Algebraic Cobordism

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Outline:

- Describe the setting of "oriented cohomology over a field k"
- Describe the fundamental properties of algebraic cobordism
- Sketch the construction of algebraic cobordism
- Give an application to Donaldson-Thomas invariants

Oriented cohomology

k: a field. $\mathbf{Sm}/k$: smooth quasi-projective varieties over k.

What should "cohomology of smooth varieties over k" be?

This should be at least the following:

D1. An additive contravariant functor A^* from $\mathbf{Sm}/k$ to graded (commutative) rings:

$$X \mapsto A^*(X);$$
$$(f : Y \to X) \mapsto f^* : A^*(X) \to A^*(Y).$$

D2. For each projective morphisms $f : Y \to X$ in $\mathbf{Sm}/k$, a push-foward map ($d = \mathrm{codim} f$)

$$f_* : A^*(Y) \to A^{*+d}(X)$$

These should satisfy some compatibilities and additional axioms. For instance, we should have

A1. $(fg)_* = f_* g_*$; $\mathrm{id}_* = \mathrm{id}$

A2. For $f : Y \to X$ projective, f_* is $A^*(X)$-linear:

$$f_*(f^*(x) \cdot y) = x \cdot f_*(y).$$

A3. Let

$$\begin{CD} W @>{f'}>> Y \\ @V{g'}VV @VV{g}V \\ Z @>>{f}> X \end{CD}$$

be a transverse cartesian square in $\mathbf{Sm}/k$, with g projective. Then

$$f^* g_* = g'_* f'^*.$$

Examples

- *singular cohomology:* $(k \subset \mathbb{C})$ $X \mapsto H^{2*}_{sing}(X(\mathbb{C}), \mathbb{Z})$.
- *topological K-theory:* $X \mapsto K^{2*}_{top}(X(\mathbb{C}))$
- *complex cobordism:* $X \mapsto MU^{2*}(X(\mathbb{C}))$
- *the Chow ring:* $X \mapsto \mathrm{CH}^*(X)$.
- *algebraic K_0:* $X \mapsto K_0(X)[\beta, \beta^{-1}]$
- *algebraic cobordism:* $X \mapsto MGL^{*,*}(X)$

Chern classes

Once we have f^* and f_*, we have the 1st Chern class of a line bundle $L \to X$:

Let $s : X \to L$ be the zero-section. Define

$$c_1(L) := s^*(s_*(1_X)) \in A^1(X).$$

If we want to extend to a good theory of A^*-valued Chern classes of vector bundles, we need two additional axioms.

Axioms for oriented cohomology

PB:
Let $E \to X$ be a rank n vector bundle,
$\mathbb{P}(E) \to X$ the projective-space bundle,
$O_E(1) \to \mathbb{P}(E)$ the tautological quotient line bundle.
$\xi := c_1(O_E(1)) \in A^1(\mathbb{P}(E))$.

Then $A^*(\mathbb{P}(E))$ is a free $A^*(X)$-module with basis $1, \xi, \ldots, \xi^{n-1}$.

EH:
Let $p : V \to X$ be an affine-space bundle. Then $p^* : A^*(X) \to A^*(V)$ is an isomorphism.

Higher Chern classes

Once we have these two axioms, use Grothendieck's method to construct Chern classes:

Let $E \to X$ be a vector bundle of rank n. By (PB), there are unique elements $c_i(E) \in A^i(X)$, $i = 0, \ldots, n$, with $c_0(E) = 1$ and

$$\sum_{i=0}^{n} (-1)^i c_i(E) \xi^{n-i} = 0 \in A^*(\mathbb{P}(E)),$$

$\xi := c_1(O_E(1))$.

The proof of the Whitney product formula uses the *splitting principle* and additional facts which rely on (PB) and (EH).

Recap:

Definition k a field. An *oriented cohomology theory A over k* is a functor

$$A^* : \mathbf{Sm}/k^{\mathrm{op}} \to \textbf{GrRing}$$

together with push-forward maps

$$g_* : A^*(Y) \to A^{*+d}(X)$$

for each projective morphism $g : Y \to X$,
$d = \mathrm{codim} g$, satisfying the axioms A1-3, PB and EV:

- functoriality of push-forward,
- projection formula,
- compatibility of f^* and g_* in transverse cartesian squares,
- projective bundle formula,
- homotopy.

The formal group law

A: an oriented cohomology theory.

The projective bundle formula yields:

$$A^*(\mathbb{P}^\infty) := \varprojlim_n A^*(\mathbb{P}^n) = A^*(k)[[u]]$$

where the variable u maps to $c_1(\mathcal{O}(1))$ at each finite level. Similarly

$$A^*(\mathbb{P}^\infty \times \mathbb{P}^\infty) = A^*(k)[[u, v]].$$

where

$$u = c_1(\mathcal{O}(1, 0)),\ v = c_1(\mathcal{O}(0, 1))$$

$$\mathcal{O}(1, 0) = p_1^*\mathcal{O}(1);\ \mathcal{O}(0, 1) = p_2^*\mathcal{O}(1).$$

Let $\mathcal{O}(1,1) = p_1^*\mathcal{O}(1) \otimes p_2^*\mathcal{O}(1) = \mathcal{O}(1,0) \otimes \mathcal{O}(0,1)$. There is a unique

$$F_A(u,v) \in A^*(k)[[u,v]]$$

with

$$F_A(c_1(\mathcal{O}(1,0)), c_1(\mathcal{O}(0,1))) = c_1(\mathcal{O}(1,1))$$

in $A^1(\mathbb{P}^\infty \times \mathbb{P}^\infty)$.

Since $\mathcal{O}(1)$ is the universal line bundle, we have

$$F_A(c_1(L), c_1(M)) = c_1(L \otimes M) \in A^1(X)$$

for *any* two line bundles $L, M \to X$. (Jouanolou's trick+ axiom (EH)).

Properties of $F_A(u,v)$

These all follow from properties of $\otimes$:

- $1 \otimes L \cong L \cong L \otimes 1$
  $\Rightarrow F_A(0,u) = u = F_A(u,0)$.

- $L \otimes M \cong M \otimes L \Rightarrow F_A(u,v) = F_A(v,u)$.

- $(L \otimes M) \otimes N \cong L \otimes (M \otimes N)$
  $\Rightarrow F_A(F_A(u,v),w) = F_A(u,F_A(v,w))$.

So: $F_A(u,v)$ defines a *formal group law* (commutative, rank 1) over $A^*(k)$.

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⚠ c_1 *is not necessarily additive!*

Topological background: $\mathbb{C}$-oriented theories

The axioms for an oriented cohomology theory on $\mathbf{Sm}/k$ are abstracted from Quillen's notion of a *$\mathbb{C}$-oriented cohomology theory* on the category of differentiable manifolds. This is a cohomology theory $M \mapsto E^*(M)$ plus pushforward maps f_* for proper "$\mathbb{C}$-oriented" maps f, satisfying the analog of our axioms (with shift of $2\dim_{\mathbb{C}}$ instead of $\dim_{\mathbb{C}}$).

A $\mathbb{C}$-oriented theory E has a formal group law with coefficients in $E^*(pt)$.

Examples

1. $H^*(-,\mathbb{Z})$ has the additive formal group law $(u+v,\mathbb{Z})$.

2. K^*_{top} has the multiplicative formal group law $(u+v-\beta uv, \mathbb{Z}[\beta,\beta^{-1}])$, $\beta =$ Bott element in $K^{-2}_{top}(pt)$.

The Lazard ring and Quillen's theorem

There is a universal formal group law $F_{\mathbb{L}}$, with coefficient ring the *Lazard ring* $\mathbb{L}$. For an oriented theory A on $\mathbf{Sm}/k$, let

$$\phi_A : \mathbb{L} \to A^*(k);\ \phi(F_{\mathbb{L}}) = F_A.$$

be the ring homomorphism classifying F_A. In the setting of a topological $\mathbb{C}$-oriented theory E, we have instead $\phi_E : \mathbb{L} \to E^*(pt)$.

Theorem 1 (Quillen) *(1) Complex cobordism MU^* is the universal $\mathbb{C}$-oriented theory.*

(2) $\phi_{MU} : \mathbb{L} \to MU^*(pt)$ is an isomorphism, i.e., F_{MU} is the universal group law.

The Conner-Floyd theorem

Note. Let $\phi : \mathbb{L} = MU^*(pt) \to R$ classify a group law F_R over R. If ϕ satisfies the "Landweber exactness" conditions, form the $\mathbb{C}$-oriented spectrum $MU \wedge_\phi R$, with

$$(MU \wedge_\phi R)(X) = MU^*(X) \otimes_{MU^*(pt)} R$$

and formal group law F_R.

Theorem 2 (Conner-Floyd)

$K^*_{top} = MU \wedge_\times \mathbb{Z}[\beta, \beta^{-1}]$; *$K^*_{top}$ is the universal multiplicative oriented cohomology theory.*

Algebraic cobordism

The main theorem

Theorem 3 (L.-Morel) *Let k be a field of characteristic zero. There is a universal oriented cohomology theory Ω over k, called* algebraic cobordism*. Ω has the additional properties:*

Formal group law. *The classifying map $\phi_\Omega : \mathbb{L} \to \Omega^*(k)$ is an isomorphism, so F_Ω is the universal formal group law.*

Localization. *Let $i : Z \to X$ be a closed codimension d embedding of smooth varieties with complement $j : U \to X$. The sequence*

$$\Omega^{*-d}(Z) \xrightarrow{i_*} \Omega^*(X) \xrightarrow{j^*} \Omega^*(U) \to 0$$

is exact.

For an arbitrary formal group law $\phi : \mathbb{L} = \Omega^*(k) \to R$, $F_R := \phi(F_{\mathbb{L}})$, we have the oriented theory

$$X \mapsto \Omega^*(X) \otimes_{\Omega^*(k)} R := \Omega^*(X)_\phi.$$

$\Omega^*(X)_\phi$ is universal for theories whose group law factors through ϕ. Let

$$\Omega^*_\times := \Omega^* \otimes_{\mathbb{L}} \mathbb{Z}[\beta, \beta^{-1}]$$
$$\Omega^*_+ := \Omega^* \otimes_{\mathbb{L}} \mathbb{Z}.$$

The Conner-Floyd theorem extends to the algebraic setting:

Theorem 4 *The canonical map*

$$\Omega^*_\times \to K_0^{alg}[\beta, \beta^{-1}]$$

is an isomorphism, i.e., $K_0^{alg}[\beta, \beta^{-1}]$ *is the universal multiplicative theory over* k*.*

There is an additive version as well:

Theorem 5 *The canonical map*

$$\Omega^*_+ \to \mathrm{CH}^*$$

is an isomorphism, i.e., CH^* *is the universal additive theory over* k.

Remark

Define "connective algebraic K_0", $k_0^{alg} := \Omega^* \otimes_{\mathbb{L}} \mathbb{Z}[\beta]$.

$$k_0^{alg}/\beta = \mathrm{CH}^*$$
$$k_0^{alg}[\beta^{-1}] = K_0^{alg}[\beta, \beta^{-1}].$$

This realizes $K_0^{alg}[\beta, \beta^{-1}]$ as a deformation of CH^*.

Relation with motivic homotopy theory

$$\mathrm{CH}^n(X) \cong H^{2n}(X, \mathbb{Z}(n)) = H^{2n,n}(X)$$

$$K_0(X) \cong K^{2n,n}(X)$$

The universality of Ω^* gives a natural map

$$\nu_n(X) : \Omega^n(X) \to MGL^{2n,n}(X).$$

Conjecture 1 *$\Omega^n(X) \cong MGL^{2n,n}(X)$ for all n, all $X \in \mathbf{Sm}/k$.*

Note. (1) $\nu_n(X)$ is surjective, and an isomorphism after $\otimes\mathbb{Q}$.

(2) $\nu_n(k)$ is an isomorphism.

The construction of algebraic cobordism

The idea

We build $\Omega^*(X)$ following roughly Quillen's basic idea, defining generators and relations. The original description of Levine-Morel was rather complicated, but necessary for proving all the main properties of Ω^*. Following a suggestion of Pandharipande, we now have a very simple presentation, with the same kind of generators as for complex cobordism. The relations are also similar, but need to allow for "double-point degenerations".

The simplified presention requires the base-field k to have characteristic zero.

This is joint work with R. Pandharipande.

Generators

$\mathbf{Sch}_k$:= finite type k-schemes.

Definition Take $X \in \mathbf{Sch}_k$.

1. $\mathcal{M}(X)$:= the set of isomorphism classes of projective morphisms $f : Y \to X$, with $Y \in \mathbf{Sm}/k$.

2. Grade $\mathcal{M}(X)$:

$$\mathcal{M}_n(X) := \{f : Y \to X \in \mathcal{M}(X) \mid n = \dim_k Y\}.$$

3. $\mathcal{M}_*(X)$ is a graded monoid under $\coprod$; let $\mathcal{M}_*^+(X)$ be the group completion.

Explicitly: $\mathcal{M}_n^+(X)$ is the free abelian group on $f : Y \to X$ in $\mathcal{M}(X)$ with Y irreducible and $\dim_k Y = n$.

Double point degenerations

Definition Let C be a smooth curve, $c \in C$ a k-point. A morphism $\pi : Y \to C$ in $\mathbf{Sm}/k$ is a *double-point degeneration at* c if

$$\pi^{-1}(c) = S \cup T$$

with

1. S and T smooth,
2. S and T intersecting transversely on Y.

We allow the special cases $S \cap T = \emptyset$, or $T = \emptyset$.

The codimension two smooth subscheme $D := S \cap T$ is called the *double-point locus* of the degeneration.

The degeneration bundle

Let $\pi : Y \to C$ be a double-point degeneration at $c \in C(k)$, with

$$\pi^{-1}(c) = S \cup T; \; D := S \cap T.$$

Set $N_{D/S}$:= the normal bundle of D in S.

Set

$$\mathbb{P}(\pi, c) := \mathbb{P}(\mathcal{O}_D \oplus N_{D/S}),$$

a $\mathbb{P}^1$-bundle over D.

Let $N_{D/T}$:= the normal bundle of D in T.

$$N_{D/S} = \mathcal{O}_Y(T) \otimes \mathcal{O}_D; \;\; N_{D/T} = \mathcal{O}_Y(S) \otimes \mathcal{O}_D.$$

Since $\mathcal{O}_Y(S+T) \otimes \mathcal{O}_D \cong \mathcal{O}_D$,

$$N_{D/S} \cong N_{D/T}^{-1}.$$

So the definition of $\mathbb{P}(\pi, c)$ does not depend on the choice of S or T:

$$\mathbb{P}(\pi, c) = \mathbb{P}_D(\mathcal{O}_D \oplus N_{D/S}) = \mathbb{P}_D(\mathcal{O}_D \oplus N_{D/T}).$$

Double-point cobordisms

We impose the relation of *double point cobordism*:

Definition Let $f : Y \to X \times \mathbb{P}^1$ be a projective morphism with $Y \in \mathbf{Sm}/k$. Call f a *double-point cobordism* if

1. $p_2 \circ f : Y \to \mathbb{P}^1$ is a double-point degeneration at $0 \in \mathbb{P}^1$.

2. $(p_2 \circ f)^{-1}(1)$ is smooth.

Double-point relations

Let $f : Y \to X \times \mathbb{P}^1$ be a double-point cobordism.

Write $(p_2 \circ f)^{-1}(0) = Y_0 = S \cup T$, $(p_2 \circ f)^{-1}(1) = Y_1$, giving elements

$$[S \to X], [T \to X], [\mathbb{P}(p_2 \circ f, 0) \to X], [Y_1 \to X]$$

of $\mathcal{M}(X)$. The element

$$R(f) := [Y_1 \to X] - [S \to X] - [T \to X] + [\mathbb{P}(p_1 \circ f, 0) \to X]$$

is the *double-point relation* associated to the double-point cobordism f.

A presentation of algebraic cobordism

Definition For $X \in \mathbf{Sch}_k$, $\Omega_*^{dp}(X)$ (double-point cobordism) is the quotient of $\mathcal{M}_*^+(X)$ by the subgroup of generated by relations $\{R(f)\}$given by double-point cobordisms:

$$\Omega_*^{dp}(X) := \mathcal{M}_*^+(X)/<\{R(f)\}>$$

for all double-point cobordisms $f : Y \to X \times \mathbb{P}^1$. In other words, we impose all double-point cobordism relations

$$[Y_1 \to X] = [S \to X] + [T \to X] - [\mathbb{P}(p_2 \circ f, 0) \to X]$$

We have the homomorphism

$$\phi : \mathcal{M}_*^+ \to \Omega_*$$

sending $f : Y \to X$ to $f_*(1_Y) \in \Omega_*(X)$.

Theorem 6 (L.-Pandharipande) *The map ϕ descends to an isomorphism*

$$\phi : \Omega_*^{dp} \to \Omega_*$$

If we evaluate at $\mathrm{Spec}\, k$, we have the isomorphism

$$\phi(k) : \Omega_*^{dp}(k) \to \Omega_*(k).$$

Since $\Omega_*(k) = \mathbb{L}$ and the class of a smooth projective variety X in $\mathbb{L}$ is completely determined by the Chern numbers of X, the fact that $\phi(k)$ is an isomorphism can be expressed as:

Let $\psi : \mathcal{M}_d^+(k) \to \mathbb{Z}$ be a homomorphism that sends all double point relations $R(f)$ to zero. Then for each smooth projective variety X of dimension d, $\psi(X)$ depends only on the Chern numbers of X.

Dually, if X and Y are smooth projective varieties of the same dimension, and if X and Y have the same Chern numbers, then there exist double point cobordisms $f_k : W_k \to \operatorname{Spec} k \times \mathbb{P}^1$ and integers r_k such that

$$X - Y = \sum_k r_k R(f_k),$$

as a formal sum of irreducible smooth projective varieties over k.

Elementary structures in Ω_*^{dp}

- For $g : X \to X'$ projective, we have

$$g_* : \mathcal{M}_*(X) \to \mathcal{M}_*(X')$$
$$g_*(f : Y \to X) := (g \circ f : Y \to X')$$

- For $g : X' \to X$ smooth of dimension d, we have

$$g^* : \mathcal{M}_*(X) \to \mathcal{M}_{*+d}(X')$$
$$g^*(f : Y \to X) := (p_2 : Y \times_X X' \to X')$$

- Products over k induce an external product

$$\Omega_*^{dp}(X) \otimes \Omega_*^{dp}(Y) \to \Omega_*^{dp}(X \times Y).$$

• For $L \to X$ a globally generated line bundle, we have the *1st Chern class operator*

$$\tilde{c}_1(L) : \Omega^{dp}_*(X) \to \Omega^{dp}_{*-1}(X)$$
$$\tilde{c}_1(L)(f : Y \to X) := (f \circ i_D : D \to X)$$

$D :=$ the divisor of a general section of f^*L.

Ω^*_{dp} as oriented cohomology

It is not at all apparant that $\Omega^*_{dp}(X) := \Omega^{dp}_{\dim X - *}(X)$ has the structures/satisfies the axioms of an oriented theory on $\mathbf{Sm}/k$.

Ω_* was constructed as the "universal Borel-Moore functor of geometric type" on $\mathbf{Sch}_k$, a more elementary structure than an oriented cohomology theory.

To relate Ω_* and Ω^{dp}_*, we show that Ω^{dp}_* is a Borel-Moore functor of geometric type

B-M functors of geometric type

This is a "weak homology theory" A_*: $A_*(X)$ is a graded abelian group for each $X \in \mathbf{Sch}_k$ with

1. push-forward for projective morphisms
2. pull-back (with a shift) for smooth maps
3. external products, unit element $1 \in A_0(k)$
4. 1st Chern class operators $\tilde{c}_1(L) : A_*(X) \to A_{*-1}(X)$ for each line bundle $L \to X$.
5. Ring homomorphism $\phi_A : \mathbb{L}_* \to A_*(k)$, i.e., a formal group law F_A over $A_*(k)$

These satisfy some axioms:

(Dim) For $X \in \mathbf{Sm}/k$, set $1_X := p_X^*(1)$. Then

$$\tilde{c}_1(L)^{\dim X+1}(1_X) = 0.$$

(Sect) Let $i : D \to X$ be a smooth divisor on $X \in \mathbf{Sm}/k$. Then

$$i_*(1_D) = \tilde{c}_1(O_X(D))(1_X).$$

(FGL) For line bundles $L, M \to X$, $X \in \mathbf{Sm}/k$,

$$F_A(\tilde{c}_1(L), \tilde{c}_1(M))(1_X) = \tilde{c}_1(L \otimes M)(1_X).$$

in addition to standard compatibilities of f_*, f^* and $\tilde{c}_1$.

$\Omega_* = \Omega_*^{dp}$

The proof that $\Omega_* = \Omega_*^{dp}$ goes by showing that the 1st Chern class operators in Ω_*^{dp} (defined only for globally generated line bundles!) satisfy a formal group law.

This permits the extension of operators $\tilde{c}_1$ on Ω_*^{dp} to all L. The axioms (Dim), (Sect) and (FGL) are then not hard to verify.

The universality of Ω_* gives a surjective map $\Omega_* \to \Omega_*^{dp}$.

The double-point cobordism relation is satisfied in Ω_*, giving a surjective map $\Omega_*^{dp} \to \Omega_*$.

We give a sketch of the proof that the $\tilde{c}_1$ satisfy a formal group law at the end of the lecture, time permitting.

A conjecture of MNOP

Donaldson-Thomas invariants

Let X be a smooth projective 3-fold over $\mathbb{C}$ and let Hilb(X, n) be the Hilbert scheme of length n closed subschemes of X.

Maulik, Nekrasov, Okounkov and Pandharipande construct a natural "virtual fundamental class"

$$[\mathrm{Hilb}(X, n)]^{vir} \in \mathrm{CH}_0(\mathrm{Hilb}(X, n))$$

and define the "partition function"

$$Z(X, q) := 1 + \sum_{n \geq 1} \deg([\mathrm{Hilb}(X, n)]^{vir})q^n$$

MNOP conjecture (1st proved by Jun Li):

Conjecture 2 *Let $M(q)$ be the MacMahon function:*

$$M(q) = \prod_{n \geq 1} \frac{1}{(1-q^n)^n} = 1 + q + 3q^2 + 6q^3 + 13q^4 + \dots.$$

Then

$$Z(X,q) = M(q)^{\deg(c_3(T_X \otimes K_X))}$$

for all smooth projective X over $\mathbb{C}$.

Note. The MacMahon function has a combinatorial origin as the generating function for the number of 3-dimensional partitions of size n, i.e., 3-dimensional Young diagrams with n cubes.

Proof of the MNOP conjecture

MNOP verify:

Proposition 1 (Double point relation) *Let $\pi : Y \to C$ be a double-point degeneration (over $\mathbb{C}$) at $0 \in C$ of relative dimension 3. Let $c \in C$ be a regular value of π. Write $\pi^{-1}(0) = S \cup T$, $\pi^{-1}(c) = X$. Then*

$$Z(X,q) = \frac{Z(S,q) \cdot Z(T,q)}{Z(\mathbb{P}(\pi,0),q)}$$

In other words, sending a smooth projective X to $Z(X,q)$ descends to a homomorphism

$$Z(-,q) : \Omega^{-3}(\mathbb{C}) \to (1 + \mathbb{Z}[[q]])^{\times}.$$

It follows from general principles that, for $P(c_1, \ldots, c_n)$ a weighted homogeneous polynomial in the Chern classes $c_1, \ldots, c_n$ (with $\mathbb{Z}$-coefficients) sending a smooth projective variety X over $\mathbb{C}$ to $\deg P(c_1, \ldots, c_n)(T_X)$ descends to a homomorphism

$$P : \Omega^{-n}(\mathbb{C}) \to \mathbb{Z}$$

For example: $X \mapsto \deg(c_3(T_X \otimes K_X))$. Thus $X \mapsto M(q)^{\deg(c_3(T_X \otimes K_X))}$ descends to

$$M(q)^? : \Omega^{-3}(\mathbb{C}) \to (1 + \mathbb{Z}[[q]])^\times.$$

Next we have the result of MNOP:

Proposition 2 *The degree 0 conjecture is true for* $X = \mathbb{CP}^3$, $\mathbb{CP}^1 \times \mathbb{CP}^2$, *and* $(\mathbb{CP}^1)^3$.

To finish, we use the well-known fact from topology:

Proposition 3 *The rational Lazard ring* $\mathbb{L}^* \otimes \mathbb{Q} = MU^{2*}(pt) \otimes \mathbb{Q}$ *is a polynomial ring over* $\mathbb{Q}$ *with generators the classes* $[\mathbb{CP}^n]$, $n = 0, 1, \ldots$, *with* $[\mathbb{CP}^n]$ *in degree* $* = -n$.

Since $(1 + \mathbb{Z}[[q]])^\times$ is torsion free, $M(q)^?$ and $Z(-, q)$ factor through $\Omega^{-3}(\mathbb{C})_\mathbb{Q} = \mathbb{L}^{-3}_\mathbb{Q}$ and agree on as $\mathbb{Q}$-basis, hence are equal.

The formal group law for Ω_*^{dp}

The strategy Quillen gave a geometric contruction for the formal group law for MU^* (or in our case Ω^*) by using the projective bundle formula to write $c_1(O_{\mathbb{P}^n \times \mathbb{P}^m}(1,1))$ as

$$c_1(O_{\mathbb{P}^n \times \mathbb{P}^m}(1,1)) = u + v + \sum_{i \geq 1, j \geq 1} a_{ij} u^i v^j$$

where $u = c_1(O(1,0)$, $v = c_1(O(0,1))$ and the a_{ij} are in $\Omega^*(k)$. Passing to the limit over n, m defines the power series

$$F(u,v) := u + v + \sum_{ij} a_{ij} u^i v^j.$$

Properties of the tensor product of line bundles shows that $F(u,v)$ defines a formal group law.

We don't have the projective bundle formula for Ω_*^{dp}, but if we can write $c_1(O_{\mathbb{P}^n \times \mathbb{P}^m}(1,1))$ as above "by hand", we have a hope of getting a formal group law for Ω_*^{dp}.

Extending the double point relation

Lemma 1 *Let* X *be in* $\mathbf{Sm}/k$. *Suppose we have smooth divisors* S, T *and* W *such that* $S+T+W$ *is a reduced strict normal crossing divisor and* $W \sim_\ell S+T$. *Let* $D = S \cap T$, $E = S \cap T \cap W$. *Then*

$$[W \to X] = [S \to X] + [T \to X] - [\mathbb{P}_1 \to X] + [\mathbb{P}_2 \to X] - [\mathbb{P}_3 \to X]$$

where

$$\mathbb{P}_1 := \mathbb{P}_D(O_D(S) \oplus O_D), \ \mathbb{P}_E := \mathbb{P}(O_E(-T) \oplus O_E(-W))$$
$$\mathbb{P}_2 = \mathbb{P}_{\mathbb{P}_E}(O \oplus O(1)), \ \mathbb{P}_3 = \mathbb{P}_E(O_E(-T) \oplus O_E(-W) \oplus O_E)$$

Proof. Blow-up X along $(S \cup T) \cap W$ to form a morphism

$$f : X' \to \mathbb{P}^1$$

with $f^{-1}(0) = S + T$. $f^{-1}(\infty) = W$. Blow up X' along S forming X''. This resolves the singularties of X', leaves W and T alone and blows up S along E. In addition, this gives a double-point cobordism with total space X'' smooth fiber W and singular fiber $S' \cup T$.

Deformation to the normal bundle of E in S also gives a double-point cobordism with smooth fiber S and singular fiber $S' \cup \mathbb{P}_3$. Putting these together gives the results.

Let $H_{n,m} \subset \mathbb{P}^n \times \mathbb{P}^m$ be the divisor of a general section of $O(1,1)$. This gives us the normal crossing divisor $H_{n,m} + \mathbb{P}^n \times \mathbb{P}^{m-1} + \mathbb{P}^{n-1} \times \mathbb{P}^m$, with

$$H_{n,m} \sim_\ell \mathbb{P}^n \times \mathbb{P}^{m-1} + \mathbb{P}^{n-1} \times \mathbb{P}^m$$

Applying the extended double point in this case gives a start of the relation we seek, but the "coefficients" are non-constant projective space bundles.

We need to iterate, making the $\mathbb{P}^n$-bundles eventually into products. When we apply the extended double-point relation again, we get two-term towers of $\mathbb{P}^n$-bundles.

Admissible towers An *admissible tower* over X is a morphism $Y \to X$ that can be factored as

$$Y = \mathbb{P}_N \to \mathbb{P}_{N-1} \to \ldots \to \mathbb{P}_1 \to \mathbb{P}_0 = X$$

where $\mathbb{P}_{i+1} = \mathbb{P}_{\mathbb{P}_i}(\oplus_j L_j)$ with the L_j line bundles on $\mathbb{P}_i$.

Lemma 2 *Let $Y \to X$ be an admissible tower. Let $H_1, \ldots, H_s$ be smooth semi-ample divisors on X. For an index $I = (i_1, \ldots, i_s)$ $i_j \geq 0$, let $H^I = \cap_j H_j^{(i_j)}$. Suppose that the H^I are irreducible and that the restriction of $the H_j$ to H^I generate* Pic(H^I) *for each I. Then there are admissible towers $Y_{I,j} \to$* Spec k *such that*

$$[Y \to X] = \sum_{I,j} n_{I,j}[Y_{I,j} \times H^I \to X]$$

in $\Omega^*_{dp}(X)$.

The proof is an induction, using the extended double point relation and:

Let $E \to Z$ be a vector bundle $L \to Z$ be a line bundle and $i : D \to Z$ a smooth divisor on $Z \in \mathbf{Sm}/k$. Then

1. $\mathbb{P}(E \oplus L) + \mathbb{P}_D(i^*(E \oplus L \oplus L(D))) + \mathbb{P}(E \oplus L(D))$ is a reduced SNC divisor on $\mathbb{P}(E \oplus L \oplus L(D))$

2. $\mathbb{P}(E \oplus L) + \mathbb{P}_D(i^*(E \oplus L \oplus L(D))) \sim_\ell \mathbb{P}(E \oplus L(D))$

The formal group law We apply the proposition to the divisors $\mathbb{P}^{n-1} \times \mathbb{P}^m$, $\mathbb{P}^n \times \mathbb{P}^{m-1}$ and $H_{n,m}$ on $\mathbb{P}^n \times \mathbb{P}^m$, where $H_{n,m}$ is the divisor of a general section of $O(1,1)$. The admissible tower lemma plus an induction gives:

Proposition 4 *For each n, m there are elements $a_{i,j}^{n,m} \in \Omega_*^{dp}(k)$ such that*

$$[H_{n,m} \to \mathbb{P}^n \times \mathbb{P}^m] = [\mathbb{P}^{n-1} \times \mathbb{P}^m \to \mathbb{P}^n \times \mathbb{P}^m] + [\mathbb{P}^n \times \mathbb{P}^{m-1} \to \mathbb{P}^n \times \mathbb{P}^m] + \sum_{i \geq 1, j \geq 1} a_{i,j}^{n,m} [\mathbb{P}^{n-i} \times \mathbb{P}^{m-j} \to \mathbb{P}^n \times \mathbb{P}^m]$$

in $\Omega_{dp}^*(\mathbb{P}^n \times \mathbb{P}^m)$.

One then shows that the $a_{i,j}^{n,m}$ are independent of n, m for $n >> 0$, $m >> 0$, giving elements $a_{ij} \in \Omega_{dp}^*$.

Since $[H_{n,m} \to \mathbb{P}^n \times \mathbb{P}^m]$ represents $c_1(O(1,1))$ and $[\mathbb{P}^{n-i} \times \mathbb{P}^{m-j} \to \mathbb{P}^n \times \mathbb{P}^m]$ represents $c_1(O(1,0))^i \cdot c_1(O(0,1))^j$, this relation eventually leads to showing that

$$F(u,v) := u + v + \sum_{ij} a_{ij} u^i v^j$$

gives the formal group law for Ω^*_{dp} we were looking for.

Thank you!