A SURVEY OF ALGEBRAIC COBORDISM

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ABSTRACT. This paper is based on the author's lecture at the ICM Satellite Conference in Algebra at Suzhou University, August 30-September 2, 2002, describing a joint work with Fabien Morel.

1. Introduction

Together with Fabien Morel, we have constructed a theory of *algebraic cobordism*, which lifts the theory of complex cobordism to algebraic varieties over a field of characteristic zero, as the theory of the Chow ring lifts singular cohomology, or the theory of algebraic K_0 lifts the topological K_0. In this paper, we give an introduction to this theory for the non-expert. For those interested in more details, we refer the reader to [5, 6, 7, 8].

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2. Complex cobordism

We recall that the *Thom space* $\mathrm{Th}(E)$ of a vector bundle $E \to X$ is the quotient space $D(E)/S(E)$, where $D(E)$ and $S(E)$ are the disk bundle and sphere bundle

$$D(E) := \{v \in E \mid ||v|| \leq 1\},$$
$$S(E) := \{v \in E \mid ||v|| = 1\},$$

with respect to a chosen metric on E. It is easy to see that $\mathrm{Th}(E)$ is independent of choice of metric; in fact, one can define $\mathrm{Th}(E)$ without reference to a metric as

$$\mathrm{Th}(E) := \mathbb{P}(E \oplus e_{\mathbb{C}})/\mathbb{P}(E),$$

where $\mathbb{P}$ is the associated bundle of projective spaces, and $e_{\mathbb{C}}$ is the trivial complex line bundle.

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Taking the example of the universal rank n complex vector bundle $\mathcal{U}_n \to \mathbf{Gr}(n,\infty)$, we have the *2nth universal Thom space*

$$MU_{2n} := \mathrm{Th}(\mathcal{U}_n).$$

The bundle $\mathcal{U}_n \oplus e_{\mathbb{C}} \to \mathbf{Gr}(n,\infty)$ is classified by an inclusion $i_n : \mathbf{Gr}(n,\infty) \to \mathbf{Gr}(n+1,\infty)$, giving the isomorphism

$$\mathcal{U}_n \oplus e_{\mathbb{C}} \cong i_n^*\mathcal{U}_{n+1}.$$

This in turn yields the map of Thom spaces $\mathrm{Th}(\mathcal{U}_n \oplus e_{\mathbb{C}}) \to \mathrm{Th}(\mathcal{U}_{n+1})$. In addition, one has the homeomorphism $\mathrm{Th}(\mathcal{U}_n \oplus e) \cong S^2 \wedge \mathrm{Th}(\mathcal{U}_n)$, which yields the connecting maps

$$S^2 \wedge MU_{2n} \xrightarrow{\epsilon_n} MU_{2n+2}.$$

We set $MU_{2n+1} := S^1 \wedge MU_{2n}$. The sequence of spaces

$$MU_0 = pt., MU_1, MU_2, \ldots, MU_{2n}, MU_{2n+1}, \ldots$$

with attaching maps

$$S^1 \wedge MU_{2n} = M_{2n+1} \xrightarrow{\mathrm{id}} M_{2n+1}$$
$$S^1 \wedge MU_{2n+1} = S^2 \wedge MU_{2n} \xrightarrow{\epsilon_n} MU_{2n+2}$$

defines the *Thom spectrum* MU; for a topological space X, the *complex cobordism* of X is defined as the set of stable homotopy classes of pointed maps

$$MU^n(X) := \lim_{N\to\infty}[\Sigma^N X_+, MU_{N+n}].$$

Sending X to the graded group $MU^*(X)$ evidently defines a contravariant functor. In fact, this satisfies the axioms of a cohomology theory on topological spaces.

2.1. **Quillen's construction.** Restricting to differentiable manifolds, the cohomology theory MU^* was given a more geometric flavor by Quillen [9], following work of Thom. In [9] Quillen describes $MU^n(X)$ as generated (for n even) by the set of *complex oriented* proper maps $f : Y \to X$ of codimension n. Here a complex orientation is given by factoring f through a closed embedding $i : Y \to E$, where $E \to X$ is a complex vector bundle, together with a complex structure on the normal bundle N_i of Y in E (for n even). For n odd, one puts a complex structure on $N_i \oplus e_{\mathbb{R}}$. One then imposes the *cobordism relation* by identifying $f^{-1}(X\times 0)$ and $f^{-1}(X\times 1)$, if $f : Y \to X\times\mathbb{R}$ is a proper complex oriented map, transverse to $X \times \{0,1\}$.

From this definition, it becomes apparant that $MU^*(X)$ has natural push-forward maps $f_* : MU^n(X) \to MU^{n-d}(X')$ for a proper complex-oriented map $f : X \to X'$ of relative dimension d. Pull-back is defined by noting that, given a differentiable map $g : X' \to X$, and a complex-oriented map $f : Y \to X$, one can change f by a homotopy to make f transverse to g. One then defines $g^*(f)$ as the projection $Y \times_X X' \to X'$. One also has the compatibility $g^* f_* = f'_* g'^*$ for cartesian squares

$$\begin{array}{ccc} X' \times_X Y & \xrightarrow{g'} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow[g]{} & X \end{array}$$

with f proper and complex-oriented, and g transverse to f.

Disjoint union defines the addition in $MU^*(X)$, and "reversing" the orientation defines the minus. Taking products of maps defines external products $MU^n(X) \otimes MU^m(Y) \to MU^{n+m}(X \times Y)$. Taking $X = Y$ and pulling back by the diagonal defines cup products on $MU^*(X)$, making $MU^*(X)$ a graded commutative ring with identity $1_X = \mathrm{id}_X$.

2.2. **Chern classes and the projective bundle formula.** Let $L \to X$ be a complex line bundle on a differentiable manifold X, and let $s : X \to L$ be the zero section. Define $c_1(L) \in MU^2(X)$ by

$$c_1(L) = s^* s_*(1_X).$$

One has the projective bundle formula: Let $E \to X$ be a rank $n+1$ vector bundle on X, $L \to \mathbb{P}(E)$ the tautological line bundle on $\mathbb{P}(E)$, and let $\xi = c_1(L)$. Then $MU^*(\mathbb{P}(E))$ is a free $MU^*(X)$-module, with basis $1, \xi, \ldots, \xi^n$. In fact, MU^* is the universal cohomology theory with Chern classes and a projective bundle formula.

2.3. **The formal group law.** It is not the case that $c_1(L \otimes M) = c_1(L) + c_1(M)$! To make this failure precise, one considers the universal case of the tautological complex line bundle L_n on $\mathbb{P}^n$ and the limit bundle L_∞ on $\mathbb{P}^\infty$. Letting $\xi_n = c_1(L_n)$, sending u to ξ_n defines an isomorphism

$$MU^*(\mathbb{P}^n) \cong MU^*(pt.)[u]/u^{n+1}.$$

Taking limits gives $MU^*(\mathbb{P}^\infty) \cong MU^*(pt.)[[u]]$, with u mapping to ξ_∞. Similarly, we have $MU^*(\mathbb{P}^\infty \times \mathbb{P}^\infty) \cong MU^*(pt.)[[u, v]]$, with u going to $c_1(p_1^* L_\infty)$, and v to $c_1(p_2^* L_\infty)$. We have the power series $F(u, v) \in MU^*(pt.)[[u, v]]$ defined as the element corresponding to $c_1(p_1^* L_\infty \otimes p_2^* L_\infty)$. Thus, for any two line bundles L, M, we have

$$c_1(L \otimes M) = F(c_1(L), c_1(M)).$$

From the elementary properties of tensor product, we see that F defines a commutative *formal group law* on $MU^*(pt.)$, that is

$$F(u,0) = F(0,u) = u,$$
$$F(u,v) = F(v,u),$$
$$F(u,F(v,w)) = F(F(u,v),w).$$

In fact, Quillen [9] has shown this is the universal formal group law, so the failure of c_1 to be additive is as complete as it can possibly be.

2.4. **The Lazard ring.** The coefficient ring of the universal formal group was first studied by Lazard [4], and is thus known as the *Lazard ring* $\mathbb{L}$. The Lazard ring is known to be a polynomial ring over $\mathbb{Z}$ in infinitely many variables

$$\mathbb{L} = \mathbb{Z}[x_1, x_2, \ldots].$$

$\mathbb{L}$ is naturally a graded ring with $\deg(x_i) = -i$.

Explicitly, one constructs $\mathbb{L}$ and the universal group law $F_{\mathbb{L}}$ as follows: Let $\tilde{\mathbb{L}} = \mathbb{Z}[\{A_{ij} \mid i,j \geq 1\}]$, where we give A_{ij} degree $-i-j+1$, and let $F \in \tilde{\mathbb{L}}[[u,v]]$ be the power series $F = u + v + \sum_{ij} A_{ij}u^iv^j$. Let

$$\mathbb{L} = \tilde{\mathbb{L}}/F(u,v) = F(v,u), F(F(u,v),w) = F(u,F(v,w)).$$

and let $F_{\mathbb{L}}$ be the image of F in $\mathbb{L}[[u,v]]$. Then $(F_{\mathbb{L}}, \mathbb{L})$ is evidently the universal commutative dimension 1 formal group; $\mathbb{L}$ is thus the Lazard ring.

3. Oriented cohomology theories

We abstract the properties of MU^* in an algebraic setting. Fix a base field k and let $\mathbf{Sm}_k$ denote the category of smooth quasi-projective k-schemes.

Definition 3.1. An *oriented cohomology theory* on $\mathbf{Sm}_k$ is given by

- D1. A contravarient functor A^* from $\mathbf{Sm}_k$ to graded rings.
- D2. For each projective morphism $f : X \to Y$ in $\mathbf{Sm}_k$ of relative codimension d an $A^*(Y)$-linear push-forward homomorphism $f_* : A^*(X) \to A^{*+d}(Y)$.

These satisfy:

- A1. $(f \circ g)_* = f_* \circ g_*$. $\mathrm{id}_* = \mathrm{id}$.

A2. Let

$$\begin{array}{ccc} W & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & Z \end{array}$$

be a cartesian square, with X, Y, Z and W in $\mathbf{Sm}_k$, and with f projective. Then

$$g^* f_* = f'_* g'^*.$$

A3. *Projective bundle formula.* For a line bundle L on $X \in \mathbf{Sm}_k$ with zero-section $s : X \to L$, define

$$c_1(L) := s^* s_*(1_X) \in A^1(X).$$

Let $E \to X$ be a rank $n+1$ vector bundle over $X \in \mathbf{Sm}_k$, and $\mathbb{P}(E) \to X$ the associated projective bundle. Let $\xi = c_1(O(1))$. Then $A^*(\mathbb{P}(E))$ is a free module over $A^*(X)$ with basis $1, \xi, \ldots, \xi^n$.

A4. *Homotopy.* Let $p : V \to X$ be an $\mathbb{A}^n$ bundle over $X \in \mathbf{Sm}_k$. Then $p^* : A^*(X) \to A^*(V)$ is an isomorphism.

Remark 3.2. The reader should note that an oriented cohomology theory as defined above is not a cohomology theory in the usual sense, as there is no requirement of a Mayer-Vietoris property. One should perhaps call the above data an oriented *pre*-cohomology theory, but we have chosen not to use this terminology.

4. The formal group law

Let A^* be an oriented cohomology theory. We have

$$\varprojlim_{n,m} A^*(\mathbb{P}^n \times \mathbb{P}^m) \cong A^*(k)[[u, v]],$$

the isomorphism sending u to $c_1(p_1^* O(1))$ and v to $c_1(p_2^* O(1))$. The class of $c_1(p_1^* O(1) \otimes p_2^* O(1))$ thus gives a power series $F_A(u, v) \in A^*(k)[[u, v]]$ with

$$c_1(p_1^* O(1) \otimes p_2^* O(1)) = F_A(c_1(p_1^* O(1)), c_1(p_2^* O(1))).$$

By naturality, we have, for $X \in \mathbf{Sm}_k$ with line bundles L, M, the identity

$$c_1(L \otimes M) = F_A(c_1(L), c_1(M)).$$

In addition, $F_A(u, v) = u + v \mod uv$, $F_A(u, v) = F_A(v, u)$, and $F_A(F_A(u, v), w) = F_A(u, F_A(v, w))$. Thus, F_A gives a formal group law with coefficients in $A^*(k)$. In particular, for each oriented cohomology theory A, there is a canonical ring homomorphism $\phi_A : \mathbb{L} \to A^*(k)$ classifying the group law F_A.

Note that $c_1 : \mathrm{Pic}(X) \to A^1(X)$ is a group homomorphism if and only if $F_A(u,v) = u+v$; we call such a theory *ordinary*.

Examples 4.1. (1) The Chow ring of algebraic cycles modulo rational equivalence, CH^*, and étale cohomology $H^{2*}_{\text{ét}}(-,\mathbb{Z}/n(*))$ (also with $\mathbb{Z}_l(*)$ or $\mathbb{Q}_l(*)$ coefficients). These are all ordinary theories. Similarly, if $\sigma : k \to \mathbb{C}$ is an embedding, and X is in $\mathbf{Sm}_k$, let $X^\sigma(\mathbb{C})$ be the complex manifold of $\mathbb{C}$-points on $X \times_k \mathbb{C}$. We have the ordinary theory

$$X \mapsto H^*(X^\sigma(\mathbb{C}), \mathbb{Z})$$

where $H^*(-,\mathbb{Z})$ is singular cohomology.

(2) For $X \in \mathbf{Sm}_k$, we have the Grothendieck group of algebraic vector bundles on X, $K_0^{\mathrm{alg}}(X)$. For a projective morphism $f : Y \to X$, we have the pushforward $f_* : K_0^{\mathrm{alg}}(Y) \to K_0^{\mathrm{alg}}(X)$, defined by sending E to the alternating sum $\sum_i (-1)^i [R^i f_*(E)]$. Here, we need to identify $K_0^{\mathrm{alg}}(X)$ with the Grothendieck group of coherent sheaves on X, for which we require X to be regular (e.g. smooth over k).

This does not define an oriented cohomology theory, since there is no natural grading on K_0^{alg} which respects the pushforward maps in the proper manner. To correct this, we adjoin a variable β (of degree -1), and its inverse β^{-1}, and define

$$f_*([E]\beta^n) := f_*([E])\beta^{n-d}$$

for $f : Y \to X$ projective, $d = \dim_k X - \dim_k Y$. This defines the oriented cohomology theory theory $K_0[\beta, \beta^{-1}]$.

$K_0[\beta, \beta^{-1}]$ is not an ordinary theory, in fact, its formal group law is the *multiplicative group*

$$F_{K_0}(u,v) = u + v - \beta uv.$$

To see this, it follows from the definition of c_1 that $c_1(L) = \beta^{-1}(1 - L^\vee)$, where $L^\vee$ is the dual line bundle. If $L = \mathcal{O}_X(D)$, $M = \mathcal{O}_X(E)$ for smooth transverse divisors D and E, we have the exact sequences

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$
$$0 \to \mathcal{O}_X(-E) \to \mathcal{O}_X \to \mathcal{O}_E \to 0$$

and

$$0 \to \mathcal{O}_X(-D-E) \to \mathcal{O}_X(-D) \oplus \mathcal{O}_X(-E) \to \mathcal{O}_X \to \mathcal{O}_D \otimes_{\mathcal{O}_X} \mathcal{O}_E \to 0.$$

From these, one finds the relation in K_0^{alg}

$$[1]-[(L\otimes M)^\vee] = ([1]-[L^\vee]) + ([1]-[M^\vee]) - ([1]-[L^\vee])\cdot([1]-[M^\vee]),$$

which yields the stated group law.

(3) As in (1), let $\sigma : k \to \mathbb{C}$ be an embedding. We have the oriented cohomology theories

$$X \mapsto K^0_{\mathrm{top}}(X^\sigma(\mathbb{C}))[\beta, \beta^{-1}]$$
$$X \mapsto MU^*(X^\sigma(\mathbb{C}))$$

These are both extraordinary theories (i.e., not ordinary). The group law for K^0_{top} is the multiplicative group, and for MU^* the universal group law.

5. Algebraic cobordism

Let $P\mathbf{Sch}_k$ be the category with objects finite type k-schemes, and with morphisms the projective maps $Y \to X$; let $P\mathbf{Sm}_k$ be the full subcategory of $P\mathbf{Sch}_k$ with objects the smooth quasi-projective schemes over k. We can now state our main result on algebraic cobordism:

Theorem 5.1 ([5, 6, 7])**.** *Let k be a field of characteristic zero.*

1. *There is a universal oriented cohomology theory Ω^* on $\mathbf{Sm}_k$.*
2. *The homomorphism $\phi_\Omega : \mathbb{L} \to \Omega^*(k)$ is an isomorphism*
3. *For X of dimension d, write $\Omega_n(X) := \Omega^{d-n}(X)$. Then the covarient functor Ω_* on $P\mathbf{Sm}_k$ extends to a covariant functor on $P\mathbf{Sch}_k$ satisfying*
   (a) *Ω_* has pull-back homomorphisms for smooth quasi-projective morphisms, compatible in cartesian squares.*
   (b) *Let $i : Z \to X$ be a closed embedding with open complement $j : U \to X$. Then the sequence*

$$\Omega_*(Z) \xrightarrow{i_*} \Omega_*(X) \xrightarrow{j^*} \Omega_*(U) \to 0.$$

is exact.

Idea of construction: For a finite type k-scheme X, let $\mathcal{M}(X)$ be the set of isomorphism classes of morphisms $f : Y \to X$, with $Y \in \mathbf{Sm}_k$ and f projective. $\mathcal{M}(X)$ is a graded monoid under disjoint union, with $f : Y \to X$ in degree $\dim_k(Y)$, let $\mathcal{M}(X)^+$ be the group completion. Composition with a projective morphism $g : X \to X'$ makes $\mathcal{M}^+$ a functor on $P\mathbf{Sm}_k$

We construct $\Omega_*(X)$ as a quotient of $\mathcal{M}(X)^+$ in three steps:

(1) Impose the relation of "classical cobordism": $f^{-1}(0) = f^{-1}(1)$ for $f : W \to X \times \mathbb{A}^1$, $W \in \mathbf{Sm}_k$, f projective, and f transverse with respect to the inclusion $X \times \{0, 1\} \to X \times \mathbb{A}^1$.

(2) For $L \to X$ a globally generated line bundle and $f : Y \to X$ in $\mathcal{M}(X)$, let $i : D \to Y$ be the zero locus of a general section of f^*L. Set $c_1(L)(f) := f \circ i : D \to X$. One checks that this is well-defined modulo the relation of classical cobordism.

Impose the universal formal group law:

$$c_1(L \otimes M)(f) = F_{\mathbb{L}}(c_1(L), c_1(M))(f)$$

for globally generated line bundles L and M on X and $f : Y \to X$ in $\mathcal{M}(X)$.

(3) Impose the "Gysin relation", by identifying $c_1(\mathcal{O}_W(D))(\mathrm{id}_W)$ with the class of $i : D \to W$ for D a smooth divisor on W.

Remarks 5.2. (1) The above gives the rough outline of a somewhat simplified version of the actual construction. We refer the reader to [7, 8] for more details.

(2) The restriction to characteristic zero in Theorem 5.1 arises from a heavy use of resolution of singularities [3]. In addition, the weak factorization theorem of [1] is used in an essential way in the proof of Theorem 5.1(2).

6. Degree formulas

In the paper [10], Rost made a number of conjectures based on the theory of algebraic cobordism in the Morel-Voevodsky stable homotopy category; assuming these conjecture, Rost is able to construct the so-called *splitting varieties* which play a crucial role in Voevodsky's approach to proving the Bloch-Kato conjecture. Many of Rost's conjectures have been proved by homotopy-theoretic means (*cf.* [2]); our construction of an algebro-geometric cobordism gives an alternate proof of these results, and settles many remaining open questions as well. We give a sampling of some of these results.

6.1. **The generalized degree formula.** All the degree formulas follow from the "generalized degree formula". Before stating this result, we first define the degree homomorphism

$$\deg : \Omega_*(X) \to \Omega_*(k).$$

We assume the base-field k has characteristic zero.

Let X be an irreducible finite type k-scheme and let $i_x : x \to X$ be the generic point of X, with structure map $p_x : x \to \operatorname{Spec} k$. By

Theorem 5.1, we have the commutative diagram

$$\begin{array}{ccc} \Omega_*(k) & \xrightarrow{p_x^*} & \Omega_*(k(x)) \\ {\scriptstyle \phi_{\Omega/k}} \nwarrow & & \nearrow {\scriptstyle \phi_{\Omega/k(x)}} \\ & \mathbb{L} & \end{array}$$

with $\phi_{\Omega/k}$ and $\phi_{\Omega/k(x)}$ isomorphisms. Thus the base-change homomorphism $p_x^* : \Omega_*(k) \to \Omega_*(k(x))$ is also an isomorphism.

Let $f : Y \to X$ be in $\mathcal{M}(X)$, with X irreducible. Define $\deg f \in \Omega_*(k)$ to be the element mapping to $f_x : Y \times_X x \to x$ in $\Omega_*(k(x))$ under the isomorphism $p_x^* : \Omega_*(k) \to \Omega_*(k(x))$. More generally, if η in any element of $\Omega_*(X)$, let $\deg(\eta) \in \Omega_*(k)$ be the element with

$$p_x^*(\deg(\eta)) = i_x^* \eta \in \Omega_*(k(x)).$$

Theorem 6.2 (Generalized degree formula)**.** *Let X be an irreducible finite type k-scheme, and let η be in $\Omega_*(X)$. Let $f_0 : B_0 \to X$ be a resolution of singularities of X, with B_0 quasi-projective over k. Then there are $a_i \in \Omega_*(k)$, $f_i : B_i \to X$ in $\mathcal{M}(X)$, $i = 1, \ldots, s$, such that*

1. *$f_i : B_i \to f(B_i)$ is birational and $f(B_i)$ is a proper closed subset of X, $i = 1, \ldots, s$.*
2. *$\eta - (\deg \eta)[f_0] = \sum_i a_i[f_i]$ in $\Omega_*(X)$.*

Proof. It follows from the definitions of Ω_* that, for X an irreducible finite type k-scheme, we have

$$\Omega_*(k(X)) = \varinjlim_{U} \Omega_*(U),$$

where the limit is over dense open subschemes U of X, and $\Omega_*(k(X))$ is the value at $\operatorname{Spec} k(X)$ of the functor Ω_* on finite type $k(X)$-schemes. Thus, there is a smooth open subscheme $j : U \to X$ of X such that $j^*\eta = (\deg \eta)[U]$ in $\Omega_*(U)$. Shrinking U if necessary, we may assume that $B_0 \to X$ is an isomorphism over U. Thus, $j^*(\eta - (\deg \eta)[f_0]) = 0$ in $\Omega_*(U)$.

Let $W = X \setminus U$. From the localization sequence

$$\Omega_*(W) \xrightarrow{i_*} \Omega_*(X) \xrightarrow{j^*} \Omega_*(U)$$

we find an element $\eta_1 \in \Omega_*(W)$ with $i_*(\eta_1) = \eta - (\deg \eta)[f_0]$, and noetherian induction completes the proof. □

Remarks 6.3. (1) If X is in $\mathbf{Sm}_k$, we can take $f_0 = \mathrm{id}_X$, giving the formula

$$\eta - (\deg \eta)[\mathrm{id}_X] = \sum_{i=1}^{m} a_i[f_i]$$

in $\Omega_*(X)$.

(2) If X is in $\mathbf{Sm}_k$, and $\eta = [f]$ for some $f : Y \to X$ in $\mathcal{M}(X)$, we have

$$[f] - (\deg f)[\mathrm{id}_X] = \sum_{i=1}^{m} a_i [f_i]$$

in $\Omega_*(X)$.

(3) If $f : Y \to X$ is in $\mathcal{M}(X)$, and $\dim Y = \dim X$, then $\deg f$ is an integer, namely, the usual degree of f if f is dominant, and zero if f is not dominant. Indeed, the map $\Omega_*(k(X)) \to \Omega_*(\overline{k(X)})$ is an isomorphism ($\overline{k(X)}$ the algebraic closure of $k(X)$), and it is clear that the image of $[f]$ in $\Omega_*(\overline{k(X)})$ is $[k(Y) : k(X)] \cdot [\mathrm{Spec}\, \overline{k(X)}]$ if f is dominant, and zero if not.

6.4. **Classical cobordism and algebraic cobordism.** From the the universal property of Ω^*, one sees that a homomorphism of fields $\sigma : k \to \mathbb{C}$ yields a natural homomorphism $\Re_\sigma : \Omega^*(X) \to MU^{2*}(X^\sigma(\mathbb{C}))$, with $f : Y \to X$ going to the class of the map of complex manifolds $f^\sigma : Y^\sigma(\mathbb{C}) \to X^\sigma(\mathbb{C})$.

Let $P = P(c_1, \ldots, c_d)$ be a degree d (weighted) homogeneous polynomial. If X is smooth and projective of dimension d over k, we have the *Chern number*

$$P(X) := \deg(P(c_1(\Theta_{X^\sigma(\mathbb{C})}), \ldots, c_d(\Theta_{X^\sigma(\mathbb{C})}))).$$

$P(X)$ is in fact independent of the choice of σ.

Let s_d be the polynomial which corresponds to $\sum_i \xi_i^d$, where $\xi_1, \ldots$ are the Chern roots. The following divisibility is known: If $d = p^n - 1$ for some prime p, and $\dim X = d$, then $s_d(X)$ is divisible by p.

In addition, for integers $d = p^n - 1$ and $r \geq 1$, there are mod p characteristic classes $t_{d,r}$, with $t_{d,1} = s_d/p \mod p$. The s_d and the $t_{d,r}$ have the following properties:

(6.1)

(1) $s_d(X) \in p\mathbb{Z}$ is defined for X smooth and projective of dimension $d = p^n - 1$. $t_{d,r}(X) \in \mathbb{Z}/p$ is defined for X smooth and projective of dimension $rd = r(p^n - 1)$.

(2) s_d and $t_{d,r}$ extend to homomorphisms $s_d : \Omega^{-d}(k) \to p\mathbb{Z}$, $t_{d,r} : \Omega^{-rd}(k) \to \mathbb{Z}/p$.

(3) If X and Y are smooth projective varieties with $\dim X, \dim Y > 0$, $\dim X + \dim Y = d$, then $s_d(X \times Y) = 0$.

(4) If $X_1, \ldots, X_s$ are smooth projective varieties with $\sum_i \dim X_i = rd$, then $t_{d,r}(\prod_i X_i) = 0$ unless $d | \dim X_i$ for each i.

Theorem 6.5. *Let $f : Y \to X$ be a morphism of smooth projective k-schemes of dimension d, $d = p^n - 1$ for some prime p. Then there is a zero-cycle η on X such that*

$$s_d(Y) - (\deg f)s_d(X) = p \cdot \deg(\eta).$$

Theorem 6.6. *Let $f : Y \to X$ be a morphism of smooth projective k-schemes of dimension rd, $d = p^n - 1$ for some prime p. Suppose that X admits a sequence of surjective morphisms*

$$X = X_0 \to X_1 \to \ldots \to X_{r-1} \to X_r = \operatorname{Spec} k$$

such that,

(1) $\dim X_i = d(r - i)$.
(2) *Let η be a zero-cycle on $X_i \times_{X_{i+1}} \operatorname{Spec} k(X_{i+1})$. Then $p | \deg(\eta)$.*

Then

$$t_{d,r}(Y) = \deg(f)t_{d,r}(X).$$

Proof. These two theorems follow easily from the generalized degree formula Theorem 6.2 and the amplifications of Remark 6.3. Indeed, for Theorem 6.5, we have the identity

$$[Y \to X] - (\deg f)[\mathrm{id}_X] = \sum_{i=1}^{m} a_i[B_i \to X]$$

in $\Omega_*(X)$ with the a_i in $\Omega_*(k)$ and with $\dim(B_i) < d$ for all i. Pushing forward to $\Omega_*(k)$ gives the identity

$$[Y] - (\deg f)[X] = \sum_{i=1}^{m} a_i[B_i]$$

in $\Omega_d(k)$. We can express each a_i as a sum

$$a_i = \sum_l n_{il}[Y_{il}],$$

where the Y_{il} are smooth projective varieties over k. Applying s_d gives

$$s_d(Y) - \deg(f)s_d(X) = \sum_{i,l} n_{il}s_d(Y_{il} \times B_i).$$

As $\dim(B_i) < d$ for all i, we have $\dim(Y_{il}) > 0$ for all i, l.

Since s_d vanishes on non-trivial products, only the terms with B_i a point z_i of X survive in this last sum. Rewriting the sum, this gives

$$s_d(Y) - \deg(f)s_d(X) = \sum_j m_j s_d(Y_j) \deg(z_j)$$

for smooth projective dimension d k-schemes Y_j, integers m_j and points z_j of X. Since $s_d(Y_j) = pn_j$ for suitable integers n_j, we have

$$s_d(Y) - \deg(f)s_d(X) = p\deg(\sum_j m_j n_j z_j),$$

proving Theorem 6.5.

For Theorem 6.6, we have as before

$$[Y \to X] - (\deg f)[\mathrm{id}_X] = \sum_{i=1}^m a_i[B_i \to X]$$

in $\Omega_*(X)$. We then decompose each $B_j \to X_2$ using Theorem 6.2, giving

$$[B_j] = a_{0j}[X_2] + \sum_i n_{ij}a_{ij}[B_{ij}].$$

in $\Omega_*(X_2)$. Iterating, we have the identity in $\Omega_*(k)$

$$[Y] - (\deg f)[X] = \sum_{I=(i_0,\ldots,i_r)} n_I[\prod_{j=0}^r Y_{ij}],$$

where the Y_{ij} are smooth projective k-schemes. In addition, the conditions on the tower imply that for each product $\prod_{j=0}^r Y_{ij}$ such that $d|\dim Y_{ij}$ for all j has $p|n_I$. Thus, arguing as above, we see that $t_{d,r}(Y) = \deg(f)t_{d,r}(X)$. □

7. Comparison results

In this last section, we explain how one can recover both the Chow ring $\mathrm{CH}^*(X)$ and $K_0^{\mathrm{alg}}(X)$ from $\Omega^*(X)$.

Suppose we have a formal group law (f, R), giving the canonical homomorphism $\phi_f : \mathbb{L} \to R$. Let $\Omega^*_{(f,R)}$ be the functor

$$\Omega^*_{(f,R)}(X) = \Omega^*(X) \otimes_{\mathbb{L}} R,$$

where $\Omega^*(X)$ is an $\mathbb{L}$-algebra via the isomorphism $\phi_\Omega : \mathbb{L} \to \Omega^*(k)$. For X a finite type k-scheme, define $\Omega_*^{(f,R)}(X)$ similarly.

Since $\otimes$ is right exact, the theories $\Omega^*_{(f,R)}$ and $\Omega_*^{(f,R)}$ have the same formal properties as Ω^* and Ω_*. In particular, $\Omega^*_{(f,R)}$ is an oriented cohomology theory, and $\Omega_*^{(f,R)}$ satisfies localization. The universal property of Ω^* gives the analogous universal property for $\Omega^*_{(f,R)}$.

In particular, let Ω^*_+ be the theory with $(f(u,v), R) = (u+v, \mathbb{Z})$, and let $\Omega^*_\times$ be the theory with $(f(u,v), R) = (u+v-\beta uv, \mathbb{Z}[\beta,\beta^{-1}])$. We thus have the canonical natural transformations of oriented theories

$$\Omega^*_+ \to \mathrm{CH}^*; \quad \Omega^*_\times \to K_0^{\mathrm{alg}}[\beta,\beta^{-1}]. \tag{7.1}$$

Theorem 7.1. *The natural transformations* (7.1) *are isomorphisms.*

Proof. We define maps backwards:

$$\mathrm{CH}^* \to \Omega^*_+; \quad K_0^{\mathrm{alg}}[\beta, \beta^{-1}] \to \Omega^*_\times.$$

For CH^*, we first note that $\Omega^n(k)_+ = 0$ for $n \neq 0$, since $\mathbb{L} \cong \Omega^*(k)$, and $\mathbb{L}$ is generated by the coefficients of the universal group law. To map CH^* to Ω^*_+, send a subvariety $Z \subset X$ to the map $\tilde{Z} \to X$, where $\tilde{Z} \to Z$ is a resolution of singularities of Z. It follows from localization that the class of $\tilde{Z}$ in $\Omega^*_+(X)$ is independent of the choice of the resolution. A similar argument shows that the relations defining CH^* go to zero. It is evident that the composition $\mathrm{CH}^* \to \Omega^*_+ \to \mathrm{CH}^*$ is the identity. Finally, the generalized degree formula Theorem 6.2 shows that the map $\mathrm{CH}^* \to \Omega^*_+$ is surjective, proving the result.

For $K_0^{\mathrm{alg}}[\beta, \beta^{-1}]$, we use a "Chern character" to define the backwards map. In fact, sending a line bundle L to $\mathrm{ch}(L) := \sum_i c_1^{\Omega_\times}(L)^i$ is easily seen to satisfy

$$\mathrm{ch}(L \otimes M) = \mathrm{ch}(L)\mathrm{ch}(M).$$

Defining $\mathrm{ch}(\oplus_i L_i) = \sum_i \mathrm{ch}(L_i)$ and using the splitting principle defines the ring homomorphism $\mathrm{ch} : K_0^{\mathrm{alg}}[\beta, \beta^{-1}] \to \Omega^*_\times$. One calculates the associated Todd genus as follows: The respective projective bundle formulas give isomorphisms

$$\lim_{\leftarrow} K_0^{\mathrm{alg}}[\beta, \beta^{-1}](\mathbb{P}^n) \cong \mathbb{Z}[\beta, \beta^{-1}][[u]]; \quad \lim_{\leftarrow} \Omega^*_\times(\mathbb{P}^n) \cong \mathbb{Z}[\beta, \beta^{-1}][[v]],$$

with u going to $1 - O(-1)$, and v going to the class of a hyperplane H. Thus, one can write $\mathrm{ch}(1 - O(-1))$ as $\phi([H])$ in $\Omega_\times(\mathbb{P}^n)$ for a unique power series $\phi(v) = av + \ldots$. Then $\mathrm{Todd}(v)^{-1} = \phi(v)/v$. But one easily computes $\phi(v) = v$, giving $\mathrm{Todd}(v) = 1$. The Riemann-Roch formula then gives

$$\mathrm{ch}(\mathcal{O}_Z) = [Z] \in \Omega_\times(X)$$

for $Z \to X$ a smooth closed subscheme of $X \in \mathbf{Sm}_k$. This and localization implies that $\mathrm{ch} : K_0^{\mathrm{alg}}[\beta, \beta^{-1}] \to \Omega^*_\times$ is surjective; one easily computes that the composition

$$K_0^{\mathrm{alg}}[\beta, \beta^{-1}] \to \Omega^*_\times \to K_0^{\mathrm{alg}}[\beta, \beta^{-1}]$$

is the identity, completing the proof. □

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