finite type k-scheme Y, and let W be a closed subscheme of Y. Then*

i) the natural map

$$\tilde{S}K_*^W(Y; D_1, \ldots, D_s) \to \tilde{S}K_*^{W \times \mathbb{A}^1}(Y \times \mathbb{A}^1; D_1 \times \mathbb{A}^1, \ldots, D_s \times \mathbb{A}^1)$$

is an isomorphism.

ii) for all $p \geq 0$, there is a natural isomorphism of λ-rings (without unit)

$$\tilde{S}K_p^{W \times \mathbb{A}^1}(Y \times \mathbb{A}^1; D_1 \times \mathbb{A}^1, \ldots, D_s \times \mathbb{A}^1, Y \times \{0, 1\}) \to \tilde{S}K_{p+1}^W(Y; D_1, \ldots, D_s).$$

Proof. The second assertion follows from the first by the usual long exact relativization sequence, and the naturality of the λ-ring operations. The first assertion follows from the homotopy invariance of Quillen K-theory and homotopy invariance for the functors Pic and $\Gamma(-, \mathcal{O}^*)$, for regular schemes, together with theorem 7.4. □

Lemma 7.6. *Let $D_1, \ldots, D_s$ define a reduced normal crossing subscheme of an irreducible finite type k-scheme Y, and let W be a closed subscheme of Y. Suppose that W has pure codimension $q \geq 2$ on Y, and that $W \cap D_I$ has codimension q on D_I for each multi-index I. Let U be the complement of W in Y, and $E_i = D_i \cap U$. Let x be in kernel of the map*

$$K_0^W(Y; D_1, \ldots, D_s) \to K_0(Y; D_1, \ldots, D_s)$$

Then there is an element η in $\tilde{S}K_1(U; E_1, \ldots, E_s)$ mapping to x via the boundary map in the long exact localization sequence

$$\to K_1(U; E_1, \ldots, E_s) \to K_0^W(Y; D_1, \ldots, D_s) \to K_0(Y; D_1, \ldots, D_s)$$

Proof. Since Y is regular and irreducible, and W has codimension at least 2, the map

$$\Gamma(Y, \mathcal{O}_Y^*) \to \Gamma(U, \mathcal{O}_U^*)$$

is an isomorphism. This implies that the map

$$K_1^{(1)}(Y; D_1, \dots, D_s) \to K_1^{(1)}(U; E_1, \dots, E_s)$$

is an isomorphism as well. Similarly, we have the identities

$$\tilde{K}_0^W(Y; D_1, \dots, D_s) = K_0^W(Y; D_1, \dots, D_s); \qquad \tilde{K}_1(U; E_1, \dots, E_s) = K_1(U; E_1, \dots, E_s).$$

The lemma then follows from the commutative diagram with exact rows and split exact columns

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\tilde{S}K_1(Y; D_1, \dots, D_s) & \to & \tilde{S}K_1(U; E_1, \dots, E_s) & \to & \tilde{S}K_0^W(Y; D_1, \dots, D_s) \\
\downarrow & & \downarrow & & \downarrow \\
\tilde{K}_1(Y; D_1, \dots, D_s) & \to & \tilde{K}_1(U; E_1, \dots, E_s) & \to & \tilde{K}_0^W(Y; D_1, \dots, D_s) \\
\downarrow & & \downarrow & & \downarrow \\
K_1^{(1)}(Y; D_1, \dots, D_s) & \to & K_1^{(1)}(U; E_1, \dots, E_s) & \to & K_0^{(1)W}(Y; D_1, \dots, D_s) \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

$\square$

It follows from Remark 4.6(iii) that the functor $\tilde{S}K_0(-)$ has the natural structure of a functor from schemes to λ-rings (without unit), and that the map

$$\tilde{S}K_p(-) \to \tilde{K}_p(-)$$

is a map of functors to λ-rings (without unit). Similarly, the identity

$$\det(\lambda^k(\rho - \mathrm{id})) = \lambda^k(\det(\rho)),$$

for a representation

$$\rho\colon G \to \mathrm{GL}_n(R)$$

implies that the map

$$\tilde{K}_p(-) \to K_p^{(1)}(-)$$

is a map of functors to λ-rings without unit, where we give $K_p^{(1)}(-)$ the operations

$$\lambda^k = \begin{cases} \mathrm{id} & \text{for } k = 1, \\ 0 & \text{for } k > 1. \end{cases}$$

Lemma 7.7. *Let Y be a k-scheme of finite type. Then*

$$\tilde{S}K_p(Y) = F_\gamma^2 K_p(Y)$$

for all p.

Proof. The identity

$$\gamma_1 + \ldots + \gamma_{n+1} = 0$$

in the representation ring of SL_n, together with Remark 4.6(iii), implies the inclusion

$$\tilde{S}K_p(Y) \subset F^2_\gamma K_p(Y)$$

for all p. Conversely, since $\lambda^k = 0$ on $K_p^{(1)}(Y)$ for all $k > 1$, it follows that

$$F^2_\gamma K_p^{(1)}(Y) = 0,$$

hence

$$F^2_\gamma K_p(Y) \subset \ker(\tilde{K}_p(Y) \to K_p^{(1)}(Y)) = \tilde{S}K_p(Y).$$

□

§8. K-theory and motivic cohomology

We conclude with an application of the results of the two previous sections to a comparison of K-theory and motivic cohomology, the latter defined as Bloch's higher Chow groups $\mathrm{CH}^q(X,p)$. In [L], we have shown that there is a natural isomorphism

$$\mathrm{CH}^q(X,p)_{\mathbb{Q}} \cong K_p(X)^{(q)}_{\mathbb{Q}} \tag{8.1}$$

for X smooth and quasi-projective over a field.

The isomorphism (8.1) uses a cubical version $\mathcal{Z}^q(X,*)_c$ of Bloch's simplicial complex $\mathcal{Z}^q(X,*)$. In ([L], §3) it is shown that there is a natural isomorphism

$$\mathcal{Z}^q(X,*)_c \to \mathcal{Z}^q(X,*)$$

in the derived category, giving the natural isomorphism

$$H_*(\mathcal{Z}^q(X,*)_c) \cong \mathrm{CH}^q(X,p)$$

We let $Z_p(\mathcal{Z}^q(X,*)_c)$ denote the (homological) cycles of dimension p in the complex $\mathcal{Z}^q(X,*)_c$.

We refine the isomorphism (8.1) to the following result:

Theorem 8.1. *Let X be a smooth, quasi-projective variety over a field k, of dimension d over k. Let $(p \geq 0, q \geq 0)$ be a pair of integers, with $(p,q) \neq (0,0), (0,1)$.*

i) Let

$$M_{p,q,d} = \begin{cases} d+p-q+1 & \text{for } q > 1 \\ 1 & \text{for } q = 0, 1;\ p > 0. \end{cases}$$

Then there are natural maps

$$\mathrm{cl}^{q,p}\colon \mathrm{CH}^q(X,p)[\frac{1}{M_{p,q,d}!}] \to K_p(X)[\frac{1}{M_{p,q,d}!}].$$

ii) Let

$$N_{p,d} = \begin{cases} \max(d+p-q+1, q-1) & \text{for } q > 1 \\ 1 & \text{for } q = 0,1;\ p > 0. \end{cases}$$

Then

$$\mathrm{cl}^{q,p}(\mathrm{CH}^q(X,p)[\frac{1}{N_{p,q,d}!}]) \subset F^q_\gamma K_p(X)[\frac{1}{N_{p,d}!}],$$

and the induced map

$$\mathrm{cl}^{q,p}\colon \mathrm{CH}^q(X,p)[\frac{1}{N_{p,q,d}!}] \to \mathrm{gr}^q_\gamma K_p(X)[\frac{1}{N_{p,d}!}]$$

is an isomorphism.

□

As immediate corollary, we have

Corollary 8.2. *Let X be a smooth, quasi-projective variety of dimension d over a field k. Then there are natural graded isomorphisms*

$$\oplus_q \mathrm{CH}^q(X,p)[\frac{1}{(d+p-1)!}] \to \oplus_q \mathrm{gr}^q_\gamma K_p(X)[\frac{1}{(d+p-1)!}].$$

Proof. For $(p,q) \neq (0,0),(0,1)$, this follows from theorem 8.1. From theorem 7.4 and lemma 7.7, (this is of course well-known) we have the natural isomorphism

$$\mathrm{gr}^1_\gamma K_0(X) \cong \mathrm{Pic}(X);$$

by definition of F^1, we have

$$\mathrm{gr}^0_\gamma K_0(X) \cong H^0(X,\mathbb{Z}).$$

The result then follows from the natural isomorphisms

$$\mathrm{CH}^1(X,0) = \mathrm{CH}^1(X) \cong \mathrm{Pic}(X); \quad \mathrm{CH}^0(X,0) \cong H^0(X,\mathbb{Z}).$$

□

The proof of theorem 8.1 is essentially the same as the proof of the rational result in [L]; we merely use the results of §6 and §7 to make the refinement.

For the convenience of the reader, we give a sketch of the proof of the isomorphism (8.1):

Let $\square^n$ be the affine space $\mathbb{A}^n_k$, let $\partial\square^n$ be the collection of divisors

$$D_i^\epsilon : t_i = \epsilon; i = 1, \ldots, n; \epsilon = 0, 1,$$

and let

$$\partial_0\square^n = \partial\square^n \setminus D_n^0.$$

The group $\mathcal{Z}^q(X, n)_c$ is the subgroup of the group of codimension q cycles on $X \times \square^n$, consisting of those cycles Z such that

i) each irreducible component W of Z intersectes each "face":

$$X \times [D_{i_1}^{\epsilon_1} \cap \ldots \cap D_{i_s}^{\epsilon_s}]$$

properly.

ii)

$$Z \cdot D_i^\epsilon = 0$$

for $\epsilon = 1$, $i = 1, \ldots, n$ and for $\epsilon = 0$, $i = 1, \ldots, n-1$.

Identifying $\square^{n-1}$ with D_n^0 in the obvious way, the restriction to D_n^0 defines the map

$$d_{n-1} \colon \mathcal{Z}^q(X, n)_c \to \mathcal{Z}^q(X, n-1)_c,$$

giving the complex $(\mathcal{Z}^q(X, *)_c, d)$. We denote the homology $H_p(\mathcal{Z}^q(X, *)_c)$ by $\mathrm{CH}^q(X, p)_c$. As mentioned above, there is a canonical isomorphism

$$\mathrm{CH}^q(X, p)_c \cong \mathrm{CH}^q(X, p). \tag{8.2}$$

Step 1: Let T be a smooth k-scheme, $D_1, \ldots, D_n$ subschemes forming a reduced normal crossing divisor, and W a closed, codimension q subset which intersects each D_I properly. By lemma 7.1, we have the isomorphism

$$K_0^W(T)^{(q)}_{\mathbb{Q}} \cong \mathcal{Z}^q_W(T)_{\mathbb{Q}}$$

and the exact sequence

$$0 \to K_0^W(T; D_1 \ldots, D_n)^{(q)}_{\mathbb{Q}} \to \mathcal{Z}^q_W(T)_{\mathbb{Q}} \xrightarrow{\cap} \oplus_{i=1}^n \mathcal{Z}^q_{W \cap D_i}(D_i)_{\mathbb{Q}}, \tag{8.3}$$

where $\cap$ is the map

$$Z \mapsto \sum_{i=1}^n Z \cdot D_i.$$

Let $K_r^{[q]}(X \times \square^p; \partial\square^p)$ be the direct limit

$$K_r^{[q]}(X \times \square^p; X \times \partial\square^p) := \lim_{\rightarrow} K_r^W(X \times \square^p; X \times \partial\square^p)$$

over closed subsets W of $X \times \square^p$ of codimension q, such that each component of W intersects each face of $X \times \square^p$ in codimension q. Similarly, let $K_r^{[q]}(X \times \square^p; \partial_0\square^p)$ be the direct limit

$$K_r^{[q]}(X \times \square^p; X \times \partial_0\square^p) := \lim_{\rightarrow} K_r^W(X \times \square^p; X \times \partial_0\square^p)$$

over the same set of W. By (8.3), we have the canonical isomorphisms

$$\begin{aligned} K_0^{[q]}(X \times \square^p; X \times \partial\square^p)_{\mathbb{Q}}^{(q)} &\cong Z_p(\mathcal{Z}^q(X,*)_{c\mathbb{Q}}), \\ K_0^{[q]}(X \times \square^{p+1}; X \times \partial_0\square^{p+1})_{\mathbb{Q}}^{(q)} &\cong \mathcal{Z}^q(X, p+1)_{c\mathbb{Q}}, \end{aligned} \tag{8.4}$$

In particular, the natural map

$$K_0^{[q]}(X \times \square^p; X \times \partial\square^p)_{\mathbb{Q}}^{(q)} \to K_0(X \times \square^p; X \times \partial\square^p)_{\mathbb{Q}}^{(q)}$$

combined with (8.4) defines the map

$$\mathrm{cyc}^{q,p}\colon Z_p(\mathcal{Z}^q(X,*)_{c\mathbb{Q}}) \to K_0(X \times \square^p; X \times \partial\square^p)_{\mathbb{Q}}^{(q)}. \tag{8.5}$$

As

$$K_0(X \times \square^{p+1}; X \times \partial_0\square^{p+1}) = 0,$$

$\mathrm{cyc}^{q,p}$ descends to

$$\mathrm{cl}^{q,p}\colon \mathrm{CH}^q(X,*)_{c\mathbb{Q}} \to K_0(X \times \square^p; X \times \partial\square^p)_{\mathbb{Q}}^{(q)} \cong K_p(X)_{\mathbb{Q}}^{(q)}. \tag{8.6}$$

Step 2: Let T be a smooth k-scheme, $D_1, \ldots, D_n$ a reduced normal crossing divisor on T. Form the *n-fold double of T along* $D_1, \ldots, D_n$ by gluing together 2^n copies of T (indexed by the subsets of $\{1, \ldots, n\}$) by identifying T_I with T_J along the common subset $\cap_{j \in J \setminus I} D_i$, for all $I \subset J$. We denote the n-fold double by $\mathcal{D}(T; D_1, \ldots, D_n)$. Let $\mathcal{D}^i$ denote the subscheme of $\mathcal{D}(T; D_1, \ldots, D_n)$ formed as the union of the T_I with $i \in I$. Then restriction to the component $T_\emptyset$ of $\mathcal{D}$ defines the map of relative K-groups

$$K_p(\mathcal{D}; \mathcal{D}^1, \ldots, \mathcal{D}^n) \to K_p(T; D_1, \ldots, D_n)$$

In addition, the map

$$K_p(\mathcal{D}; \mathcal{D}^1, \ldots, \mathcal{D}^n) \to K_p(\mathcal{D}) \tag{8.7}$$

is a split injection for all p, with splitting given by a certain sequence of splittings of maps of schemes (see [L], §1). In particular, the splitting (8.7) defines a splitting of the natural maps

$$\tilde{S}K_p(\mathcal{D};\mathcal{D}^1,\ldots,\mathcal{D}^n)\to\tilde{S}K_p(\mathcal{D}).$$

By ([L], corollary 1.11) the restriction map

$$K_0(\mathcal{D};\mathcal{D}^1,\ldots,\mathcal{D}^n)\to K_0(T;D_1,\ldots,D_n) \tag{8.8}$$

is an isomorphism, if there is a map

$$T\to S$$

to a smooth affine k-scheme S, such that the divisors $D_1,\ldots,D_n$ are pull backs of reduced normal crossing divisors $E_1,\ldots,E_n$ on S, and the collection of divisors $E_1,\ldots,E_n$ is *locally split* on S (see the statement of [L], corollary 1.11 for a precise definition). In this case, one has the following result similar to lemma 7.3 ([L], theorem 2.3),

Lemma. *Let x be an element of $K_0(T;D_1,\ldots,D_n)^{(q)}_{\mathbb{Q}}$. Then there is a closed, codimension q subset W of T, intersecting each D_I properly, such that x is in the image*

$$K_0^W(T;D_1,\ldots,D_n)^{(q)}_{\mathbb{Q}}\to K_0(T;D_1,\ldots,D_n)^{(q)}_{\mathbb{Q}}.$$

□

From this lemma, one easily shows that the map (8.6) is surjective.

Step 3: To show that (8.6) is injective, suppose

$$\mathrm{cyc}^{q,p}(Z)=0$$

for some Z. Let $W\subset X\times\square^n$ be the support of Z, and let $U=X\times\square^n\backslash W$. By the localization sequence

$$K_1(U;U\times\partial\square^n)^{(q)}_{\mathbb{Q}}\xrightarrow{\delta}K_0^W(X\times\square^n,X\times\partial\square^n)^{(q)}_{\mathbb{Q}}\to K_0(X\times\square^n,X\times\partial\square^n)^{(q)}_{\mathbb{Q}},$$

there is an element $\eta\in K_1(U;U\times\partial\square^n)^{(q)}_{\mathbb{Q}}$ with

$$\delta(\eta)=\mathrm{cyc}^{q,p}(Z). \tag{8.9}$$

By the homotopy property, we identify the above K_1 with

$$K_0(U\times\mathbb{A}^1;U\times\partial\square^n\times\mathbb{A}^1,U\times\{0,1\})^{(q)}_{\mathbb{Q}}$$

giving the element $\tilde{\eta}$ corresponding to η. We then use the lemma of Step 2 to find a closed, codimension q subset W_U of $U\times\mathbb{A}^1$, intersecting all "faces" in $U\times\partial\square^p\times\mathbb{A}^1,U\times\{0,1\}$

properly, and an element b of

$$K_0^{W_U}(U\times\mathbb{A}^1;U\times\partial\square^p\times\mathbb{A}^1,U\times\{0,1\})^{(q)}_{\mathbb{Q}}$$

lifting $\tilde{\eta}$. One checks that the closure W of W_U in $X\times\square^{n+1}$ intersects all faces in $X\times\partial\square^{p+1}$ properly. The element b thus defines, via the isomorphism (8.2), an element B of $(\mathcal{Z}^q(X,n+1)_{c\mathbb{Q}}$. The relation (8.9) implies (see [L], lemma 2.6)

$$d_p(B)=Z,$$

which gives the injectivity of the map (8.6). $\square$

We now show how one modifies this argument to prove theorem 8.1. The cases $q=0,1$ are well known (see [Bl], section 6), so we assume that $q\geq 2$.

First of all, let $(T;D_1,\ldots,D_n)$ be as in (8.3), and let $d_T=\dim_k(T)$. By lemma 7.1, we have the isomorphism

$$K_0^W(T)[\frac{1}{(d_T-q+1)!}](q)\cong\mathcal{Z}^q_W(T)[\frac{1}{(d_T-q+1)!}]$$

and the exact sequence

$$0\to K_0^W(T;D_1\ldots,D_n)[\frac{1}{(d_T-q+1)!}]^{(q)}\to\mathcal{Z}^q_W(T)[\frac{1}{(d_T-q+1)!}]$$
$$\xrightarrow{\cap}\oplus_{i=1}^n\,\mathcal{Z}^q_{W\cap D_i}(D_i)[\frac{1}{(d_T-q+1)!}],$$

as in (8.3). As in step 1, this gives us the isomorphism

$$K_0^{[q]}(X\times\square^p;X\times\partial\square^p)[\frac{1}{(d+p-q+1)!}]^{(q)}\cong Z_p(\mathcal{Z}^q(X,*)_c[\frac{1}{(d+p-q+1)!}])\qquad(8.4)'$$

and thus the map

$$\mathrm{cyc}'^{q,p}\colon Z_p(\mathcal{Z}^q(X,*)_c[\frac{1}{(d+p-q+1)!}])\to K_0(X\times\square^p;X\times\partial\square^p)[\frac{1}{(d+p-q+1)!}]^{(q)}.\qquad(8.5)'$$

Next, we note the following

Lemma 8.3. *For all n, p, q and X, the map*

$$K_p^{[q]}(X\times\square^n;X\times\partial_0\square^n)\to K_p^{[q]}(X\times\square^n)$$

is injective.

Proof. Let $E_1, \ldots, E_{2n-1}$ be the divisors

$$E_i = \begin{cases} D_i^1 & i = 1, \ldots, n \\ D_{n-i}^0 & i = n+1, \ldots 2n-1. \end{cases}$$

It suffices to show that the maps

$$K_p^{[q]}(X \times \square^n; X \times \{E_1, \ldots, E_{i+1}\}) \to K_p^{[q]}(X \times \square^n; X \times \{E_1, \ldots, E_i\})$$

is injective for all i.

Suppose first that $i \leq n$. The inclusion

$$\iota_i^1 \colon \square^{n-1} \cong D_i^1 \to \square^n$$

is split by the projection p_i:

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n).$$

As p_i sends D_j^1 to $D_i^1 \cap D_j^1$, and p_i is flat, p_i defines a splitting in the relativization sequence

$$\begin{aligned} \ldots \to K_{p+1}^{[q]}&(X \times D_i^1; X \times \{D_i^1 \cap D_1^1, \ldots, D_i^1 \cap D_{i-1}^1\}) \\ &\to K_p^{[q]}(X \times \square^n; X \times \{D_1^1, \ldots, D_i^1\}) \\ &\quad \to K_p^{[q]}(X \times \square^n; X \times \{D_1^1, \ldots, D_{i-1}^1\}) \to \ldots \end{aligned}$$

giving the injectivity in this case. If $n < i \leq 2n-1$, then the inclusion

$$D_i^0 \to \square^n$$

is split by the flat map q_i:

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{i-1}, 0, (1-x_i)(x_{i+1}-1)+1, x_{i+2}, \ldots, x_n).$$

Then

$$q_i(D_j^1) = \begin{cases} D_i^0 \cap D_j^1 & \text{for } j \neq i, \\ D_i^0 \cap D_{i+1}^i & \text{for } j = i, \end{cases}$$

$$q_i(D_j^0) = D_i^0 \cap D_j^1 \quad \text{for } 1 \leq j < i.$$

Thus q_i defines a splitting in the relativization sequence

$$\begin{aligned} \ldots \to K_{p+1}^{[q]}&(X \times E_{n+i}; X \times \{E_{n+i} \cap E_1, \ldots, E_{n+i} \cap E_{n+i-1}\}) \\ &\to K_p^{[q]}(X \times \square^n; X \times \{E_1, \ldots, E_{n+i}\}) \\ &\quad \to K_p^{[q]}(X \times \square^n; X \times \{E_1, \ldots, E_{n+i-1}\}) \to \ldots \end{aligned}$$

$\square$

Lemma 8.4. *The map (8.5)′ descends to the map*

$$\mathrm{cl}'^{q,p}\colon \mathrm{CH}^q(X,*)_c[\frac{1}{(d+p-q+1)!}] \to K_0(X\times\square^p; X\times\partial\square^p)[\frac{1}{(d+p-q+1)!}]^{(q)}. \qquad (8.6)'$$

Proof. Fix a prime $l > d+p-q+1$, and let k be a generator of $\mathbb{Z}/l^\times$. Using lemma 8.3, the map sending a subvariety Z to the K_0-class of $\mathcal{O}_Z$ defines the map

$$\alpha^{q,p}\colon \mathcal{Z}^q(X,p+1)_c \to K_0^{[q]}(X\times\square^{p+1}; X\times\partial_0\square^{p+1})$$

We let

$$\beta^{q,p+1,l}\colon \mathcal{Z}^q(X,p+1)_c\otimes\mathbb{Z}_{(l)} \to K_0^{[q]}(X\times\square^{p+1}; X\times\partial_0\square^{p+1})\otimes\mathbb{Z}_{(l)}$$

be the map

$$\prod_{i=q+1}^{d+p}\frac{\psi^k - k^i\mathrm{id}}{k^q-k^i}\circ\alpha^{q,p}$$

Then the image of $\beta^{q,p+1,l}(W)$ in $K_0^{[q]}(X\times\square^p; X\times\partial\square^p)\otimes\mathbb{Z}_{(l)}$ under the restriction to the face $t_n=0$, followed by the canonical map

$$K_0^{[q]}(X\times\square^p; X\times\partial\square^p)\otimes\mathbb{Z}_{(l)} \to K_0(X\times\square^p; X\times\partial\square^p)\otimes\mathbb{Z}_{(l)}$$

is equal to $\mathrm{cyc}'^{q,p}(d_p(W))$. As

$$K_0(X\times\square^{p+1}; X\times\partial_0\square^{p+1}) = 0,$$

we have

$$\mathrm{cyc}'^{q,p}(d_p(W)) = 0.$$

$\square$

This proves the first part of theorem 8.1. To prove the injectivity of the map $\mathrm{cl}'^{q,p}$, we proceed as in Step 3. Let l be a prime with

$$l \geq N_{d,p,q};$$

we tensor everything with $\mathbb{Z}_{(l)}$ and fix a generator k of $\mathbb{Z}/l^\times$. Suppose

$$\mathrm{cyc}'^{q,p}(Z) = 0$$

Take W, U and $\eta\in K_1(U; U\times\partial\square^n)\otimes\mathbb{Z}_{(l)}$ as in Step 3, with

$$\delta(\eta) = \mathrm{cyc}'^{q,p}(Z), \qquad (8.9)'$$

in the obvious modification of the localization sequence of Step 3.

Passing to an infinite prime to l extension of k and changing notation, we may assume that k is infinite.

As W has codimension at least two, we may assume, by lemma 7.6, that η is in $\tilde{S}K_1(U; U \times \partial\square^n) \otimes \mathbb{Z}_{(l)}$. By corollary 7.5, this gives us the corresponding element

$$\tilde{\eta} \in \tilde{S}K_0(U \times \mathbb{A}^1; U \times \partial\square^n \times \mathbb{A}^1, U \times \{0,1\}) \otimes \mathbb{Z}_{(l)}$$

Let $\mathcal{D}$ be the iterated double of $U \times \mathbb{A}^1$ with respect to the divisors $U \times \partial\square^p \times \mathbb{A}^1$, $U \times \{0,1\}$. Using the isomorphism (8.8), and the splitting (8.7), we have an element

$$\tau \in \tilde{S}K_0(\mathcal{D}),$$

mapping to $\tilde{\eta}$. By lemma 7.7, τ is in $F^2_\gamma K_0(\mathcal{D})$.

Let τ_q be the element of $K_0(\mathcal{D})$:

$$\tau_q := \prod_{i=q+1}^{d+p} \prod_{i=2}^{q-1} \frac{\psi^k - k^i \mathrm{id}}{k^q - k^i}(\tilde{\eta}).$$

By lemma 7.2, τ_q in in $F^q_\gamma K_0(\mathcal{D})$.

By lemma 7.1, we have the injective map in $\mathcal{A}_{(l)}$

$$(K_0^{[q]}(X \times \square^p; X \times \partial\square^p) \otimes \mathbb{Z}_{(l)})^{(q)} \to \oplus_{w \in (X \times \square^p)^{(q)}} (K_0^w(\mathcal{O}_{X \times \square^p, w}) \otimes \mathbb{Z}_{(l)})^{(q)}$$

As ψ^k acts as $k^q\mathrm{id}$ on $(K_0^w(\mathcal{O}_{X \times \square^p, w}) \otimes \mathbb{Z}_{(l)})^{(q)}$, the same holds true for $(K_0^{[q]}(X \times \square^p; X \times \partial\square^p) \otimes \mathbb{Z}_{(l)})^{(q)}$. In particular, $\mathrm{cyc}'^{q,p}(Z)$ satisfies

$$\prod_{i=q+1}^{d+p+q} \prod_{i=2}^{q-1} \frac{\psi^k - k^i \mathrm{id}}{k^q - k^i}(\mathrm{cyc}'^{q,p}(Z)) = \mathrm{cyc}'^{q,p}(Z).$$

Thus, we may replace η with

$$\prod_{i=q+1}^{d+p} \prod_{i=2}^{q-1} \frac{\psi^k - k^i \mathrm{id}}{k^q - k^i}(\eta),$$

and assume from the start that $\tau = \tau_q$.

Using lemma 7.3, we find a closed, codimension q subset $\mathcal{T}$ of $\mathcal{D}$, intersecting all the iterated doubles of the "faces" in $U \times \partial\square^p \times \mathbb{A}^1, U \times \{0,1\}$ properly, such that τ is in the image of

$$K_0^{\mathcal{T}}(\mathcal{D}) \to K_0(\mathcal{D}).$$

Projecting back to $U \times \mathbb{A}^1$ by the splitting (8.7) (see [L], proof of theorem 2.3, for details), we find a closed, codimension q subset T_U of $U \times \mathbb{A}^1$, intersecting all "faces" in $U \times \partial\square^p \times \mathbb{A}^1, U \times \{0,1\}$ properly, and an element $\tilde{\eta}^{T_U}$ of

$$K_0^{T_U}(U \times \mathbb{A}^1; U \times \partial\square^p \times \mathbb{A}^1, U \times \{0,1\}) \otimes \mathbb{Z}_{(l)}$$

lifting $\tilde{\eta}$.

Let T denote the closure of T_U in $X \times \square^{p+1}$. It is shown in ([L], proof of theorem 2.7, pg. 274) that T intersects all faces of $X \times \partial\square^{p+1}$ properly. By using the splitting of lemma 8.3, as well as lemma 8.3 itself, (see [L], pg. 274) we may extend the image of $\tilde{\eta}^{T_U}$ in

$$K_0^{T_U}(U \times \mathbb{A}^1; U \times \partial\square^p \times \mathbb{A}^1, U \times 1) \otimes \mathbb{Z}_{(l)}$$

to an element ρ of

$$K_0^T(X \times \square^{p+1}; X \times \partial\square^{p+1}) \otimes \mathbb{Z}_{(l)}.$$

Let T^0 be the set of generic points of T. Taking the image of ρ in

$$K_0^{T^0}(\mathcal{O}_{X \times \square^{p+1}, T}) \otimes \mathbb{Z}_{(l)} \cong \mathcal{Z}^q_{T_0}(T) \otimes \mathbb{Z}_{(l)}$$

we get a cycle B on $X \times \square^{p+1}$, with $\mathbb{Z}_{(l)}$ coefficients. One easily sees that B is in fact in $\mathcal{Z}^q(X, n+1) \otimes \mathbb{Z}_{(l)}$. By ([L], lemma 2.6), the relation $(8.9)'$ implies that

$$d_p(B) = Z,$$

yielding the injectivity of the map $(8.6)'$.

To show that the image of $\mathrm{cl}'^{q,p}$ is contained in $F^q_\gamma K_p(X)$, first suppose that $p > 1$. Then the γ-filtration on

$$K_p(X)/F^{q+1}_\gamma K_p(X) = F^2_\gamma K_p(X)/F^{q+1}_\gamma K_p(X)$$

is split into eigenspaces for the Adams operators, after inverting $(q-1)!$. As the image of $\mathrm{cl}'^{q,p}$ in $K_p(X)/F^{q+1}_\gamma K_p(X)$ must land in the weight q eigenspace by construction, this forces the image in this quotient to be zero. For $p = 0, 1$, $q \geq 2$, it follows easily from our construction, together with theorem 7.4 and lemma 7.7, that he image of $\mathrm{cl}'^{q,p}$ is contained in $F^2_\gamma K_p(X)$; the proof then proceeds as above. For $p = 1$, $q = 1$, we have $K_1(X) = F^1_\gamma K_1(X)$, whence the result.

Similarly, the restriction of gamma filtration to $F^q_\gamma K_p(X)$ is split into eigenspaces for the ψ^k, after inverting $(d + p - q + 1)!$. By lemma 7.3, and the argument of Step 2, $\mathrm{cl}'^{q,p}$ is surjective; as we have just proved the injectivity, the proof of theorem 8.1 is complete.

$\square$

Appendix A. Closed simplicial model categories

Suppose we are given a class of morphisms $\mathcal{F}$ in a category $\mathcal{C}$. A morphism $q\colon A \to B$ has the *left lifting property* (LLP) with respect to $\mathcal{F}$ if there is a lifting $h\colon B \to Z$ for each commutative diagram

$$\begin{array}{ccc} A & \to & Z \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ B & \to & W \end{array}$$

with $p\colon Z \to W$ in $\mathcal{F}$. A morphism $p\colon Z \to W$ has the *right lifting property* (RLP) with respect to $\mathcal{F}$ if there is a lifting $h\colon B \to Z$ for each commutative diagram

$$\begin{array}{ccc} A & \to & Z \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ B & \to & W \end{array}$$

with $q\colon A \to B$ in $\mathcal{F}$.

We refer to [Q1] and [Q2] for the notions and basic properties of closed model categories; we recall that a closed model category is a category $\mathcal{C}$ together with three distinguished classes of morphisms: fibrations, cofibrations and weak equivalences; a morphism which is a (co)fibration and a weak equivalence is called a *trivial* (co)fibration. In addition, these classes satisfy the axioms CM1-CM5 of ([Q2], p. 233):

CM1: $\mathcal{C}$ is closed under finite projective and injective limits.

CM2: Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be maps in $\mathcal{C}$. Then if two of the maps f, g and gf are weak equivalences, so is the third.

CM3: If f is a retract of g (i.e., there are maps $a\colon f \to g$, $b\colon g \to f$ in the category of maps such that $ba = \mathrm{id}_f$) and g is a weak equivalence, a fibration, or a cofibration, then so is f.

CM4: Cofibrations have the LLP with respect to trivial fibrations, and fibrations have the RLP with respect to trivial cofibrations.

CM5: Any map f may be factored in two ways:

i) as pq, where p is a fibration, q is a cofibration, and p is a weak equivalence.

ii) as pq, where p is a fibration, q is a cofibration, and q is a weak equivalence.

These five axioms imply

i) The class of cofibrations (resp. trivial cofibrations) is closed under composition and co-base change, and contains all isomorphisms.

ii) The class of fibrations (resp. trivial fibrations) is closed under composition and base change, and contains all isomorphisms.

iii) (Axiom M6 of [Q1]) The fibrations, cofibrations and weak equivalences determine each other by

a) a map f is a fibration $\Leftrightarrow$ f has the RLP with respect to trivial cofibrations

b) a map f is a cofibration $\Leftrightarrow$ f has the LLP with respect to trivial fibrations

c) a map f is a weak equivalence $\Leftrightarrow$ f has a factorization $f = uv$ with u a trivial fibration and v a trivial cofibration.

We have the category Δ with objects the ordered sets $[0], \dots, [n], \dots$,

$$[n] := \{0 < 1 < \ldots < n\},$$

and with morphisms being order-preserving maps; we let $\mathcal{S}$ denote the category of simplicial sets: the category of functors

$$\Delta^{\mathrm{op}} \to \mathbf{Sets}$$

We have the object

$$\Delta_n := \mathrm{Hom}_\Delta(-, [n]) \colon \Delta^{\mathrm{op}} \to \mathbf{Sets}$$

of $\mathcal{S}$. Similarly, we consider the functor

$$\mathrm{Hom}_\Delta(-, -) \colon \Delta^{\mathrm{op}} \times \Delta \to \mathbf{Sets}$$

as a functor

$$\tilde{\Delta} \colon \Delta \to \mathcal{S}$$
$$\tilde{\Delta}([n]) = \Delta_n.$$

We have the category of topological spaces **Top**, and the geometric realization functor

$$| - | \colon \mathcal{S} \to \mathbf{Top}.$$

We recall that the categories $\mathcal{S}$ and **Top** are both closed model categories, with the following notions of fibrations, cofibrations and weak equivalences:

Fibrations: In **Top**, we have the classical notion of a fibration being a map $f \colon X \to Y$ with the (right) homotopy lifting property. In $\mathcal{S}$, we have the notion of a *Kan fibration*: for each n and k, $0 \le k \le n$, the sub-simplicial set $\wedge_{n,k}$ of Δ_n, where $f \colon [i] \to [n]$ in Δ is in $\wedge_{n,k}$ if $f([i])$ does not contain the subset $[n] \backslash \{k\}$ of $[n]$. A map $f \colon X \to Y$ in $\mathcal{S}$ is a Kan fibration if f has the right lifting property with respect to the map $\wedge_{n,k} \to \Delta_n$, for each n and k.

Weak equivalences: In **Top**, a map f is a weak equivalence if f induces an isomorphism on π_n for each n. In $\mathcal{S}$, a map f is a weak equivalence if the geometric realization $|f|$ is a weak equivalence in **Top**.

Cofibrations: In both **Top** and $\mathcal{S}$, cofibrations are defined by requiring the left lifting property with respect to maps which are both a fibration and a weak equivalence. In $\mathcal{S}$, a map $f \colon X \to Y$ is a cofibration if and only if the map on n-simplices, $f_n \colon X_n \to Y_n$, is injective for each n.

We recall that a *simplicial category* ([Q1], II, §1) $\mathcal{C}$ is a category enriched in simplicial sets; i.e, a category $\mathcal{C}$ together with

i) a functor $(X,Y) \mapsto \mathcal{H}om_{\mathcal{C}}(X,Y)$ from $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ to $\mathcal{S}$

ii) an associative "composition law" in $\mathcal{S}$

$$\mathcal{H}om_{\mathcal{C}}(X,Y) \times \mathcal{H}om_{\mathcal{C}}(Y,Z) \to \mathcal{H}om_{\mathcal{C}}(X,Z)$$

iii) an identification

$$u \mapsto \tilde{u}$$

of $\mathrm{Hom}_{\mathcal{C}}(-,-)$ with the 0-simplices $\mathcal{H}om_{\mathcal{C}}(-,-)_0$ of $\mathcal{H}om_{\mathcal{C}}(-,-)$ such that

$$f \circ s_0^n(\tilde{u}) = \mathcal{H}om_{\mathcal{C}}(u,Z)_n(f); \qquad s_0^n(\tilde{u}) \circ g = \mathcal{H}om_{\mathcal{C}}(W,u)_n(g)$$

for $u \in \mathrm{Hom}_{\mathcal{C}}(Y,Z)$, $f \in \mathcal{H}om_{\mathcal{C}}(X,Y)_n$ and $g \in \mathcal{H}om_{\mathcal{C}}(Y,W)_n$, where s_0^n is the map induced by the unique surjection $[n] \to [0]$.

The category $\mathcal{S}$ is a simplicial category, with

$$\mathcal{H}om_{\mathcal{S}}(X,Y) = \mathrm{Hom}_{\mathcal{S}}(X \times \tilde{\Delta}, Y).$$

Similarly, category **Top** is a simplicial category, with

$$\mathcal{H}om_{\mathbf{Top}}(X,Y) = \mathrm{Hom}_{\mathbf{Top}}(X \times |\tilde{\Delta}|, Y).$$

If $\mathcal{C}$ is a simplicial category, X an object of $\mathcal{C}$ and K a simplicial set, we say that $X \otimes K$ exists if there is a pair $(X \otimes K, \alpha)$, with $X \otimes K$ an object of $\mathcal{C}$, and α a map (evaluation)

$$\alpha: K \to \mathcal{H}om_{\mathcal{C}}(X, X \otimes K),$$

yielding an isomorphism

$$\mathcal{H}om_{\mathcal{C}}(X \otimes K, -)) \to \mathcal{H}om_{\mathcal{S}}(K, \mathcal{H}om_{\mathcal{C}}(X,-))$$

by the obvious composition; the functor $\mathcal{H}om_{\mathcal{S}}(K, \mathcal{H}om_{\mathcal{C}}(X,-))$ is thus represented by $(X \otimes K, \alpha)$. We say X^K exists if the functor $\mathcal{H}om_{\mathcal{S}}(K, \mathcal{H}om_{\mathcal{C}}(-,X))$ is similarly represented by an object X^K and an evaluation map

$$\beta: K \to \mathcal{H}om_{\mathcal{C}}(X^K, X).$$

We recall from ([Q1], II, §2) that a *closed simplicial model category* is a closed model category $\mathcal{C}$ which is also a simplicial category, and satisfies

SM0: If X is an object of $\mathcal{C}$, then the objects $X \otimes K$ and X^K exist for all finite simplicial sets K.

SM7: If $q: A \to B$ is a cofibration, and $p: X \to Y$ is a fibration, then

$$\mathcal{H}om(B,X) \xrightarrow{(q^*,p_*)} \mathcal{H}om(A,X) \times_{\mathcal{H}om(A,Y)} \mathcal{H}om(B,Y)$$

is a fibration in $\mathcal{S}$, which is trivial if either q or p is trivial. □

The closed model categories $\mathcal{S}$ and **Top** are closed simplicial model categories ([Q1], II, §3); making the obvious modifications in the definitions, the pointed categories $\mathcal{S}^*$ and **Top*** are closed simplicial model categories as well.

In ([Q1], II, §4), Quillen gives a sufficient criterion for the category $s\mathcal{A}$ of simplicial objects in a category $\mathcal{A}$ to be a simplicial closed model category, where a map f in $s\mathcal{A}$ is a fibration (resp. weak equivalence) if $\mathcal{H}om(P, f)$ is a fibration (resp. weak equivalence) for all projective objects of $\mathcal{A}$, and cofibrations are determined by the LLP with respect to trivial fibrations.

For a category $\mathcal{C}$ and a small category I, we let $\mathcal{C}^I$ denote the category of functors

$$X: I \to \mathcal{C}.$$

As an application of the criterion of Quillen mentioned above, Bousfield and Kan ([B-K], XI, §8, proof of proposition 8.1) prove the following result:

Theorem A.1. *Let I be a small category. Then the simplicial category $\mathcal{S}^I$ is a closed simplicial model category with the following notions of fibration, weak equivalence, and cofibration:*

Fibrations and weak equivalences: a map $f: X \to Y$ is a fibration (resp. weak equivalence) in $\mathcal{S}^I$ if and only if the map $f(i): X(i) \to Y(i)$ is a fibration (resp. weak equivalence) in $\mathcal{S}$ for each $i \in I$.

Cofibrations: A map $f: X \to Y$ is a cofibration in $\mathcal{S}^I$ if and only if f has the left lifting property with respect to maps g which are a fibration and a weak equivalence.

□

We conclude with a few additional results which we need in the body of the paper.

Let $\mathcal{A}$ be a closed model category, with initial object $\emptyset$ and final object $*$. An object X of $\mathcal{A}$ is called *cofibrant* (resp. fibrant) if the canonical map $\emptyset \to X$ (resp. $X \to *$) is a cofibration (resp. fibration). Call X *bifibrant* if X is fibrant and cofibrant. If A is an object of $\mathcal{A}$, a *bifibrant model of A* is a diagram of weak equivalences

$$A = A_0 \xrightarrow{i_0} B_1 \xleftarrow{j_1} A_1 \xrightarrow{i_1} \ldots \xleftarrow{j_n} A_n = A',$$

with A' bifibrant. Given A, we can always find a bifibrant model, e.g., first by factoring the canonical map $A \to *$ as

$$A \xrightarrow{i} B \xrightarrow{p} *$$

with i a trivial cofibration and p a fibration, then factoring the canonical map $\emptyset \to B$ as

$$\emptyset \xrightarrow{q} A' \xrightarrow{j} B$$

with q a cofibration and j a trivial fibration.

For a closed model category $\mathcal{A}$, the *homotopy category of* $\mathcal{A}$, $\mathrm{Ho}\mathcal{A}$ is the category gotten from $\mathcal{A}$ by localizing with respect to the weak equivalences; this category exists ([Q1], I, §1 theorem 1.). If $\mathcal{A}$ is a closed simplicial model category, the relation of homotopy equivalence of maps is defined by setting

$$[A, B] := \pi_0(|\mathcal{H}om_{\mathcal{A}}(A, B)|).$$

This gives the notion of a homotopy equivalence $f: A \to B$, and a strong deformation retract, as in the case of topological spaces.

Lemma A.2. *Let A be a closed simplicial model categroy. A trivial cofibration between fibrant objects, and a trivial fibration between cofibrant objects are strong deformation retracts. A weak equivalence between bifibrant objects is a homotopy equivalence*

Proof. The first pair of assertions is a part of ([Q1], II, §2, corollary to proposition 4). If $f: X \to Y$ is a weak equivalence of bifibrant objects, factor f as $f = uv$, with $v: X \to Z$ a trivial cofibration, and $u: Z \to Y$ a trivial fibration. Then Z is also bifibrant, hence u and v are both homotopy equivalences. □

Remark A. 3. We give a list of basic properties of homotopy classes of maps, and the relation with maps in the homotopy category, for a closed simplicial model category $\mathcal{A}$:

i) If A is cofibrant, and B is fibrant, then $[A, B]$ is the same as the set of maps from A to B in the homotopy category $\mathrm{Ho}\mathcal{A}$.

ii) A homotopy equivalence $f: A \to B$ is a weak equivalence; if A and B are bifibrant, then a weak equivalence $f: A \to B$ is a homotopy equivalence. In particular, if A, X and Y are bifibrant, and $f: X \to Y$ is a weak equivalence, then

$$f_*: [A, X] \to [A, Y]; \qquad f^*: [X, A] \to [Y, A]$$

are isomorphisms.

iii) $\mathrm{Ho}\mathcal{A}$ is equivalent to the category with objects the bifibrant objects of $\mathcal{A}$, with morphisms
$$\mathrm{Hom}(A, B) := [A, B].$$

Appendix B. Homotopy limits

Let I be a small category. For an object i of I, we have the *over category* I/i of maps $j \to i$ in I, as well as the *nerve* $\mathcal{N}(I/i)$. Each map $s: i \to i'$ gives the functor

$$s_*: I/i \to I/i';$$

taking the nerve gives the functor

$$\mathcal{N}(I/-): I \to \mathcal{S}. \tag{$B.1$}$$

Bousfield and Kan have defined homotopy limits of an object X of $\mathcal{S}^I$ as

$$\operatorname*{holim}_I X = \mathcal{H}om_{\mathcal{S}^I}(\mathcal{N}(I/-), X). \tag{$B.2$}$$

This gives the functor

$$\operatorname*{holim}_I: \mathcal{S}^I \to \mathcal{S}$$

having the following properties:

(B.3i) Let $f: X \to Y$ be a fibration (resp. weak equivalence). Then

$$\operatorname*{holim}_I f: \operatorname*{holim}_I X \to \operatorname*{holim}_I Y$$

is a fibration (resp. weak equivalence).

ii) $\operatorname*{holim}_I$ is right adjoint to the functor

$$\begin{aligned} &(-) \times \mathcal{N}(I/-): \mathcal{S} \to \mathcal{S}^I \\ &A \mapsto A \times \mathcal{N}(I/-) \end{aligned}$$

and hence preserves projective limits.

iii) If $f: J \to I$, $Y: J \to \mathcal{C}$ and $X: I \to \mathcal{C}$ are functors, and

$$\omega: X \circ f \to Y$$

is a natural transformation, we have the natural map

$$\omega_* \circ f^*: \operatorname*{holim}_I X \to \operatorname*{holim}_J Y$$

In addition, there is the basic construction of ([B-K], XI, §7.6):

Theorem B.1. *Let X be a fibrant object of $\mathcal{S}^{*I}$. Then there is a "fringed" spectral sequence*

$$E_2^{s,t} := \lim_{\leftarrow}{}^s \pi_{-t} X; \quad 0 \le s \le t \Longrightarrow \pi_{-s-t} \operatorname*{holim}_I X,$$

which converges if e.g. there is for each s an integer $N(s)$, $s \le N(s) < \infty$ with

$$E_r^{s,s+i} = E_{N(s)}^{s,s+i}$$

for all $r \ge N(s)$.

□

This is extended by Dwyer and Kan to function complexes. Let aI be the *twisted arrow category* with objects maps $i \to j$ in I, and maps $(i \to j) \to (i' \to j')$ commutative diagrams

$$\begin{array}{ccc} i & \leftarrow & i' \\ \downarrow & & \downarrow \\ j & \rightarrow & j' \end{array}$$

For X and Y in $\mathcal{S}^I$, we have the functor

$$\begin{aligned} &\mathcal{H}om(X,Y)\colon aI \to \mathcal{S} \\ &\mathcal{H}om(X,Y)(i \to j) = \mathcal{H}om_{\mathcal{S}}(X(i), Y(j)) \end{aligned}$$

with the obvious maps. In addition, there is the obvious isomorphism

$$\mathcal{H}om_{\mathcal{S}^I}(X,Y) \cong \lim_{\substack{\leftarrow \\ aI}} \mathcal{H}om(X,Y) \tag{B.4}$$

The main result of [D-K] (Theorem 3.3) is

Theorem B.2. *Suppose that X is cofibrant and Y is fibrant. Then the natural map*

$$\lim_{\substack{\leftarrow \\ aI}} \mathcal{H}om(X,Y) \to \operatorname*{holim}_{aI} \mathcal{H}om(X,Y)$$

is a weak equivalence. The same holds in the pointed setting. □

Actually, the pointed case is not considered in [D-K], but the same proof gives the result. As consequence, we have

Corollary B.3. *Let X be a cofibrant object of $\mathcal{S}^I$, and Y a pointed fibrant object of $\mathcal{S}^I$. There is a "fringed" spectral sequence*

$$E_2^{s,t} := \lim_{\substack{\leftarrow \\ aI}}{}^s \pi_{-t} \mathcal{H}om(X,Y); \quad 0 \le s \le t \Longrightarrow \pi_{-s-t} \mathcal{H}om_{\mathcal{S}^I}(X,Y).$$

This spectral sequence converges if e.g. the condition of theorem B.1 are satisfied (for the object $\mathcal{H}om(X,Y)$ of $\mathcal{S}^{aI}$). If X is pointed as well, the same holds for the pointed $\mathcal{H}om$.

Proof. This follows directly from the isomorphism (B.4), theorem B.2, and theorem B.1, applied to the fibrant object $\mathcal{H}om(X,Y)$ of $\mathcal{S}^{aI}$; the proof in the pointed case is the same. $\square$

Corollary B.4. *Let $f: X \to X'$ be a map of cofibrant objects of $\mathcal{S}^I$, $g: Y' \to Y$ a map of fibrant objects of $\mathcal{S}^I$. If the maps*

$$f(i)^* g(j)_*: \mathcal{H}om_{\mathcal{S}}(X'(i), Y'(j)) \to \mathcal{H}om_{\mathcal{S}}(X(i), Y(j))$$

are weak equivalences for all pairs $i, j \in I$ for which there is a map $i \to j$ in I, then

$$f^* g_*: \mathcal{H}om_{\mathcal{S}}(X', Y') \to \mathcal{H}om_{\mathcal{S}}(X, Y)$$

is a weak equivalence. The same holds in the pointed case as well.

Proof. The map in $\mathcal{S}^{aI}$

$$f^* g_*: \mathcal{H}om(X', Y') \to \mathcal{H}om(X, Y)$$

is a weak equivalence of fibrant objects of $\mathcal{S}^{aI}$. By ([B-K], XI, 5.5, 5.6), the map

$$\operatorname*{holim}_{aI} f^* g_*: \operatorname*{holim}_{aI} \mathcal{H}om(X', Y') \to \operatorname*{holim}_{aI} \mathcal{H}om(X, Y)$$

is a weak equivalence of fibrant objects of $\mathcal{S}$. By theorem B.2, the natural maps

$$\mathcal{H}om_{\mathcal{S}^I}(X', Y') \cong \varprojlim_{aI} \mathcal{H}om(X', Y') \to \operatorname*{holim}_{aI} \mathcal{H}om(X', Y')$$
$$\mathcal{H}om_{\mathcal{S}^I}(X, Y) \cong \varprojlim_{aI} \mathcal{H}om(X, Y) \to \operatorname*{holim}_{aI} \mathcal{H}om(X, Y)$$

are weak equivalences, whence the result. $\square$

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