The slice tower and the Adams-Novikov spectral sequence

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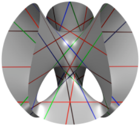


Homotopy theory

Basic structures

$$
\begin{array}{ccccc}
 & & \mathbf{Spc}[WE^{-1}] & & \\
 & & \| & & \\
\mathbf{Spc} & \longrightarrow & \mathcal{H} & \xrightarrow[\text{invert } (-)\wedge S^1]{\Sigma^\infty} & \mathrm{SH}
\end{array}
$$

Important objects and operations:

$$I = [0,1], S^1 = I/\{0,1\}, \Sigma(X) := X \wedge S^1, S^n := (S^1)^{\wedge n}$$

$$\pi_n(X,x) = \mathrm{Map}(S^n, X)/\mathrm{htpy}, WE := \{f : X \to Y \mid \pi_n(f) \text{ is an iso}\}$$

$$\Sigma^\infty : \mathcal{H} \to \mathrm{SH} \text{ inverts } \Sigma.$$

For $E \in \mathrm{SH}$, X a space, define

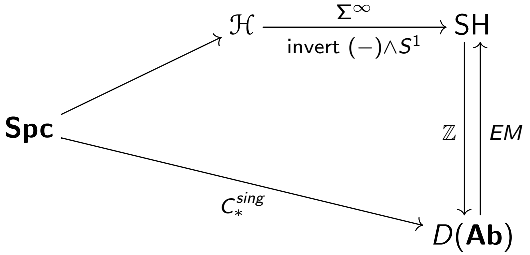
$$E^n(X) := [\Sigma^\infty X_+, \Sigma^n E]_{\mathrm{SH}}$$

SH is constructed so that the functor $X \mapsto E^*(X)$ satisfies the Eilenberg-Steenrod axioms of a *generalized cohomology theory*.

Examples:
The sphere spectrum $\mathbb{S} := \Sigma^\infty S^0$
Topological K-theory K_{top}
Complex cobordism MU

Homotopy theory
Ordinary cohomology

Usual (singular) cohomology is also represented in SH by use of the *Eilenberg-MacLane functor*



$$\pi_n(EM(A_*)) = H_n(A_*).$$

Motivic homotopy theory

1. Replace *spaces* with *diagrams of spaces* parametrized by the category $\mathbf{Sm}/k$ of smooth varieties over a base-field k:

$$\mathbf{Spc}(k) := \text{ the category of presheaves of spaces on } \mathbf{Sm}/k.$$

2. Define the $\mathbb{A}^1$-weak equivalences $WE_{\mathbb{A}^1}$ to be generated by
   i) maps $f : \mathcal{X} \to \mathcal{Y}$ that are a weak equivalence at each Nisnevich stalk for $x \in X \in \mathbf{Sm}/k$.
   ii) maps of the form $\mathcal{X} \times \mathbb{A}^1 \to \mathcal{X}$ (make $\mathbb{A}^1 = [0, 1]$).

Define:

$$\mathcal{H}(k) := \mathbf{Spc}(k)[WE_{\mathbb{A}^1}^{-1}].$$

The $\mathbb{P}^1$-spectrum category $\mathrm{SH}(k)$ inverts $\Sigma_{\mathbb{P}^1}$ (replace S^1 with $\mathbb{P}^1$)

Motivic homotopy theory

$$\begin{array}{ccccccc}
\mathbf{Sm}/k & & & & \mathbf{Spc}(k)[WE_{\mathbb{A}^1}^{-1}] & & \\
& \searrow^{\text{representable}} & & & \| & & \\
\mathbf{Spc} & \xrightarrow[\text{const.}]{} & \mathbf{Spc}(k) & \longrightarrow & \mathcal{H}(k) & \xrightarrow[\text{invert } (-)\wedge\mathbb{P}^1]{\Sigma^\infty_{\mathbb{P}^1}} & \mathrm{SH}(k)
\end{array}$$

One big difference: there is a two-variable family of spheres

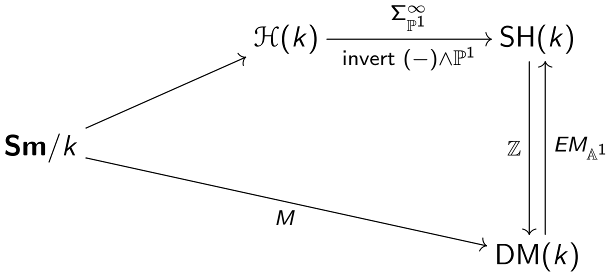
$$S^{a,b} := S^{a-b} \wedge \mathbb{G}_m^{\wedge b}; \quad \mathbb{P}^1 \cong S^{2,1}.$$

Consequence: cohomology is bi-graded: For $\mathcal{E} \in \mathrm{SH}(k)$, $X \in \mathbf{Sm}/k$, define

$$\mathcal{E}^{a,b}(X) := [\Sigma^\infty_{\mathbb{P}^1} X_+, \Sigma^{a,b}\mathcal{E}]_{\mathrm{SH}(k)}$$

Motivic homotopy theory
motivic (ordinary) cohomology

The analog of singular cohomology is *motivic cohomology*. The motivic ordinary cohomology theories are represented by objects in Voevodsky's category of motives over k, $\mathrm{DM}(k)$.



Examples: the motivic sphere spectrum over k: $\mathbb{S}_k := \Sigma^\infty_{\mathbb{P}^1}\mathrm{Spec}\, k_+$.

Algebraic K-theory K_{alg}.

Algebraic cobordism MGL.

Motivic cohomology $M\mathbb{Z} := EM_{\mathbb{A}^1}(\mathbb{Z})$.

The most basic structure constants in SH are the *stable homotopy groups of spheres*: $\pi_n^s(S^0) = \pi_n(\mathbb{S}) := \mathbb{S}^{-n}(pt)$.

$$\pi_0(\mathbb{S}) = \mathbb{Z},\ \pi_1(\mathbb{S}) = \mathbb{Z}/2, \ldots$$

There is no formula for $\pi_n(\mathbb{S})$ in general, however:

Theorem (Serre)

$\pi_n(\mathbb{S})$ is a finite abelian group for all $n > 0$.

Consequence: $\mathrm{SH}_\mathbb{Q} \cong D(\mathbf{Ab})_\mathbb{Q}$.

The analog in SH(k) of $\pi_n(\mathbb{S})$ is the sheaf $\pi_{a,b}(\mathbb{S}_k)$, associated to the presheaf

$$U \mapsto \mathbb{S}_k^{-a,-b}(U) = [\Sigma^{a,b}\Sigma^\infty_{\mathbb{P}^1} U_+, \mathbb{S}_k]_{\mathrm{SH}(k)}.$$

Theorem (Morel)

1. $\pi_{n,n}(\mathbb{S}_k)$ is the Milnor-Witt sheaf $\mathcal{K}^{MW}_{-n}$
2. $\pi_{0,0}(\mathbb{S}_k) = K_0^{MW}$ is the sheaf of Grothendieck-Witt groups GW.

Theorem (Cisinski-Deglise)

Suppose k has finite (Galois) cohomological dimension. Then $\mathrm{SH}(k)_\mathbb{Q} \cong \mathrm{DM}(k)_\mathbb{Q}$.

Note: $GW(\mathbb{R}) = \mathbb{Z} \oplus \mathbb{Z}$, whereas $M\mathbb{Z}^{0,0}(\mathrm{Spec}\,\mathbb{R}) = \mathbb{Z}$, so the Cisinski-Deglise theorem cannot hold for $k = \mathbb{R}$.

One can decompose an arbitrary spectrum E into Eilenberg-MacLane spectra by using the *Postnikov tower*:

$$\ldots \to E{<}n+1{>} \to E{<}n{>} \to \ldots \to E.$$

Here $E{<}n{>} \to E$ is the $n-1$-connected cover of E, that is $\pi_m E{<}n{>} = 0$ for $m < n$ and $\pi_m E{<}n{>} = \pi_m E$ for $m \geq n$.

Thus, the nth layer in this tower is the Eilenberg-MacLane spectrum $\Sigma^n EM(\pi_n E)$.

Applying the homological functor $[\Sigma^\infty X_+, -]$ to the Postnikov tower yields the *Atiyah-Hirzebruch spectral sequence*:

$$\boxed{E_2^{p,q} := H^p(X, \pi_{-q}E) \Longrightarrow E^{p+q}(X)}$$

which shows how to compute E-cohomology from ordinary cohomology.

To pass from the classical case to the motivic case, we replace S^1 with $\mathbb{P}^1$.

This gives us the $n-1$ $\mathbb{P}^1$-connected cover of $\mathcal{E}$:

$$f_n\mathcal{E} \to \mathcal{E}$$

Assembling these gives Voevodsky's *slice tower*

$$\ldots \to f_{n+1}\mathcal{E} \to f_n\mathcal{E} \to \ldots \to \mathcal{E}$$

with nth layer $s_n\mathcal{E}$.

The Postnikov (slice) tower

The motivic case

Theorem (Voevodsky, Levine)

$$s_0(\mathbb{S}_k) \cong \mathrm{EM}_{\mathbb{A}^1}(\mathbb{Z}).$$

Theorem (Röndigs-Østvær, Pelaez)

There is a canoncally defined nth homotopy motive $\pi^{\mu}_n\mathcal{E} \in \mathrm{DM}(k)$ *and isomorphism*

$$s_n\mathcal{E} \cong \Sigma^n_{\mathbb{P}^1}\mathrm{EM}_{\mathbb{A}^1}(\pi^{\mu}_n\mathcal{E}).$$

The slice tower breaks up a $\mathbb{P}^1$-spectrum into motives and yields the motivic Atiyah-Hirzebruch spectral sequence

$$\boxed{E_2^{p,q} = H^{p-q}(X, \pi^{\mu}_{-q}(n-q)) \Longrightarrow \mathcal{E}^{p+q,n}(X).}$$

In the classical case, the computation of the layers in the Postnikov tower is just the compution of the homotopy groups of the spectrum.

- $\pi_n EM(A) = A$ for $n = 0$, 0 else (by construction)
- $\pi_n K_{top} = \mathbb{Z}$ for n even, 0 for n odd (Bott periodicity)
- $\pi_{odd} MU = 0$. The ring $\pi_{2*} MU$ is the coefficient ring for the universal formal group law, the Lazard ring $\mathbb{L}_*$ (Thom, Milnor, Quillen).

The motivic case is parallel to the classical case, although the computations are considerably more difficult.

- $\pi^\mu_n \mathrm{EM}_{\mathbb{A}^1}(A) = A$ for $n = 0$, 0 else
- $\pi^\mu_n K_{alg} = \mathbb{Z}$ for all n (Voevodsky, Levine)
- $\pi^\mu_* \mathrm{MGL} = MU_{2*} = \mathbb{L}_*$ (after inverting the characteristic of k). (announced by Hopkins-Morel, proof supplied by Hoyois).

This gives us the motivic Atiyah-Hirzebruch spectral sequences for K-theory and algebraic cobordism

$$E_2^{p,q} = H^{p-q}(X, \mathbb{Z}(-q)) \Longrightarrow K_{-p-q}(X)$$

$$E_2^{p,q} = H^{p-q}(X, MU_{2q}(n-q)) \Longrightarrow \mathrm{MGL}^{p+q,n}(X).$$

To compute $\pi_n(\mathbb{S})$, Adams introduced a technique analogous to the descent spectral sequence in algebraic geometry.

Let E be a commutative ring spectrum and $\mathbb{S} \to E$ the unit map.

This extends to the augmented cosimplicial spectrum

$$\mathbb{S} \to E \overset{\rightarrow}{\underset{\rightarrow}{\leftarrow}} E \wedge E \overset{\rightarrow}{\underset{\rightarrow}{\overset{\leftarrow}{\underset{\leftarrow}{\rightarrow}}}} E \wedge E \wedge E \overset{\rightleftarrows}{\underset{\leftrightarrows}{\vdots}} \cdots$$

which is an aproximation of $\mathbb{S}$.

Taking π_* and applying the "stupid filtration" yields the Adams-type spectral sequence

$$\boxed{E_2^{p,q} := \mathrm{Ext}^{p,q}_{E_*(E)}(E_*, E_*) \Longrightarrow \pi_{q-p}(\mathbb{S})^{\wedge_E}}$$

Popular choices:

- $E = EM(\mathbb{Z}/p)$: the Adams spectral sequence
- $E = MU$: the Adams-Novikov spectral sequence
- $E = BP$: the p-local Adams-Novikov spectral sequence

(1) is related to the structure of mod p dual Steenrod algebra $EM(\mathbb{Z}/p)_*(EM(\mathbb{Z}/p)) = H_*(EM(\mathbb{Z}/p), \mathbb{Z}/p)$.

(2) and (3) are closely related to the theory of formal groups via the identity $MU_{2*} = \mathbb{L}_*$.

These constructions lift to the motivic setting:

$$E_2^{p,q,n} := \mathrm{Ext}^{p,q,n}_{\mathcal{E}_{*,*}(\mathcal{E})}(\mathcal{E}_{*,*}, \mathcal{E}_{*,*}) \Longrightarrow \pi_{q-p,n}(\mathbb{S}_k)^{\wedge\mathcal{E}}$$

The motivic Adams and Adams-Novikov spectral sequences have been studied (Hu-Kriz-Orsmby, Dugger-Isaksen, Isaksen).The extra weight-grading looks confusing but actually helps.

Voevodsky conjectured that the slices $s_n(\mathbb{S}_k)$ could be constructed from the (classical) Adams-Novikov spectral sequence:

Conjecture (Voevodsky 2002)

There is a natural isomorphism

$$s_n(\mathbb{S}_k) \cong \Sigma^n_{\mathbb{P}^1} \mathrm{EM}_{\mathbb{A}^1}(E_1^{*,2n}(AN))$$

This is now proven. It follows from the Hopkns-Morel-Hoyois theorem on the slices of MGL by considering the motivic Adams-Novikov resolution of $\mathbb{S}_k$.

Comparing classical and motivic theories

Theorem (Levine 2013)

Let k be an algebraically closed field of characteristic zero.
1.The constant sheaf functor $c : \mathrm{SH} \to \mathrm{SH}(k)$ is fully faithful
2. There is a canonical isomorphism $\pi_n(\mathbb{S}) \cong \pi_{n,0}(\mathbb{S}_k)(k)$

Ingredients: A theorem of Suslin-Voevodsky comparing mod n Suslin homology and mod n singular homology + convergence of the slice spectral sequence for $\pi_{*0}(\mathbb{S}_k)(k)$ and its Betti realization.

For $k = \bar{k}$, the slice spectral sequence for $\pi_{*,0}(\mathbb{S}_k)(k)$ gives a spectral sequence for $\pi_*(\mathbb{S})$ with the same E_2-term as Adams-Novikov. In fact

Theorem (Levine 2014)

For $k = \bar{k}$, the Adams-Novikov spectral sequence for $\pi_*(\mathbb{S})$ and the slice spectral sequence for $\pi_{n,0}(\mathbb{S}_k)(k)$ agree after reindexing.

Taking $k = \bar{\mathbb{Q}}$, this introduces a $\mathrm{Gal}(\mathbb{Q})$-action into the picture.

The $\mathrm{Gal}(\mathbb{Q})$ action on $\pi_{n,0}(\mathbb{S}_{\mathbb{Q}})(\bar{\mathbb{Q}})$ is trivial, but what about the $\mathrm{Gal}(\mathbb{Q})$ action on $\pi_{n,0}(f_q\mathbb{S}_{\mathbb{Q}})(\bar{\mathbb{Q}})$???

Thank you!