# Bloch's cycle complex

Recall Bloch's cycle complex $(z^q(X,*), d)$:

$$z^q(X,n) := \mathbb{Z}\{\text{irreducible, codimension } q \text{ subvarieties } W \subset X \times \Delta^n \text{ in good position}\}$$

with differential the alternating sum of intersections with the codimension one faces.

The higher Chow groups of $X$ are

$$\mathrm{CH}^q(X,n) := H_n(z^q(X,*), d).$$

# Bloch's cycle complex

Universal integral cohomology is $H^p(X, \mathbb{Z}(q)) := \mathrm{CH}^q(X, 2q-p)$. To reflect the re-indexing, set

$$\Gamma_{Bl}(q)^*(X) := z^q(X, 2q-p),$$

giving the complex of sheaves on $X_{\mathrm{Zar}}$

$$U \mapsto \Gamma_{Bl}(q)^*(U).$$

Since the higher Chow groups have a Mayer-Vietoris property, we have

$$H^p(X, \mathbb{Z}(q)) := H^p(\Gamma_{Bl}(q)^*(X)) \cong \mathbb{H}^p(X_{\mathrm{Zar}}, \Gamma_{Bl}(q)).$$

Recall the Dold-Thom theorem:

**Theorem (Dold-Thom)**

*Let $(T,*)$ be a pointed CW complex. There is a natural isomorphism*

$$H_n(T,*) \cong \pi_n(\mathrm{Sym}^\infty T).$$

Here

$$\mathrm{Sym}^\infty T = \varinjlim[T \to \mathrm{Sym}^2 T \to \ldots \to \mathrm{Sym}^n T \to \ldots]$$

with $\mathrm{Sym}^n T \to \mathrm{Sym}^{n+1} T$ the map "add $*$ to the sum".

### Definition

For $X, Y$ varieties, $X$ smooth and irreducible, set

$$z_{\mathrm{fin}}(Y)(X) := \mathbb{Z}[\{\text{irreducible, reduced } W \subset X \times_k Y \text{ with } W \to X \text{ finite and surjective}\}].$$

### Definition

For a $k$-scheme $Y$, the Suslin complex of $Y$, $C_*^{\mathrm{Sus}}(Y)$, is the complex associated to the simplicial abelian group

$$n \mapsto z_{\mathrm{fin}}(Y)(\Delta_k^n).$$

The Suslin homology of $Y$ is

$$H_n^{\mathrm{Sus}}(Y, A) := H_n(C_*^{\mathrm{Sus}}(Y) \otimes A).$$

Since a finite cycle $W \subset Y \times \Delta^n$ is a cycle of codimension $d_Y = \dim Y$, intersecting all faces properly, we have the inclusion of complexes

$$C_*^{\mathrm{Sus}}(Y) \hookrightarrow z^{d_Y}(Y, *).$$

For $Y$ smooth and projective, this inclusion induces an isomorphism

$$H_n^{\mathrm{Sus}}(Y, \mathbb{Z}) \cong H^{2d_Y - n}(Y, \mathbb{Z}(d_Y)).$$

(Poincare' duality).

One can recover all the universal cohomology groups from the Suslin homology construction, properly modified. For this, we recall how the Dold-Thom theorem gives a model for cohomology.

Since $S^n$ has only one non-trivial reduced homology group, $H_n(S^n, \mathbb{Z}) = \mathbb{Z}$, the Dold-Thom theorem tells us that $\mathrm{Sym}^\infty S^n$ is a $K(\mathbb{Z}, n)$, i.e.

$$\pi_m(\mathrm{Sym}^\infty S^n) = \begin{cases} 0 & \text{for } m \neq n \\ \mathbb{Z} & \text{for } m = n. \end{cases}$$

Obstruction theory tells us that

$$H^m(X, \mathbb{Z}) = \pi_{n-m}(Maps(X, Sym^\infty S^n)).$$

for $m \leq n$.

To rephrase this in the algebraic setting, we need a good replacement for the $n$-spheres. The correct choice is governed by the Gysin morphism:

Let $i : A \to B$ be a closed immersion of manifolds, $d = \mathrm{codim}_{\mathbb{R}} i$, $N_{A/B}$ the normal bundle. The Gysin morphism $i_* : H^n(A) \to H^{n+d}(B)$ is defined via

$$H^n(A) \cong H^{n+d}(Th(N_{A/B}), *) \cong H^{n+d}(T_\epsilon(A), \partial T_\epsilon(A)) \\ = H^{n+d}(B, B \setminus T_\epsilon(A)) \to H^{n+d}(B)$$

where $Th(N_{A/B}) := \mathbb{P}(N_{A/B} \oplus 1)/\mathbb{P}(N_{A/B})$ is the Thom space.

If $A = pt$, then $N_{A/B} = \mathbb{R}^d$ and $Th(N_{A/B}) = \mathbb{RP}^d/\mathbb{RP}^{d-1} = S^d$.

In the algebraic setting, let $i : X \to Y$ be a closed immersion of smooth varieties, $N_{X/Y}$ the normal bundle. Formally, the algebraic Thom space is $Th(N_{X/Y}) := \mathbb{P}(N_{X/Y} \oplus 1)/\mathbb{P} * (N_{X/Y})$.
If $X = pt$, the $N_{X/Y} = \mathbb{A}^d$ and

$$Th(N_{X/Y}) = \mathbb{P}^d/\mathbb{P}^{d-1} =: S^{2d,d}.$$

For $d = 1$,

$$S^{2,1} = \mathbb{P}^1 = \mathbb{A}^1 \cup_{\mathbb{A}^1-\{0\}} \mathbb{A}^1 \sim S^1 \wedge (\mathbb{A}^1 - \{0\}) \neq S^1 \wedge S^1 = S^2.$$

We should use the $2d$ sphere of weight $d$, $S^{2d,d} = \mathbb{P}^d/\mathbb{P}^{d-1}$, if we want to have a Gysin map in our cohomology.

The quotient $\mathbb{P}^q/\mathbb{P}^{q-1}$ doesn't make much sense, but since we are going to apply this to finite cycles, we just take a quotient by the cycles "at infinity" as groups:

$$z_{\mathrm{fin}}(S^{2q,q})(X) = z_{\mathrm{fin}}(\mathbb{P}^q/\mathbb{P}^{q-1})(X)$$
$$:= z_{\mathrm{fin}}(\mathbb{P}^q)(X)/z_{\mathrm{fin}}(\mathbb{P}^{q-1})(X)$$

This leads to

### Definition

The Friedlander-Suslin weight $q$ cycle complex of $X$ is

$$\Gamma_{FS}(q)^*(X) := z_{\mathrm{fin}}(S^{2q,q})(X \times \Delta^{2q-*}).$$

This gives us the complex of sheaves $U \mapsto \Gamma_{FS}(q)(U)^*$.

Restriction from $X \times \Delta^n \times \mathbb{P}^q \to X \times \Delta^n \times \mathbb{A}^q$ defines the inclusion

$$\Gamma_{FS}(q)^*(X) \hookrightarrow z^q(X \times \mathbb{A}^q, 2q - *) = \Gamma_{Bl}(q)^*(X \times \mathbb{A}^q)$$

**Theorem (Friedlander-Suslin-Voevodsky)**

*For $X$ smooth and quasi-projective, the maps*

$$\Gamma_{FS}(q)^*(X) \to \Gamma_{Bl}(q)^*(X \times \mathbb{A}^q) \xleftarrow{p^*} \Gamma_{Bl}(q)^*(X)$$

*are quasi-isomorphisms. In particular, we have natural isomorphisms*

$$\mathbb{H}^p(X_{\mathrm{Zar}}, \Gamma_{FS}(q)) \cong H^p(\Gamma_{FS}(q)(X)^*) \cong H^p(X, \mathbb{Z}(q)).$$

Since

$$X \mapsto \Gamma_{FS}(q)^*(X) := z_{\text{fin}}(\mathbb{P}^q/\mathbb{P}^{q-1})(X \times \Delta^{2q-*})$$

is functorial in $X$, the Friedlander-Suslin complex gives a functorial model for Bloch's cycle complex.

Products for $\Gamma_{FS}(q)$ are similarly defined on the level of complexes.

This completes the Beilinson-Lichtenbaum program, with the exception of the vanishing conjectures.

# Categories of motives

Marc Levine

June 4, 2008

# Outline

- Grothendieck motives
- Beilinson's conjectures
- Triangulated categories
- Triangulated categories of motives

# Grothendieck motives

How to construct the category of pure motives for an adequate equivalence relation $\sim$.

An additive category $\mathcal{C}$ is *abelian* if every morphism $f : A \longrightarrow B$ has a (categorical) kernel and cokernel, and the canonical map $\mathrm{coker}(\ker f) \longrightarrow \ker(\mathrm{coker} f)$ is always an isomorphism.

An additive category $\mathcal{C}$ is *pseudo-abelian* if every idempotent endomorphism $p : A \longrightarrow A$ has a kernel:

$$A \cong \ker p \oplus \ker 1 - p.$$

For an additive category $\mathcal{C}$, there is a universal additive functor to a pseudo-abelian category $\psi : \mathcal{C} \to \mathcal{C}^\natural$.

$\mathcal{C}^\natural$ has objects $(A, p)$ with $p : A \to A$ an idempotent endomorphism,

$$\mathrm{Hom}_{\mathcal{C}^\natural}((A, p), (B, q)) = q\mathrm{Hom}_{\mathcal{C}}(A, B)p.$$

and $\psi(A) := (A, \mathrm{id})$, $\psi(f) = f$.

Note. If $p_1, \ldots, p_r$ are commuting mutually orthogonal idempotents on $A$ with $\sum_i p_i = \mathrm{id}_A$, then

$$\psi(A) = (A, p_1) \oplus \ldots \oplus (A, p_r)$$

in $\mathcal{C}^\natural$.

The category $\mathrm{Cor}_\sim(k)$ has the same objects as $\mathbf{SmProj}/k$. Morphisms (for $X$ irreducible) are

$$\mathrm{Hom}_{\mathrm{Cor}_\sim}(X, Y) := A^{d_X}_\sim(X \times Y)_\mathbb{Q}$$

with composition the composition of correspondences:

$$W' \circ W := p_{13*}(p^*_{12}(W) \cdot p^*_{23}(W'))$$

for $W \in \mathrm{Hom}_{\mathrm{Cor}_\sim}(X, Y)$, $W' \in \mathrm{Hom}_{\mathrm{Cor}_\sim}(Y, Z)$.
In general, take the direct sum over the components of $X$.
Write $X$ (as an object of $\mathrm{Cor}_\sim(k)$) $= h_\sim(X)$ or just $h(X)$. For $f : Y \to X$, set $h(f) := {}^t\Gamma_f$. This gives a functor

$$h_\sim : \mathbf{SmProj}/k^{\mathrm{op}} \to \mathrm{Cor}_\sim(k).$$

1. $\mathrm{Cor}_\sim(k)$ is an additive category with $h(X) \oplus h(Y) = h(X \amalg Y)$.
2. $\mathrm{Cor}_\sim(k)$ is a tensor category with $h(X) \otimes h(Y) = h(X \times Y)$. For $a \in A_\sim^{d_X}(X \times Y)_\mathbb{Q}$, $b \in A_\sim^{d_{X'}}(X' \times Y')_\mathbb{Q}$

$$a \otimes b := t^*(a \times b)$$

with $t : (X \times X') \times (Y \times Y') \to (X \times Y) \times (X' \times Y')$ the exchange.
3. $h_\sim$ is a symmetric monoidal functor.

The composition law for correspondences:

$$W' \circ W := p_{13*}(p_{12}^*(W) \cdot p_{23}^*(W'))$$

requires

- That $Y$ is proper (for $p_{13*}$ to be defined)
- That we work modulo an adequate equivalence relation (for $p_{12}^*(W) \cdot p_{23}^*(W')$ to be defined).

From the point of view of "higher cycle" this is bad, as we lose the choice of equivalences between cycles. Voevodsky's use of "finite correspondences" solves both problems.

**Definition**
$M_\sim^{\text{eff}}(k) := \mathrm{Cor}_\sim(k)^\natural$.

Explicitly, $M_\sim^{\text{eff}}(k)$ has objects $(X, \alpha)$ with $X \in \mathbf{SmProj}/k$ and $\alpha \in A_\sim^{d_X}(X \times X)_\mathbb{Q}$ with $\alpha^2 = \alpha$ (as correspondence mod $\sim$).

$M_\sim^{\text{eff}}(k)$ is a tensor category with unit $\mathbb{1} = (\mathrm{Spec}\, k, [\mathrm{Spec}\, k])$.

Set $\mathfrak{h}_\sim(X) := (X, \Delta_X)$, for $f : Y \to X$, $\mathfrak{h}_\sim(f) := {}^t\Gamma_f$.
This gives the symmetric monoidal functor

$$\mathfrak{h}_\sim : \mathbf{SmProj}(k)^{\text{op}} \to M_\sim^{\text{eff}}(k).$$

Grothendieck constructed the category of motives to give a universal geometric cohomology theory for smooth projective varieties.
To explain: Take a "reasonable" (i.e. Weil) cohomology theory on **SmProj**$/k$: $X \mapsto H^*(X)$ (e.g. $H^*(X) = H^*_{\text{sing}}(X(\mathbb{C}), \mathbb{Q})$) admiting a good theory of cycle class

$$Z \mapsto \gamma_X(Z) \in H^{2q}(X); \quad Z \in z^q(X).$$

Then $Z \in \text{Cor}_{\text{rat}}(X, Y)$ gives $Z_* : H^*(X) \to H^*(Y)$ by

$$Z_*(\alpha) := p_{Y*}(p_X^*(\alpha) \cup \gamma_{X\times Y}(Z))$$

and $(Z \circ W)_* = Z_* \circ W_*$.
Thus, we can think of $\mathfrak{h}_{\text{rat}}(X) \in M^{\text{eff}}_{\text{rat}}(k)$ as a formal version of the total cohomology $H^*(X)$: sending $X$ to $H^*(X)$ extends to a functor

$$H : M^{\text{eff}}_{\text{rat}}(k) \to \text{Gr}\mathbf{Ab}$$

Standard conjecture 1. $H : M^{\text{eff}}_{\text{rat}}(k) \to \mathrm{Gr}\mathbf{Ab}$ descends to $H : M^{\text{eff}}_{\text{num}}(k) \to \mathrm{Gr}\mathbf{Ab}$ (automatically faithful).

For each $n$ we have the projection of $H^*(X)$ on $H^n(X)$, giving the commuting mutually orthogonal idempotent endomorphisms $p_n$ of $H^*(X)$

Standard conjecture 2. Assume SC1. Then for each $X \in \mathbf{SmProj}/k$ and each $n$, $p_n(H^*(X))$ lifts to an idempotent endomorphism $\Pi_n$ of $\mathfrak{h}_{\text{num}}(X)$. Set $\mathfrak{h}^n(X) := (\mathfrak{h}_{\text{num}}(X), \Pi_n)$.

Standard conjecture 3. Assume SC1. Then $M^{\text{eff}}_{\text{num}}(k)$ is a semi-simple abelian category.

If we assume SC1-3, then $\mathfrak{h}(X) = \oplus_{n=0}^{2\dim X} \mathfrak{h}^n(X)$ and $\mathfrak{h}^n(X) \in M^{\text{eff}}_{\text{num}}(k)$ can be thought of as a universal construction of the $n$th cohomology of $X$. This could help explain the mysterious parallels between different cohomology theories on $\mathbf{SmProj}/k$.

Examples. 1. $\Delta_{\mathbb{P}^1} \sim_{\text{rat}} \mathbb{P}^1 \times 0 + 0 \times \mathbb{P}^1 \rightsquigarrow$

$$\mathfrak{h}(\mathbb{P}^1) = (\mathbb{P}^1, 0 \times \mathbb{P}^1) + (\mathbb{P}^1, \mathbb{P}^1 \times 0) = \mathfrak{h}^0(\mathbb{P}^1) \oplus \mathfrak{h}^2(\mathbb{P}^1).$$

$(\mathbb{P}^1, 0 \times \mathbb{P}^1) \cong \mathfrak{h}_{\text{rat}}(pt) = \mathbb{1}$, the remaining factor is the Lefschetz motive $\mathbb{L} := (\mathbb{P}^1, \mathbb{P}^1 \times 0)$.

2. Let $C$ be a smooth projective curve over $k$, $0 \in C(k)$. Then

$$\begin{aligned}\mathfrak{h}(C) &= (C, 0 \times C) \oplus (C, \Delta_C - 0 \times C - C \times 0) \oplus (C, C \times 0)\\ &= \mathfrak{h}^0(C) \oplus \mathfrak{h}^1(C) \oplus \mathfrak{h}^2(C)\\ &= \mathbb{1} \oplus \mathfrak{h}^1(C) \oplus \mathbb{L}.\end{aligned}$$

$\mathfrak{h}^1(C) \neq 0$ iff $g(C) > 0$.

The coarsest equivalence is $\sim_{\text{num}}$, so $M_{\text{num}}(k)$ should be the most simple category of motives.

In fact, at least one part of Grothendieck's program has been verified.

**Theorem (Jannsen)**

*$M_{\text{num}}(k)$ is a semi-simple abelian category. If $M_\sim(k)$ is semi-simple abelian, then $\sim=\sim_{\text{num}}$.*

The proof is surprisingly easy, relying on the Lefschetz trace formula and the fact that a nilpotent matrix has zero trace.

# Beilinson's conjectures

Pure motives describe the cohomology of smooth projective varieties over an algebraically closed field.

Mixed motives should describe the cohomology of *arbitrary* varieties.

Weil cohomology is replaced by *Bloch-Ogus* cohomology: Mayer-Vietoris for open covers and a purity isomorphism (with twists) for cohomology with supports.

# Beilinson's conjectures

Beilinson conjectured that the semi-simple abelian category of pure motives $M_{\mathrm{num}}(k)_{\mathbb{Q}}$ should admit a full embedding as the semi-simple objects in an abelian tensor category of *mixed motives* $MM(k)_{\mathbb{Q}}$.

This can be thought of as a universal version of the category of mixed Hodge structures MHS: the category of pure Hodge structures is a semi-simple abelian category and there is a functorial exact weight filtration $W_*$ on MHS such that $\mathrm{gr}^n_W H$ is a pure Hodge structure for each MHS $H$.

$MM(k)$ should have the following structures and properties:

# Beilinson's conjectures

- a natural finite exact weight filtration $W_*$ on $MM(k)_{\mathbb{Q}}$ such that for each $M \in MM(k)$, the graded pieces $\mathrm{gr}_n^W M_{\mathbb{Q}}$ are in $M_{\mathrm{num}}(k)_{\mathbb{Q}}$.
- A functor $R\mathfrak{h} : \mathbf{Sch}_k^{\mathrm{op}} \to D^b(MM(k))$
- An embedding $M_{\mathrm{rat}}(k)_{\mathbb{Z}} \hookrightarrow D^b(MM(k))$; in particular Tate/Lefschetz motives $\mathbb{Z}(n)$ ($\mathbb{L}^{\otimes n} = \mathbb{Z}(n)[2n]$).
- A natural isomorphism

$$\mathrm{Hom}_{D^b(MM(k))}(\mathbb{Z}, R\mathfrak{h}(X)(q)[p])_{\mathbb{Q}} \cong K_{2q-p}^{(q)}(X),$$

  in particular $\mathrm{Ext}^p_{MM(k)}(\mathbb{Z}, \mathbb{Z}(q))_{\mathbb{Q}} \cong K_{2q-p}^{(q)}(k)$.
- All "universal properties" of the cohomology of algebraic varieties should be reflected by identities in $D^b(MM(k))$ of the objects $R\mathfrak{h}(X)$.

### Definition

$H^p_{\mathrm{mot}}(X, \mathbb{Z}(q)) := \mathrm{Hom}_{D^b(MM(k))}(\mathbb{Z}, R\mathfrak{h}(X)(q)[p])$.

I.e., universal integral cohomology should be motivic cohomology.

The category $MM(k)$ has not been constructed.

In fact, the existence of $MM(k)$ would prove the Beilinson-Soulé vanishing conjectures!

However, there are now a number of (equivalent) constructions of *triangulated tensor categories* that satisfy all the structural properties expected of the derived categories $D^b(MM(k))$, except those which exhibit these as a derived category of an abelian category ($t$-structure).

There are at present various attempts to extend this to the triangulated version of Beilinson's vision of motivic sheaves over a base $S$.

We give a discussion of the construction of various versions of triangulated categories of mixed motives over $k$ due to Voevodsky.

# Triangulated categories

A *translation* on an additive category $\mathcal{A}$ is an equivalence $T : \mathcal{A} \to \mathcal{A}$. We write $X[1] := T(X)$.
Let $\mathcal{A}$ be an additive category with translation. A *triangle* $(X, Y, Z, a, b, c)$ in $\mathcal{A}$ is the sequence of maps

$$X \xrightarrow{a} Y \xrightarrow{b} Z \xrightarrow{c} X[1].$$

A morphism of triangles

$$(f, g, h) : (X, Y, Z, a, b, c) \to (X', Y', Z', a', b', c')$$

is a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{a} & Y & \xrightarrow{b} & Z & \xrightarrow{c} & X[1] \\ f\downarrow & & g\downarrow & & h\downarrow & & f[1]\downarrow \\ X' & \xrightarrow[a']{} & Y' & \xrightarrow[b']{} & Z' & \xrightarrow[c']{} & X'[1]. \end{array}$$

Verdier has defined a *triangulated category* as an additive category $\mathcal{A}$ with translation, together with a collection $\mathcal{E}$ of triangles, called the *distinguished triangles* of $\mathcal{A}$, which satisfy some axioms (which we won't specify).

A graded functor $F : \mathcal{A} \to \mathcal{B}$ of triangulated categories is called *exact* if $F$ takes distinguished triangles in $\mathcal{A}$ to distinguished triangles in $\mathcal{B}$.

**Remark** Suppose $(\mathcal{A}, T, \mathcal{E})$ is a triangulated category. If $(X, Y, Z, a, b, c)$ is in $\mathcal{E}$, and $A$ is an object of $\mathcal{A}$, then the sequences

$$\ldots \xrightarrow{c[-1]_*} \mathrm{Hom}_{\mathcal{A}}(A, X) \xrightarrow{a_*} \mathrm{Hom}_{\mathcal{A}}(A, Y) \xrightarrow{b_*} \mathrm{Hom}_{\mathcal{A}}(A, Z) \xrightarrow{c_*} \mathrm{Hom}_{\mathcal{A}}(A, X[1]) \xrightarrow{a[1]_*} \ldots$$

and

$$\ldots \xrightarrow{a[1]^*} \mathrm{Hom}_{\mathcal{A}}(X[1], A) \xrightarrow{c^*} \mathrm{Hom}_{\mathcal{A}}(Z, A) \xrightarrow{b^*} \mathrm{Hom}_{\mathcal{A}}(Y, A) \xrightarrow{a^*} \mathrm{Hom}_{\mathcal{A}}(X, A) \xrightarrow{c[-1]^*} \ldots$$

are exact.

A triangulated category is a machine for generating natural long exact sequences.

Let $\mathcal{A}$ be an additive category, $C^?(\mathcal{A})$ the category of cohomological complexes and $K^?(\mathcal{A})$ the homotopy category: the same objects as $C^?(\mathcal{A})$ and morphisms are chain homotopy classes of degree 0 maps of complexes.

For a complex $(A, d_A)$, let $A[1]$ be the complex

$$A[1]^n := A^{n+1}; \quad d^n_{A[1]} := -d^{n+1}_A.$$

For a map of complexes $f : A \to B$, we have the *cone sequence*

$$A \xrightarrow{f} B \xrightarrow{i} \mathrm{Cone}(f) \xrightarrow{p} A[1]$$

where $\mathrm{Cone}(f) := A^{n+1} \oplus B^n$ with differential

$$d(a, b) := (-d_A(a), f(a) + d_B(b))$$

$i$ and $p$ are the evident inclusions and projections.

We make $K^?(\mathcal{A})$ a triangulated category by declaring a triangle to be exact if it is isomorphic to the image of a cone sequence.

### Definition

Suppose $\mathcal{A}$ is both a triangulated category and a tensor category (with tensor operation $\otimes$) such that $(X \otimes Y)[1] = X[1] \otimes Y$. Suppose that, for each distinguished triangle $(X, Y, Z, a, b, c)$, and each $W \in \mathcal{A}$, the sequence

$$X \otimes W \xrightarrow{a \otimes \mathrm{id}_W} Y \otimes W \xrightarrow{b \otimes \mathrm{id}_W} Z \otimes W \xrightarrow{c \otimes \mathrm{id}_W} X[1] \otimes W = (X \otimes W)[1]$$

is a distinguished triangle. Then $\mathcal{A}$ is a *triangulated tensor category*.

**Example** If $\mathcal{A}$ is a tensor category, then $K^?(\mathcal{A})$ inherits a tensor structure, by the usual tensor product of complexes, and becomes a triangulated tensor category. (For $? = \emptyset$, $\mathcal{A}$ must admit infinite direct sums).

We form new triangulated categories from old ones by localizing.

### Definition

A full triangulated subcategory $\mathcal{B}$ of a triangulated category $\mathcal{A}$ is *thick* if $\mathcal{B}$ is closed under taking direct summands.

If $\mathcal{B}$ is a thick subcategory of $\mathcal{A}$, the set of morphisms $s : X \to Y$ in $\mathcal{A}$ which fit into a distinguished triangle $X \xrightarrow{s} Y \to Z \to X[1]$ with $Z$ in $\mathcal{B}$ forms a *saturated multiplicative system* of morphisms.

The intersection of thick subcategories of $\mathcal{A}$ is a thick subcategory of $\mathcal{A}$, So, for each set $\mathcal{T}$ of objects of $\mathcal{A}$, there is a smallest thick subcategory $\mathcal{B}$ containing $\mathcal{T}$, called the thick subcategory *generated* by $\mathcal{T}$.

Let $\mathcal{B}$ be a thick subcategory of a triangulated category $\mathcal{A}$. Let $\mathcal{S}$ be the saturated multiplicative system of map $A \xrightarrow{s} B$ with "cone" in $\mathcal{B}$.

Form the category $\mathcal{A}[\mathcal{S}^{-1}] = \mathcal{A}/\mathcal{B}$ with the same objects as $\mathcal{A}$, with

$$\mathrm{Hom}_{\mathcal{A}[\mathcal{S}^{-1}]}(X, Y) = \varinjlim_{s:X' \to X \in \mathcal{S}} \mathrm{Hom}_{\mathcal{A}}(X', Y).$$

Let $Q_{\mathcal{B}} : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ be the canonical functor.

### Theorem (Verdier)

*(i) $\mathcal{A}/\mathcal{B}$ is a triangulated category, where a triangle $T$ in $\mathcal{A}/\mathcal{B}$ is distinguished if $T$ is isomorphic to the image under $Q_{\mathcal{B}}$ of a distinguished triangle in $\mathcal{A}$.*
*(ii) The functor $Q_{\mathcal{B}} : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ is universal for exact functors $F : \mathcal{A} \to \mathcal{C}$ such that $F(B)$ is isomorphic to 0 for all $B$ in $\mathcal{B}$.*
*(iii) $\mathcal{S}$ is equal to the collection of maps in $\mathcal{A}$ which become isomorphisms in $\mathcal{A}/\mathcal{B}$ and $\mathcal{B}$ is the subcategory of objects of $\mathcal{A}$ which becomes isomorphic to zero in $\mathcal{A}/\mathcal{B}$.*

If $\mathcal{A}$ is a triangulated tensor category, and $\mathcal{B}$ a thick subcategory, call $\mathcal{B}$ a *thick tensor subcategory* if $A$ in $\mathcal{A}$ and $B$ in $\mathcal{B}$ implies that $A \otimes B$ and $B \otimes A$ are in $\mathcal{B}$.

The quotient $Q_{\mathcal{B}} : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ of $\mathcal{A}$ by a thick tensor subcategory inherits the tensor structure, and the distinguished triangles are preserved by tensor product with an object.

**Example** The classical example is the *derived category* $D^?(\mathcal{A})$ of an abelian category $\mathcal{A}$. $\mathcal{D}^?(\mathcal{A})$ is the localization of $K^?(\mathcal{A})$ with respect to the multiplicative system of *quasi-isomorphisms* $f : A \to B$, i.e., $f$ which induce isomorphisms $H^n(f) : H^n(A) \to H^n(B)$ for all $n$.

If $\mathcal{A}$ is an abelian tensor category, then $D^-(\mathcal{A})$ inherits a tensor structure $\otimes^L$ if each object $A$ of $\mathcal{A}$ admits a surjection $P \to A$ where $P$ is *flat*, i.e. $M \mapsto M \otimes P$ is an exact functor on $\mathcal{A}$. If each $A$ admits a finite flat (right) resolution, then $D^b(\mathcal{A})$ has a tensor structure $\otimes^L$ as well. The tensor structure $\otimes^L$ is given by forming for each $A \in K^?(\mathcal{A})$ a quasi-isomorphism $P \to A$ with $P$ a complex of flat objects in $\mathcal{A}$, and defining

$$A \otimes^L B := \mathrm{Tot}(P \otimes B).$$

# Triangulated categories of motives

To solve the problem of the partially defined composition of correspondences, Voevodsky introduces the notion of *finite* correspondences, for which all compositions are defined.

Recall:

**Definition**

Let $X$ and $Y$ be in $\mathbf{Sm}/k$. The group $z_{\mathrm{fin}}(Y)(X)$ is the subgroup of $z(X \times_k Y)$ generated by integral closed subschemes $W \subset X \times_k Y$ such that

1. the projection $p_1 : W \to X$ is finite
2. the image $p_1(W) \subset X$ is an irreducible component of $X$.

Write $\mathrm{Cor}_{\mathrm{fin}}(X, Y) := z_{\mathrm{fin}}(Y)(X)$. The elements of $\mathrm{Cor}_{\mathrm{fin}}(X, Y)$ are called the *finite* correspondences from $X$ to $Y$.

The following basic lemma is easy to prove:

**Lemma**

*Let $X$, $Y$ and $Z$ be in $\mathbf{Sch}_k$, $W \in \mathrm{Cor}_{\mathrm{fin}}(X, Y)$, $W' \in \mathrm{Cor}_{\mathrm{fin}}(Y, Z)$. Suppose that $X$ and $Y$ are irreducible. Then each irreducible component $C$ of $|W| \times Z \cap X \times |W'|$ is finite over $X$ and $p_X(C) = X$.*

Thus: for $W \in \mathrm{Cor}_{\mathrm{fin}}(X, Y)$, $W' \in \mathrm{Cor}_{\mathrm{fin}}(Y, Z)$, we have the composition:

$$W' \circ W := p_{XZ*}(p_{XY}^*(W) \cdot p_{YZ}^*(W')),$$

This operation yields an associative bilinear *composition law*

$$\circ : \mathrm{Cor}_{\mathrm{fin}}(Y, Z) \times \mathrm{Cor}_{\mathrm{fin}}(X, Y) \to \mathrm{Cor}_{\mathrm{fin}}(X, Z).$$

### Definition

The category $\mathrm{Cor_{fin}}(k)$ is the category with the same objects as $\mathbf{Sm}/k$, with

$$\mathrm{Hom}_{\mathrm{Cor_{fin}}(k)}(X, Y) := \mathrm{Cor_{fin}}(X, Y),$$

and with the composition as defined above.

**Remarks** (1) We have the functor $\mathbf{Sm}/k \to \mathrm{Cor_{fin}}(k)$ sending a morphism $f : X \to Y$ in $\mathbf{Sm}/k$ to the graph $\Gamma_f \subset X \times_k Y$.

(2) We write the morphism corresponding to $\Gamma_f$ as $f_*$, and the object corresonding to $X \in \mathbf{Sm}/k$ as $[X]$.

(3) The operation $\times_k$ (on smooth $k$-schemes and on cycles) makes $\mathrm{Cor_{fin}}(k)$ a tensor category. Thus, the bounded homotopy category $K^b(\mathrm{Cor_{fin}}(k))$ is a triangulated tensor category.

### Definition

The category $\widehat{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ is the localization of $K^b(\mathrm{Cor}_{\mathrm{fin}}(k))$, as a triangulated tensor category, by

- *Homotopy.* For $X \in \mathbf{Sm}/k$, invert $p_* : [X \times \mathbb{A}^1] \to [X]$
- *Mayer-Vietoris.* Let $X$ be in $\mathbf{Sm}/k$. Write $X$ as a union of Zariski open subschemes $U, V$: $X = U \cup V$.
  We have the canonical map

  $$\mathrm{Cone}([U \cap V] \xrightarrow{(j_{U,U\cap V*}, -j_{V,U\cap V*})} [U] \oplus [V]) \xrightarrow{(j_{U*}+j_{V*})} [X]$$

  since $(j_{U*} + j_{V*}) \circ (j_{U,U\cap V*}, -j_{V,U\cap V*}) = 0$. Invert this map.

The category $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$ of *effective geometric motives* is the pseudo-abelian hull of $\widehat{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ (Balmer-Schlichting).

Sending $X \in \mathbf{Sm}/k$ to the image of $[X]$ in $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$ gives the functor

$$m : \mathbf{Sm}/k \to DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$$

with

$$m(X \amalg Y) = m(X) \oplus m(Y)$$
$$m(X \times_k Y) = m(X) \otimes m(Y)$$

### Definition (The Tate motive)

$\mathbb{Z}(1)$ is the complex

$$[\mathbb{P}^1] \xrightarrow{p_*} [\mathrm{Spec}\, k]$$

with $[\mathbb{P}^1]$ in degree 2. Let $\mathbb{Z}(n) := \mathbb{Z}(1)^{\otimes n}$, $\mathbb{Z} = \mathbb{Z}(0) = m(\mathrm{Spec}\, k)$.

The cell decomposition of $\mathbb{P}^N$ yields:

$$m(\mathbb{P}^N) = \oplus_{n=0}^{N} \mathbb{Z}(n)[2n].$$

For $M \in DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$, $n \geq 0$, set

$$M(n) := M \otimes \mathbb{Z}(n).$$

**Definition**

For $X \in \mathbf{Sm}/k$, define

$$H_n^{\mathrm{mot}}(X, \mathbb{Z}) := \mathrm{Hom}_{DM_{\mathrm{gm}}^{\mathrm{eff}}(k)}(\mathbb{Z}[n], m(X))$$

and

$$H^p_{\mathrm{mot}}(X, \mathbb{Z}(q)) := \mathrm{Hom}_{DM_{\mathrm{gm}}^{\mathrm{eff}}(k)}(m(X), \mathbb{Z}(q)[p]).$$

Many structural properties of motivic homology and cohomology follows directly from the construction of $DM^{\rm eff}_{\rm gm}(k)$.For example:

- (Homotopy invariance)
  $p^*: H^p_{\rm mot}(X, \mathbb{Z}(q)) \to H^p_{\rm mot}(X \times \mathbb{A}^1, \mathbb{Z}(q))$ is an isomorphism
- (Mayer-Vietoris) If $X = U \cup V$, $U, V$ open, there is a long exact Mayer-Vietoris sequence

$$\ldots \to H^p(X, \mathbb{Z}(q)) \to H^p(U, \mathbb{Z}(q)) \oplus H^p(V, \mathbb{Z}(q)) \\ \to H^p(U \cap V, \mathbb{Z}(q)) \to H^{p-1}(X, \mathbb{Z}(q)) \to \ldots$$

For (1), use: $p : m(X \times \mathbb{A}^1) \to m(X)$ is an isomorphism in $DM^{\rm eff}_{\rm gm}(k)$.
For (2), use: we have a distinguished triangle $m(U \cap V) \to m(U) \oplus m(V) \to m(X) \to m(U \cap V)[1]$ in $DM^{\rm eff}_{\rm gm}(k)$.

It is very difficult to make computations, however, for instance, to see that one recovers the (co)homology we have defined using cycle complexes.

For this, we need a sheaf-theoretic extension of $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$. We begin with a quick review of sheaves on a Grothendieck site.

A *presheaf* $P$ on a small category $\mathcal{C}$ with values in a category $\mathcal{A}$ is a functor

$$P : \mathcal{C}^{\mathrm{op}} \to \mathcal{A}.$$

Morphisms of presheaves are natural transformations of functors. This defines the category of $\mathcal{A}$-valued presheaves on $\mathcal{C}$, $PreShv^{\mathcal{A}}(\mathcal{C})$.

**Theorem**

*(1) If $\mathcal{A}$ is an abelian category, then so is $PreShv^{\mathcal{A}}(\mathcal{C})$, with kernel and cokernel defined objectwise: For $f : F \to G$,*

$$ker(f)(x) = \ker(f(x) : F(x) \to G(x));$$
$$\text{coker}(f)(x) = \text{coker}(f(x) : F(x) \to G(x)).$$

*(2) For $\mathcal{A} = \mathbf{Ab}$, $PreShv^{\mathbf{Ab}}(\mathcal{C})$ has enough injectives.*

### Definition

Let $\mathcal{C}$ be a category. A *Grothendieck pre-topology* $\tau$ on $\mathcal{C}$ is given by defining, for $X \in \mathcal{C}$, a collection $\mathrm{Cov}_\tau(X)$ of *covering families* of $X$: a covering family of $X$ is a set of morphisms $\{f_\alpha : U_\alpha \to X\}$ in $\mathcal{C}$.

These satisfy some axioms, making a covering family the analog of coverings by a basis of open sets for a topological space, with

- a member $f_\alpha : U_\alpha \to X$ corresponding to an open subset $U_\alpha \subset T$
- fiber product $U_\alpha \times_X U_\beta$ corresponding to intersection $U_\alpha \cap U_\beta$ of open subsets

One requires

1. $\mathrm{id}_X$ is a covering of $X$
2. if $\{f_\alpha : U_\alpha \to X\}$ is a covering of $X$, and $f : Y \to X$ is a morphism, then $\{p_2 : U_\alpha \times_X Y \to Y\}$ is a covering of $Y$
3. if $\{f_\alpha : U_\alpha \to X\}$ is a covering of $X$ and $\{g_{\alpha\beta} : V_{\alpha\beta} \to U_\alpha\}$ is a covering of $U_\alpha$ for each $\alpha$, then $\{f_\alpha \circ g_{\alpha\beta} : V_{\alpha\beta} \to X\}$ is a covering of $X$.

A category with a (pre) topology is a *site*

For $S$ presheaf of abelian groups on $\mathcal{C}$ and $\{f_\alpha : U_\alpha \to X\} \in \mathrm{Cov}_\tau(X)$ for some $X \in \mathcal{C}$, we have the "restriction" morphisms

$$\begin{aligned} f_\alpha^* &: S(X) \to S(U_\alpha) \\ p_{1,\alpha,\beta}^* &: S(U_\alpha) \to S(U_\alpha \times_X U_\beta) \\ p_{2,\alpha,\beta}^* &: S(U_\beta) \to S(U_\alpha \times_X U_\beta). \end{aligned}$$

Taking products, we have the sequence of abelian groups

$$0 \to S(X) \xrightarrow{\prod f_\alpha^*} \prod_\alpha S(U_\alpha) \xrightarrow{\prod p_{1,\alpha,\beta}^* - \prod p_{2,\alpha,\beta}^*} \prod_{\alpha,\beta} S(U_\alpha \times_X U_\beta).$$

$$0 \to S(X) \xrightarrow{\prod f_\alpha^*} \prod_\alpha S(U_\alpha) \xrightarrow{\prod p_{1,\alpha,\beta}^* - \prod p_{2,\alpha,\beta}^*} \prod_{\alpha,\beta} S(U_\alpha \times_X U_\beta).$$

### Definition

A presheaf $S$ is a *sheaf* for $\tau$ if for each covering family $\{f_\alpha : U_\alpha \to X\} \in \mathrm{Cov}_\tau$, the above sequence is exact. The category $Shv_\tau^{\mathbf{Ab}}(\mathcal{C})$ of sheaves of abelian groups on $\mathcal{C}$ for $\tau$ is the full subcategory of $PreShv^{\mathbf{Ab}}(\mathcal{C})$ with objects the sheaves.

## Proposition

*(1) The inclusion* $i : Shv_\tau^{\mathbf{Ab}}(\mathcal{C}) \to PreShv_\tau^{\mathbf{Ab}}(\mathcal{C})$ *admits a left adjoint: "sheafification".*
*(2)* $Shv_\tau^{\mathbf{Ab}}(\mathcal{C})$ *is an abelian category: For* $f : F \to G$, $\ker(f)$ *is the presheaf kernel.* $\mathrm{coker}(f)$ *is the sheafification of the presheaf cokernel.*
*(3)* $Shv_\tau^{\mathbf{Ab}}(\mathcal{C})$ *has enough injectives.*

### Definition

Let $X$ be a $k$-scheme of finite type. A *Nisnevich cover* $\mathcal{U} \to X$ is an étale morphism of finite type such that, for each finitely generated field extension $F$ of $k$, the map on $F$-valued points $\mathcal{U}(F) \to X(F)$ is surjective.

Using Nisnevich covers as covering families gives us the *small Nisnevich site on* $X$, $X_{\mathrm{Nis}}$.

*Notation* $\mathrm{Sh}^{\mathrm{Nis}}(X) :=$ Nisnevich sheaves of abelian groups on $X$
For a presheaf $\mathcal{F}$ on $\mathbf{Sm}/k$ or $X_{\mathrm{Nis}}$, we let $\mathcal{F}_{\mathrm{Nis}}$ denote the associated sheaf.

We now return to motives.

The sheaf-theoretic construction of mixed motives is based on the notion of a *Nisnevich sheaf with transfer*.

### Definition

(1) The category $\mathrm{PST}(k)$ of presheaves with transfer is the category of presheaves of abelian groups on $\mathrm{Cor}_{\mathrm{fin}}(k)$ which are additive as functors $\mathrm{Cor}_{\mathrm{fin}}(k)^{\mathrm{op}} \to \mathbf{Ab}$.
(2) The category of Nisnevich sheaves with transfer on $\mathbf{Sm}/k$, $\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))$, is the full subcategory of $\mathrm{PST}(k)$ with objects those $F$ such that, for each $X \in \mathbf{Sm}/k$, the restriction of $F$ to $X_{\mathrm{Nis}}$ is a sheaf.

**Remark** A PST $F$ is a presheaf on $\mathbf{Sm}/k$ together with *transfer maps*

$$\mathrm{Tr}(a) : F(Y) \to F(X)$$

for every finite correspondence $a \in \mathrm{Cor}_{\mathrm{fin}}(X, Y)$, with:

- $\mathrm{Tr}(\Gamma_f) = f^*$
- $\mathrm{Tr}(a \circ b) = \mathrm{Tr}(b) \circ \mathrm{Tr}(a)$
- $\mathrm{Tr}(a \pm b) = \mathrm{Tr}(a) \pm \mathrm{Tr}(b)$.

### Definition

Let $F$ be a presheaf of abelian groups on $\mathbf{Sm}/k$. We call $F$ *homotopy invariant* if for all $X \in \mathbf{Sm}/k$, the map

$$p^* : F(X) \to F(X \times \mathbb{A}^1)$$

is an isomorphism.
We call $F$ *strictly homotopy invariant* if for all $q \geq 0$, the cohomology presheaf $X \mapsto H^q(X_{\mathrm{Nis}}, F_{\mathrm{Nis}})$ is homotopy invariant.

### Theorem (PST)

*Let $F$ be a homotopy invariant PST on $\mathbf{Sm}/k$. Then*

1. *The cohomology presheaves $X \mapsto H^q(X_{\mathrm{Nis}}, F_{\mathrm{Nis}})$ are PST's*
2. *$F_{\mathrm{Nis}}$ is strictly homotopy invariant:*
   $H^q(X_{\mathrm{Nis}}, F_{\mathrm{Nis}}) \cong H^q(X \times \mathbb{A}^1_{\mathrm{Nis}}, F_{\mathrm{Nis}})$ *for all $X, q$.*
3. *$F_{\mathrm{Zar}} = F_{\mathrm{Nis}}$ and $H^q(X_{\mathrm{Zar}}, F_{\mathrm{Zar}}) = H^q(X_{\mathrm{Nis}}, F_{\mathrm{Nis}})$.*

### Corollary

*Let $F$ be a homotopy invariant Nisnevich sheaf with transfers. Then all the Nisnevich cohomology sheaves $\mathcal{H}^q_{\mathrm{Nis}}(F)$ are homotopy invariant sheaves with transfers.*

### Definition

Inside the derived category $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))$, we have the full subcategory $DM^{\mathrm{eff}}_-(k)$ consisting of complexes whose cohomology sheaves are homotopy invariant.

### Proposition

*$DM^{\mathrm{eff}}_-(k)$ is a triangulated subcategory of $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))$.*

This follows from the PST theorem: $F$ a homotopy invariant sheaf with transfer $\implies$ all cohomology sheaves are homotopy invariant sheaves with transfer, so homotopy invariance "makes sense in the derived category".

We can promote the Suslin complex construction to an operation on $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))$.

### Definition

Let $F$ be a presheaf on $\mathrm{Cor}_{\mathrm{fin}}(k)$. Define the presheaf $\mathcal{C}_n^{\mathrm{Sus}}(F)$ by

$$\mathcal{C}_n^{\mathrm{Sus}}(F)(X) := F(X \times \Delta^n)$$

The *Suslin complex* $\mathcal{C}_*^{\mathrm{Sus}}(F)$ is the complex with differential

$$d_n := \sum_i (-1)^i \delta_i^* : \mathcal{C}_{n+1}^{\mathrm{Sus}}(F) \to \mathcal{C}_n^{\mathrm{Sus}}(F).$$

**Remarks** (1) If $F$ is a sheaf with transfers on $\mathbf{Sm}/k$, then $\mathcal{C}_*^{\mathsf{Sus}}(F)$ is a complex of sheaves with transfers.

(2) The homology presheaves $h_i(F) := \mathcal{H}^{-i}(\mathcal{C}_*^{\mathsf{Sus}}(F))$ are homotopy invariant. Thus, by Voevodsky's PST theorem, the associated Nisnevich sheaves $h_i^{\mathsf{Nis}}(F)$ are homotopy invariant. We thus have the functor

$$\mathcal{C}_*^{\mathsf{Sus}} : \mathsf{Sh}^{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k)) \to DM_-^{\mathrm{eff}}(k).$$

For $X \in \mathbf{Sm}/k$, we have the representable presheaf with transfers $\mathbb{Z}^{tr}(X) := \mathrm{Cor}_{\mathsf{fin}}(-, X)$. This is in fact a Nisnevich sheaf.

The Suslin complex $C_*^{\mathsf{Sus}}(X)$ is just $\mathcal{C}_*^{\mathsf{Sus}}(\mathbb{Z}^{tr}(X))(\mathrm{Spec}\, k)$.

We denote $\mathcal{C}_*^{\mathsf{Sus}}(\mathbb{Z}^{tr}(X))$ by $\mathcal{C}_*^{\mathsf{Sus}}(X)$.

For $X \in \mathbf{Sm}/k$, $\mathbb{Z}^{tr}(X)$ is the free sheaf with transfers generated by the representable sheaf of sets $\mathrm{Hom}(-, X)$. Thus: there is a canonical isomorphism

$$\mathrm{Hom}_{\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k))}(\mathbb{Z}^{tr}(X), F) = F(X)$$

and more generally: For $F \in \mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k))$ there is a canonical isomorphism

$$\mathrm{Ext}^n_{\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k))}(\mathbb{Z}^{tr}(X), F) \cong H^n(X_{\mathsf{Nis}}, F)$$

and for $C^* \in D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k)))$ there is a canonical isomorphism

$$\mathrm{Hom}_{D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k)))}(\mathbb{Z}^{tr}(X), C^*[n]) \cong \mathbb{H}^n(X_{\mathsf{Nis}}, C^*).$$

Let $\mathcal{A}$ is the localizing subcategory of $D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))$ generated by complexes

$$\mathbb{Z}^{tr}(X \times \mathbb{A}^1) \xrightarrow{p_1} \mathbb{Z}^{tr}(X); \quad X \in \mathbf{Sm}/k,$$

and let

$$Q_{\mathbb{A}^1} : D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))) \to D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))/\mathcal{A}$$

be the quotient functor.

Since $\mathbb{Z}^{tr}(X) = C_0(X)$, we have the canonical map

$$\iota_X : \mathbb{Z}^{tr}(X) \to C_*(X)$$

This acts like an "injective resolution" of $\mathbb{Z}^{tr}(X)$, with respect to the localization $\mathbb{Q}_{\mathbb{A}^1}$.

Theorem

*1. The functor*

$$\mathcal{C}_*^{\mathsf{Sus}} : \mathsf{Sh}^{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k)) \to DM_-^{\mathrm{eff}}(k).$$

*extends to an exact functor*

$$\mathbf{R}\mathcal{C}_*^{\mathsf{Sus}} : D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k))) \to DM_-^{\mathrm{eff}}(k),$$

*left adjoint to the inclusion* $DM_-^{\mathrm{eff}}(k) \to D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k)))$.

*2.* $\mathbf{R}\mathcal{C}_*^{\mathsf{Sus}}$ *identifies* $DM_-^{\mathrm{eff}}(k)$ *with* $D^-(\mathsf{Sh}_{\mathsf{Nis}}(\mathsf{Cor}_{\mathsf{fin}}(k)))/\mathcal{A}$

We define a tensor structure on $\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))$:
Set $\mathbb{Z}^{tr}(X) \otimes \mathbb{Z}^{tr}(Y) := \mathbb{Z}^{tr}(X \times Y)$.

This extends to a tensor operation on $\otimes^L$ on $D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))$.

We make $DM_-^{\mathrm{eff}}(k)$ a tensor triangulated category via the localization theorem:

$$M \otimes N := \mathbf{R}C_*(\alpha(M) \otimes^L \alpha(N)),$$

$\alpha : DM_-^{\mathrm{eff}}(k) \to D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))$ the inclusion.

**Theorem**

*There is a commutative diagram of exact tensor functors*

$$\begin{array}{ccc} K^b(\mathrm{Cor}_{\mathrm{fin}}(k)) & \xrightarrow{\mathbb{Z}^{tr}} & D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))) \\ \downarrow & & \downarrow \mathbf{R}C_* \\ DM^{\mathrm{eff}}_{\mathrm{gm}}(k) & \xrightarrow[i]{} & DM^{\mathrm{eff}}_{-}(k) \end{array}$$

*such that*

1. *$i$ is a full embedding with dense image.*
2. $\mathbf{R}\mathcal{C}^{\mathrm{Sus}}_*(\mathbb{Z}^{tr}(X)) \cong \mathcal{C}^{\mathrm{Sus}}_*(X)$.

Explanation: Sending $X \in \mathbf{Sm}/k$ to $\mathbb{Z}^{tr}(X) \in \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))$ extends to an additive functor

$$\mathbb{Z}^{tr} : \mathrm{Cor}_{\mathrm{fin}}(k) \to \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))$$

and then to an exact functor

$$\mathbb{Z}^{tr} : K^b(\mathrm{Cor}_{\mathrm{fin}}(k)) \to K^b(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))) \to D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))).$$

One shows

1. Sending $X$ to $\mathcal{C}_*^{\mathrm{Sus}}(X)$ sends the complexes

$$[X \times \mathbb{A}^1] \to [X]; \quad [U \cap V] \to [U] \oplus [V] \to [U \cup V]$$

to "zero". Thus $i$ exists.

2. Using results of Ne'eman, one shows that $i$ is a full embedding with dense image.

## Corollary

*For $X$ and $Y \in \mathbf{Sm}/k$,*

$$\mathrm{Hom}_{DM^{\mathrm{eff}}_{\mathrm{gm}}(k)}(m(Y), m(X)[n]) \cong \mathbb{H}^n(Y_{\mathrm{Nis}}, \mathcal{C}^{\mathrm{Sus}}_*(X)) \cong \mathbb{H}^n(Y_{\mathrm{Zar}}, \mathcal{C}^{\mathrm{Sus}}_*(X)).$$

Because:

$$\begin{aligned}\mathrm{Hom}_{DM^{\mathrm{eff}}_{\mathrm{gm}}(k)}(m(Y), m(X)[n]) &= \mathrm{Hom}_{DM^{\mathrm{eff}}_{-}(k)}(\mathcal{C}^{\mathrm{Sus}}_*(Y), \mathcal{C}^{\mathrm{Sus}}_*(X)[n]) \\ &= \mathrm{Hom}_{D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))}(\mathbb{Z}^{tr}(Y), \mathcal{C}^{\mathrm{Sus}}_*(X)[n]) \\ &= \mathbb{H}^n(Y_{\mathrm{Nis}}, \mathcal{C}^{\mathrm{Sus}}_*(X))\end{aligned}$$

plus the PST theorem: $\mathbb{H}^n(Y_{\mathrm{Nis}}, \mathcal{C}^{\mathrm{Sus}}_*(X)) = \mathbb{H}^n(Y_{\mathrm{Zar}}, \mathcal{C}^{\mathrm{Sus}}_*(X))$.

Taking $Y = \mathrm{Spec}\, k$, the corollary yields

$$H_n^{\mathrm{mot}}(X, \mathbb{Z}) = \mathrm{Hom}_{DM_{\mathrm{gm}}^{\mathrm{eff}}(k)}(\mathbb{Z}[n], m(X)) \cong H_n(\mathcal{C}_*^{\mathrm{Sus}}(X)(k)) = H_n(C_*^{\mathrm{Sus}}(X)) = H_n^{\mathrm{Sus}}(X, \mathbb{Z}).$$

Since $m(\mathbb{P}^q) = \oplus_{n=0}^{q}\mathbb{Z}(n)[2n]$ we have

$$\mathcal{C}_*^{\mathrm{Sus}}(\mathbb{Z}^{tr}(q)[2q])(Y) \cong C_*^{\mathrm{Sus}}(\mathbb{P}^q/\mathbb{P}^{q-1})(Y) = \Gamma_{FS}(q)(Y)[2q]$$

Applying the corollary with $X = \mathbb{Z}^{tr}(q)$ gives

$$\begin{aligned} H^p_{\mathrm{mot}}(Y, \mathbb{Z}(q)) &:= \mathrm{Hom}_{DM^{\mathrm{eff}}_{\mathrm{gm}}(k)}(m(Y), \mathbb{Z}(q)[p]) \\ &\cong \mathbb{H}^p(Y_{\mathrm{Zar}}, \mathcal{C}_*^{\mathrm{Sus}}(\mathbb{Z}(q))) = \mathbb{H}^p(Y_{\mathrm{Zar}}, \Gamma_{FS}(q)) \\ &\cong H^p(\Gamma_{FS}(q)(Y)) = H^p(Y, \mathbb{Z}(q)). \end{aligned}$$

Thus, we have identified motivic (co)homology with universal (co)homology.