Tate motives and fundamental groups

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Outline

- An overview of fundamental groups
- Categories of Tate motives
- Dg algebras and rational homotopy theory
- Tate motives via dg algebras
- Applications and open problems

$(M, 0)$: pointed topological space $\rightsquigarrow \pi_1(M, 0)$ classifying covering spaces.

The Malcev completion

$$\mathbb{Q}[\pi_1(M, 0)]^\vee := \lim_{\substack{\longleftarrow \\ n}} \mathbb{Q}[\pi_1(M, 0)]/I^n$$

classifies uni-potent local systems of $\mathbb{Q}$-vector spaces.

This part of π_1 is approachable through rational homotopy theory.

For M a manifold, the rational homotopy theory is determined by the de Rham complex.

(X, x): a k-scheme with a $\bar{k}$ point x.

$\pi_1^{alg}(X, x)$: Grothendieck fundamental group: classifies algebraic "covering spaces".

$\pi_1^{geom}(X, x) := \pi_1^{alg}(X \times_k \bar{k}, x)$: the geometric fundamental group

The fundamental exact sequence:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1^{geom}(X,x) & \longrightarrow & \pi_1^{alg}(X,x) & \longrightarrow & \pi_1^{alg}(\mathrm{Spec}\, k, \bar{x}) & \longrightarrow & 1 \\ & & & & & & \| & & \\ & & & & & & \mathrm{Gal}(\bar{k}/k) & & \end{array}$$

For $M = X(\mathbb{C})$,

$$\pi_1^{geom}(X, x) \cong \text{pro-finite completion of } \pi_1(X(\mathbb{C}), x).$$

Taking the $\mathbb{Q}_p$ Malcev completion of the p-part of $\pi_1^{geom}(X, x)$ gives a p-adic version of $\mathbb{Q}[\pi_1(X(\mathbb{C}), x)]^\vee$:

The pro-finite, pro-uni-potent completion of π_1 is "algebraic".

Suppose k is a number field and X is an open subscheme of $\mathbb{P}^1_k$

Deligne-Goncharov lift the Malcev completion $\mathbb{Q}[\pi_1(X(\mathbb{C}), x)]^\vee$ to a "pro-algebraic group over mixed Tate motives over k":

$$\mathbb{Q}[\pi_1(X(\mathbb{C}), x)]^\vee \text{ is a motive.}$$

Via the comparison isomorphism, this also gives a motivic version of the Malcev completion of $\pi_1^{geom}(X, x)$.

Question:

What about a motivic lifting of the Malcev completion of $\pi_1^{alg}(X, x)$?

Answer:

The motivic lifting is given by the Tannaka group of the category of mixed Tate motives over X (under appropriate assumptions).

Suitably interpreted, this agrees with the Deligne-Goncharov motivic π_1: Mixed Tate motives over X are uni-potent local systems on X of mixed Tate motives over k.

Tate motives

Motives over a base

Voevodsky has defined a tensor triangulated category of geometric motives, $\mathrm{DM}_{\mathrm{gm}}(k)$, over a perfect field k.

Cisinski-Deglise have extended this to a tensor triangulated category of geometric motives, $\mathrm{DM}_{\mathrm{gm}}(S)$, over a base-scheme S.

The constructions starts with the category $\mathrm{Cor}(S)$ of finite correspondences over S:

$$\mathrm{Hom}_{\mathrm{Cor}(S)}(X, Y) := \mathbb{Z}\{W \subset X \times_S Y \mid W \text{ is irreducible and } W \to X \text{ is finite and surjective}\}$$

Set $\mathrm{PST}(S) :=$ category of additive presheaves on $\mathrm{Cor}(S)$.

Motives over a base

DM(S) is formed by localizing $C(\mathrm{PST}(S))$ and inverting the Lefschetz motive. $\mathrm{DM}_{\mathrm{gm}}(S) \subset \mathrm{DM}(S)$ is generated by the motives

$$m_S(X) := \mathrm{Hom}_{\mathrm{Cor}(S)}(-, X)$$

for X smooth over S.

There are also Tate motives $\mathbb{Z}_S(n)$ and Tate twists $m_S(X)(n) := m_S(X) \otimes \mathbb{Z}_S(n)$.

For S smooth over k:

$$\begin{aligned}\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(S)}(m_S(X), \mathbb{Z}_S(n)[m]) \\ = \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)}(m_k(X), \mathbb{Z}(n)[m]) \\ = H^m(X, \mathbb{Z}(n)) = \mathrm{CH}^n(X, 2n-m).\end{aligned}$$

Tate motives

Definition

Let X be a smooth k-scheme. The triangulated category of Tate motives over X, $DTM(X) \subset \mathrm{DM}_{\mathrm{gm}}(X)_{\mathbb{Q}}$, is the full triangulated subcategory of $\mathrm{DM}_{\mathrm{gm}}(X)_{\mathbb{Q}}$ generated by objects $\mathbb{Q}_X(p)$, $p \in \mathbb{Z}$.

Note.

$$\begin{aligned}\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(X)_{\mathbb{Q}}}(\mathbb{Q}_X(0), \mathbb{Q}_X(n)[m]) \\ = H^m(X, \mathbb{Q}(n)) \cong \mathrm{CH}^n(X, 2n-m) \otimes \mathbb{Q} \\ \cong K_{2n-m}(X)^{(n)},\end{aligned}$$

so Tate motives contain a lot of information.

$W_{\leq n}DTM(X)$:= the triangulated subcategory generated by $\mathbb{Q}_X(-m)$, $m \leq n$
$W_{\geq n}DTM(X)$:= the triangulated subcategory generated by $\mathbb{Q}_X(-m)$, $m \geq n$

- There are exact truncation functors

$$W_{\leq n}, W_{\geq n} : DTM(X) \to DTM(X)$$

with $W_{\leq n}M$ in $W_{\leq n}DTM(X)$, $W_{\geq n}M$ in $W_{\geq n}DTM(X)$.

- there are canonical distinguished triangles

$$W_{\leq n}M \to M \to W_{\geq n+1}M \to W_{\leq n}M[1]$$

- There is a canonical "filtration"

$$0 = W_{\leq N-1}M \to W_{\leq N}M \to \ldots \to W_{\leq N'-1}M \to W_{\leq N'}M = M.$$

Define

$$\mathrm{gr}^W_n M := W_{\leq n} W_{\geq n} M.$$

$\mathrm{gr}^W_n M$ is in the subcategory $W_{=n}DTM(X)$ generated by $\mathbb{Q}_X(-n)$:

$$W_{=n}DTM(X) \cong D^b(\mathbb{Q}\text{-Vec})$$

since

$$\mathrm{Hom}_{DTM(X)}(\mathbb{Q}(-n), \mathbb{Q}(-n)[m]) = H^m(X, \mathbb{Q}(0)) = \begin{cases} 0 & \text{if } m \neq 0 \\ \mathbb{Q} & \text{if } m = 0 \end{cases}$$

Thus, it makes sense to take $H^p(\mathrm{gr}^W_n M)$.

Definition

Let $MTM(X)$ be the full subcategory of $DTM(X)$ with objects those M such that

$$H^p(\mathrm{gr}^W_n M) = 0$$

for $p \neq 0$ and for all $n \in \mathbb{Z}$.

Theorem

Suppose that X satisfies the $\mathbb{Q}$-Beilinson-Soulé vanishing conjectures:

$$H^p(X, \mathbb{Q}(q)) = 0$$

for $q > 0$, $p \leq 0$. Then $MTM(X)$ is an abelian rigid tensor category.

$MTM(X)$ is the category of mixed Tate motives over X.

In addition:

1. $MTM(X)$ is closed under extensions in $DTM(X)$: if $A \to B \to C \to A[1]$ is a distinguished triangle in $DTM(X)$ with $A, C \in MTM(X)$, then B is in $MTM(X)$.
2. $MTM(X)$ contains the Tate objects $\mathbb{Q}(n)$, $n \in \mathbb{Z}$, and is the smallest additive subcategory of $DTM(X)$ containing these and closed under extension.
3. The weight filtration on $DTM(X)$ induces a exact weight filtration on $MTM(X)$, with

$$\mathrm{gr}_n^W M \cong \mathbb{Q}(-n)^{r_n}$$

Finally:

$$M \in MTM(X) \mapsto \oplus_n \mathrm{gr}_n^W M \in \mathbb{Q}\text{-Vec}$$

defines an exact faithful tensor functor

$$\omega : MTM(X) \to \mathbb{Q}\text{-Vec} :$$

$MTM(X)$ is a Tannakian category. Tannakian duality gives:

Theorem

Suppose that X satisfies the $\mathbb{Q}$-Beilinson-Soulé vanishing conjectures. Let $\mathcal{G}(X) = \mathrm{Gal}(MTM(X), \omega) := \mathrm{Aut}^{\otimes}(\omega)$. Then

1. *$MTM(X)$ equivalent to the category of finite dimensional $\mathbb{Q}$-representations of $\mathcal{G}(X)$.*
2. *There is a pro-unipotent group scheme $\mathcal{U}(X)$ over $\mathbb{Q}$ with $\mathcal{G}(X) \cong \mathcal{U}(X) \ltimes \mathbb{G}_m$*

Let k be a number field. Borel's theorem tells us that k satisfies B-S vanishing.
In fact $H^p(k, \mathbb{Q}(n)) = 0$ for $p \neq 1$ ($n \neq 0$). This implies

Proposition

Let k be a number field. Then $\mathcal{L}(k) := \mathrm{Lie}\, \mathcal{U}(k)$ is the free graded pro-nilpotent Lie algebra on $\oplus_{n\geq 1} H^1(k, \mathbb{Q}(n))^*$, with $H^1(k, \mathbb{Q}(n))^*$ in degree $-n$.

Note. $H^1(k, \mathbb{Q}(n)) = \mathbb{Q}^{d_n}$ with $d_n = r_1 + r_2$ ($n > 1$ odd) or r_2 ($n > 1$ even).

$H^1(k, \mathbb{Q}(1)) = \oplus_{\mathfrak{p}\subset \mathcal{O}_k \text{ prime}} \mathbb{Q}$.

Example $\mathcal{L}(\mathbb{Q}) = \mathrm{Lie}_{\mathbb{Q}}<[2],[3],[5],\ldots,s_3,s_5,\ldots>$, with $[p]$ in degree -1 and with s_{2n+1} in degree $-(2n+1)$.

$$MTM(\mathbb{Q}) = \mathrm{GrRep}(\mathrm{Lie}_{\mathbb{Q}}<[2],[3],[5],\ldots,s_3,s_5,\ldots>)$$

Here is our main result:

Theorem

Let X be a smooth k-scheme with a k-point x. Suppose that

1. *X satisfies B-S vanishing.*
2. *$m_k(X) \in \mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ is in $DTM(k)$.*

Then there is an exact sequence of pro group schemes over $\mathbb{Q}$:

$$1 \to \pi_1^{DG}(X, x) \to \mathrm{Gal}(MTM(X), \omega) \to \mathrm{Gal}(MTM(k), \omega) \to 1$$

where $\pi_1^{DG}(X, x)$ is the Deligne-Goncharov motivic π_1.

Comments on the fundamental exact sequence:

- The k-point $x \in X(k)$ gives a splitting:

$$1 \to \pi_1^{DG}(X,x) \to \mathrm{Gal}(MTM(X),\omega) \underset{x_*}{\overset{p_*}{\rightleftarrows}} \mathrm{Gal}(MTM(k),\omega) \to 1$$

making $\pi_1^{DG}(X,x)$ a pro algebraic group over $MTM(k)$: a mixed Tate motive.

This agrees with the motivic structure of Deligne-Goncharov.

$$1 \to \pi_1^{DG}(X,x) \to \mathrm{Gal}(MTM(X),\omega) \underset{x_*}{\overset{p_*}{\rightleftarrows}} \mathrm{Gal}(MTM(k),\omega) \to 1$$

- $\pi_1^{DG}(X,x) \cong$ the pro uni-potent completion of $\pi_1^{top}(X(\mathbb{C}),x)$. So

$$\begin{aligned} &Rep_{\mathbb{Q}}(\pi_1^{DG}(X,x)) \\ &\qquad \cong \text{ uni-potent local systems of } \mathbb{Q}\text{-vector spaces on } X. \end{aligned}$$

- The splitting given by x_* defines an isomorphism

$$\mathrm{Gal}(MTM(X),\omega) \cong \pi_1^{DG}(X,x) \ltimes \mathrm{Gal}(MTM(k),\omega).$$

Thus

$$\begin{aligned} MTM(X) &\cong Rep_{\mathbb{Q}}\mathrm{Gal}(MTM(X),\omega) \\ &\qquad \cong \text{ uni-potent local systems in } MTM(k) \text{ on } X. \end{aligned}$$

DG algebras and rational homotopy theory

$(M, 0)$: a pointed manifold. The loop space ΩM has a cosimplicial model:

$$pt \rightrightarrows M \mathrel{\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}} M^2 \mathrel{\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}} M^3 \cdots$$

$F[\pi_1(M, 0)]^* = H^0(\Omega M, F)$, so we expect

$$H^0\left(C^*(pt, F) \leftarrow C^*(M, F) \leftarrow C^*(M^2, F) \cdots \right) = F[\pi_1(M, 0)]^*$$

Due to convergence problems, get instead

$$H^0\left(C^*(pt, F) \leftarrow C^*(M, F) \leftarrow C^*(M^2, F) \cdots \right) = (F[\pi_1(M, 0)]^\vee)^*$$

By the Künneth formula $C^*(M^n, F) \sim C^*(M, F)^{\otimes n}$ so

$$\begin{aligned}(F[\pi_1(M,0)]^\vee)^* \\ &\cong H^0\left(C^*(pt, F) \leftarrow C^*(M, F) \leftarrow C^*(M^2, F) \cdots \right) \\ &\cong H^0\left(F \leftarrow C^*(M, F) \leftarrow C^*(M, F)^{\otimes 2} \cdots \right) \\ &= H^0(BC^*(M, F))\end{aligned}$$

$BC^*(M, F) :=$ the reduced bar construction.

Taking $F = \mathbb{R}$, use the de Rham complex for $C^*(M, \mathbb{R})$:

the de Rham complex computes the Malcev completion $\mathbb{R}[\pi_1(M, 0)]^\vee$.

Some general theory:

Let (A, d) be a commutative differential graded algebra over a field F:

- $A = \oplus_n A^n$ as a graded-commutative $\mathbb{Q}$-algebra
- d has degree $+1$, $d^2 = 0$ and $d(xy) = dx \cdot y + (-1)^{\deg x} x \cdot dy$.

For a cdga A over F with $\epsilon : A \to F$, the reduced bar construction is:

$$B(A, \epsilon) = \mathrm{Tot}\left(F \leftarrow A \leftarrow A^{\otimes 2} \leftarrow \dots\right)$$

Some useful facts:

- $H^0(B(A,\epsilon))$ is a filtered Hopf algebra over F.
- The associated pro-group scheme $\mathcal{G}(A,\epsilon) := \mathrm{Spec}\, H^0(B(A,\epsilon))$ is pro uni-potent.
- The isomorphism

$$H^0(BC^*(M,F)) \cong (F[\pi_1(M,0)]^\vee)^*$$

  is an isomorphism of Hopf algebras: For $G = \mathrm{Spec}\, H^0(BC^*(M,F))$,

$$Rep_F G \cong \text{ uni-potent local systems of } F \text{ vector spaces on } M.$$

We have associated a pro uni-potent algebraic group $\mathcal{G}(A, \epsilon) := \mathrm{Spec}\, H^0(B(A, \epsilon))$ to an augemented cdga (A, ϵ).

We associate a cdga to a pro uni-potent algebraic group $\mathcal{G}$ by taking the cochain complex $C^*(\mathrm{Lie}(\mathcal{G}), F)$.

$C^*(\mathrm{Lie}(\mathcal{G}(A, \epsilon)), F)$ is the 1-minimal model $\tilde{A}$ of A.

We recover $\mathcal{L} = \mathrm{Lie}(\mathcal{G}(A, \epsilon))$ from $\tilde{A}$ by $\mathcal{L}^* = \tilde{A}^1$. The dual of the Lie bracket is

$$d : \tilde{A}^1 \to \Lambda^2 \tilde{A}^1 = \tilde{A}^2.$$

One can construct the abelian category of representations of $\mathcal{G}(A, \epsilon)$ without going through the bar construction by using the derived category of A-modules.

A dg module over A, (M, d) is

- $M = \oplus_n M^n$ a graded A-module
- d has degree $+1$, $d^2 = 0$ and $d_M(xm) = d_A x \cdot +(-1)^{\deg x} x \cdot d_M m$.

This gives the category $\mathbf{d.g.Mod}_A$.

Inverting quasi-isomorphisms of dg modules gives the derived category of A-modules $D(A)$. The bounded derived category is the subcategory with objects the "semi-free" finitely generated dg A-modules.

In applications, A has an Adams grading:

$$A = F \oplus \oplus_{q\geq 1} A_q = F \oplus A_+;$$

we require an Adams grading on A-modules as well.
For a semi-free A-module $M = \oplus_i A \cdot e_i$, set

$$W_{\leq n} M := \oplus_{i, |e_i| \leq n} A \cdot e_i$$

Theorem (Kriz-May)

Let A be an Adams graded cdga over F.
1. $M \mapsto W_{\leq n} M$ induces an exact weight filtration on $D^b(A)$.

2. Suppose $H^p(A_+) = 0$ for $p \leq 0$ (cohomologically connected). Then $D^b(A)$ has a t-structure with heart $\mathcal{H}(A)$ equivalent to the category of graded representations of $\mathcal{G}(A)$.

Tate motives and rational homotopy theory

We view Tate motives as dg modules over the cycle cdga:

- For a smooth scheme X, we construct a cdga $\mathcal{N}(X)$ out of algebraic cycles (Bloch, Joshua).
- The bounded derived category of dg modules is equivalent to $DTM(X)$.
- If X satisfies B-S vanishing, $\mathcal{N}(X)$ is cohomologically connected and the heart of $D^b(\mathcal{N}(X))$ is equivalent to $MTM(X)$.

$\square^1 := (\mathbb{A}^1, 0, 1)$, $\square^n := (\mathbb{A}^1, 0, 1)^n$. $\square^n$ has faces $t_{i_1} = \epsilon_1 \ldots t_{t_r} = \epsilon_r$. S_n acts on $\square^n$ by permuting the coordinates.

Definition

X: a smooth k-scheme.

$$\mathcal{C}^q(X, n) := \mathbb{Z}\{W \subset X \times \square^n \times \mathbb{A}^q \mid W \text{ is irreducible and } W \to X \times \square^n \text{ is dominant and quasi-finite.}\}$$

$$\mathcal{N}(X)^n_q := \mathcal{C}^q(X, 2q-n)^{\mathrm{Alt}_\square \mathrm{Sym}_\mathbb{A}}/degn.$$

Restriction to faces $t_i = 0, 1$ gives a differential d on $\mathcal{N}(X)^*_q$. Product of cycles (over X) makes $\mathcal{N}(X) := \mathbb{Q} \oplus \oplus_{q \geq 1} \mathcal{N}(X)^*_q$ a cdga over $\mathbb{Q}$.

Proposition

1. $H^p(\mathcal{N}(X)_q) \cong H^p(X, \mathbb{Q}(q))$.

2. $\mathcal{N}(X)$ *is cohomologically connected iff* X *satisfies the* $\mathbb{Q}$*-Beilinson-Soule' vanishing conjectures*

Theorem (Spitzweck, extended by L.)

1. There is a natural equivalence of triangulated tensor categories with weight filtrations

$$D^b(\mathcal{N}(X)) \sim DTM(X)$$

2. If X *satisfies the B-S vanishing, then the equivalence in (1) induces an equivalence of (filtered) Tannakian categories*

$$\mathcal{H}(\mathcal{N}(X)) \sim MTM(X)$$

Idea of proof. Recall: $DTM(X) \subset \mathrm{DM}(X)_{\mathbb{Q}}$: a localization of $C(\mathrm{PST}(X))_{\mathbb{Q}}$).

Sending Y to $\mathcal{N}(Y)$ defines a presheaf $\mathcal{N}_X$ of graded $\mathcal{N}(X)$-algebras in $C(\mathrm{PST}(X))_{\mathbb{Q}}$.

$\mathcal{N}_X$ gives a tilting module to relate $D^b(\mathcal{N}(X))$ and $DTM(X)$: Sending a semi-free $\mathcal{N}(X)$ module M to $\mathcal{N}_X \otimes_{\mathcal{N}(X)} M$ defines a functor

$$\phi : D^b(\mathcal{N}(X)) \to DTM(X).$$

By calculation, the Hom's agree on Tate objects $\rightsquigarrow \phi$ is an equivalence.

Corollary (Main identification)

Suppose X satisfies B-S vanishing. Then

$$\mathrm{Gal}(MTM(X), \omega) \cong \mathrm{Spec}\, H^0(B\mathcal{N}(X)).$$

We use this to prove our main result: There is a split exact sequence

$$1 \longrightarrow \pi_1^{DG}(X, x) \longrightarrow \mathrm{Gal}(MTM(X), \omega) \underset{x_*}{\overset{p_*}{\rightleftarrows}} \mathrm{Gal}(MTM(k), \omega) \longrightarrow 1$$

Thus, we need to identify $\pi_1^{DG}(X, x)$ with the kernel of

$$p_* : \mathrm{Spec}\, H^0(B\mathcal{N}(X)) \to \mathrm{Spec}\, H^0(B\mathcal{N}(k)).$$

The Deligne-Goncharov motivic π_1 is defined by:

Let $X^\bullet$ be the cosimplicial loop space of X:

$$\mathrm{Spec}\, k \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; X \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; X^2 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; X^3 \ldots$$

Then

$$\pi_1^{DG}(X, x) := \mathrm{Spec}\, \mathrm{gr}^W_* H^0(m_k(X^\bullet)^*).$$

where:

$$DTM(k) \xrightarrow{H^0} MTM(k) \xrightarrow{\mathrm{gr}^W_*} \mathbb{Q} - \mathrm{Vec}$$

Via $x^* : \mathcal{N}(X) \to \mathcal{N}(k)$, $p^* : \mathcal{N}(k) \to \mathcal{N}(X)$ define the relative bar complex

$$B(\mathcal{N}(X)/\mathcal{N}(k)) := \mathcal{N}(k) \leftarrow \mathcal{N}(X) \leftarrow \mathcal{N}(X)^{\otimes^L_{\mathcal{N}(k)} 2} \leftarrow \ldots$$

and

$$\mathcal{G}(X/k) := \mathrm{Spec}\, H^0 B(\mathcal{N}(X)/\mathcal{N}(k)).$$

The theory of augmented cdgas gives us a split exact sequence

$$1 \longrightarrow \mathcal{G}(X/k) \longrightarrow \mathcal{G}(X) \underset{x_*}{\overset{p_*}{\rightleftarrows}} \mathcal{G}(k) \longrightarrow 1$$

Thus, we need to show that

$$H^0 B(\mathcal{N}(X)/\mathcal{N}(k)) = \mathrm{gr}^W_* H^0(m_k(X^\bullet)^*).$$

Since X is assumed to be a Tate motive, we have the Künneth formula:

$$\mathcal{N}(X^n) \cong \mathcal{N}(X)^{\otimes^L_{\mathcal{N}(k)} n}$$

The Künneth formula also gives

$$m_k(X^n)^* \cong \mathcal{N}_{X^n} \cong \mathcal{N}_k \otimes^L_{\mathcal{N}(k)} \mathcal{N}(X^n).$$

This identifies

$$m_k(X^\bullet)^* \cong \mathcal{N}_k \otimes^L_{\mathcal{N}(k)} \left(\mathcal{N}(k) \leftarrow \mathcal{N}(X) \leftarrow \mathcal{N}(X)^{\otimes^L_{\mathcal{N}(k)} 2} \leftarrow \ldots\right)$$

and

$$\mathrm{gr}^W_* H^0(m_k(X^\bullet)^*) \cong H^0 B(\mathcal{N}(X)/\mathcal{N}(k)).$$

Hence

$$\pi_1^{DG}(X, x) \cong \mathcal{G}(X/k).$$

Applications and problems

Applications and problems

- Concrete computations of Hodge/étale realizations of interesting mixed Tate motives: polylog, higher polylog motives.
- Tangential base-points?
- Approach to the Deligne-Ihara conjecture via Tate motives and rational homotopy theory.
- Grothendieck-Teichmüller theory for mixed Tate motives.
- Understanding Borel's theorem.
- Extensions to mixed Artin Tate motives and elliptic motives.

Thank you!