# Algebraic Cobordism

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# Outline

- Describe "oriented cohomology of smooth algebraic varieties"
- Recall the fundamental properties of complex cobordism
- Describe the fundamental properties of algebraic cobordism
- Sketch the construction of algebraic cobordism
- Give an application to Donaldson-Thomas invariants

# Algebraic topology and algebraic geometry

# Algebraic topology and algebraic geometry

## Naive algebraic analogs:

| Algebraic topology | | Algebraic geometry |
|---|---|---|
| Singular homology $H^*(X,\mathbb{Z})$ | $\leftrightarrow$ | Chow ring $\mathrm{CH}^*(X)$ |
| Topological $K$-theory $K^*_{top}(X)$ | $\leftrightarrow$ | Grothendieck group $K_0^{alg}(X)$ |
| Complex cobordism $MU^*(X)$ | $\leftrightarrow$ | Algebraic cobordism $\Omega^*(X)$ |

# Algebraic topology and algebraic geometry

## Refined algebraic analogs:

| Algebraic topology | | Algebraic geometry |
|---|---|---|
| The stable homotopy category SH | $\leftrightarrow$ | The motivic stable homotopy category over $k$, $\mathrm{SH}(k)$ |
| Singular homology $H^*(X, \mathbb{Z})$ | $\leftrightarrow$ | Motivic cohomology $H^{*,*}(X, \mathbb{Z})$ |
| Topological $K$-theory $K^*_{top}(X)$ | $\leftrightarrow$ | Algebraic $K$-theory $K^{alg}_*(X)$ |
| Complex cobordism $MU^*(X)$ | $\leftrightarrow$ | Algebraic cobordism $MGL^{*,*}(X)$ |

# Cobordism and oriented cohomology

Complex cobordism $MU^*$ is distinguished as the universal $\mathbb{C}$-oriented cohomology theory on differentiable manifolds.

We approach algebraic cobordism by defining oriented cohomology of smooth algebraic varieties, and constructing algebraic cobordism as the universal oriented cohomology theory.

What should "oriented cohomology of smooth varieties" be?
Follow complex cobordism $MU^*$ as a model:

$k$: a field. $\mathbf{Sm}/k$: smooth quasi-projective varieties over $k$.

An oriented cohomology theory $A$ on $\mathbf{Sm}/k$ consists of:

D1. An additive contravariant functor $A^*$ from $\mathbf{Sm}/k$ to graded (commutative) rings:

$$X \mapsto A^*(X);$$
$$(f : Y \to X) \mapsto f^* : A^*(X) \to A^*(Y).$$

D2. For each projective morphism $f : Y \to X$ in $\mathbf{Sm}/k$, a push-foward map ($d = \text{codim} f$)

$$f_* : A^*(Y) \to A^{*+d}(X)$$

These should satisfy some compatibilities and additional axioms. For instance, we should have

A1. $(fg)_* = f_* g_*; \quad \mathrm{id}_* = \mathrm{id}$

A2. For $f : Y \to X$ projective, $f_*$ is $A^*(X)$-linear:

$$f_*(f^*(x) \cdot y) = x \cdot f_*(y).$$

A3. Let

$$\begin{array}{ccc} W & \xrightarrow{f'} & Y \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ Z & \xrightarrow[f]{} & X \end{array}$$

be a transverse cartesian square in $\mathbf{Sm}/k$, with $g$ projective. Then

$$f^* g_* = g'_* f'^*.$$

- *singular cohomology:* $(k \subset \mathbb{C})$ $X \mapsto H^{2*}_{sing}(X(\mathbb{C}), \mathbb{Z})$.
- *topological K-theory:* $X \mapsto K^{2*}_{top}(X(\mathbb{C}))$
- *complex cobordism:* $X \mapsto MU^{2*}(X(\mathbb{C}))$
- *the Chow ring of cycles mod rational equivalence:* $X \mapsto \mathrm{CH}^*(X)$.
- *the Grothendieck group of algebraic vector bundles:* $X \mapsto K_0(X)[\beta, \beta^{-1}]$

Once we have $f^*$ and $f_*$, we have the 1st Chern class of a line bundle $L \to X$:

Let $s : X \to L$ be the zero-section, $1_X \in A^0(X)$ the unit. Define

$$c_1(L) := s^*(s_*(1_X)) \in A^1(X).$$

If we want to extend to a good theory of $A^*$-valued Chern classes of vector bundles, we need two additional axioms.

**PB**: Let $E \to X$ be a rank $n$ vector bundle,

$\mathbb{P}(E) \to X$ the projective-space bundle,

$O(1) \to \mathbb{P}(E)$ the tautological quotient line bundle.

$\xi := c_1(O(1)) \in A^1(\mathbb{P}(E))$.

Then $A^*(\mathbb{P}(E))$ is a free $A^*(X)$-module with basis $1, \xi, \ldots, \xi^{n-1}$.

**EH**: Let $p : V \to X$ be an affine-space bundle.

Then $p^* : A^*(X) \to A^*(V)$ is an isomorphism.

**Definition** $k$ a field. An *oriented cohomology theory $A$ over $k$* is a functor

$$A^* : \mathbf{Sm}/k^{\mathrm{op}} \to \mathbf{GrRing}$$

together with push-forward maps

$$g_* : A^*(Y) \to A^{*+d}(X)$$

for each projective morphism $g : Y \to X$, $d = \mathrm{codim} g$, satisfying the axioms A1-3, PB and EV:

- functoriality of push-forward,
- projection formula,
- compatibility of $f^*$ and $g_*$ in transverse cartesian squares,
- projective bundle formula,
- homotopy.

$A$: an oriented cohomology theory. Recall that

$$c_1(L) = s^*(s_*(1))$$

⚠ $c_1$ *is not necessarily additive!* $c_1(L \otimes M) \neq c_1(L) + c_1(M)$.

Instead, there is a $F_A(u, v) \in A^*(k)[[u, v]]$ with

$$c_1(L \otimes M) = F_A(c_1(L), c_1(M)) = c_1(L) + c_1(M) + \dots$$

for line bundles $L, M$.

$F_A$ satisfies

- $F_A(u, 0) = u = F_A(0, u)$
- $F_A(u, v) = F_A(v, u)$
- $F_A(F_A(u, v), w) = F_A(u, F_A(v, w))$

so $F_A(u, v)$ is a formal group law over $A^*(k)$.

**Examples**

1. $F_{\mathrm{CH}}(u, v) = u + v$: the additive formal group law
2. $F_K(u, v) = u + v - \beta uv$: the multiplicative formal group law.

# Topological background

The axioms for an oriented cohomology theory on $\mathbf{Sm}/k$ are abstracted from Quillen's notion of a *$\mathbb{C}$-oriented cohomology theory* on the category of differentiable manifolds.

A $\mathbb{C}$-oriented theory $E$ also has a formal group law with coefficients in $E^*(pt)$: $F_E(u, v) \in E^*(pt)[[u, v]]$ with

$$c_1(L \otimes M) = F_E(c_1(L), c_1(M))$$

for continuous $\mathbb{C}$-line bundles $L, M$.

**Examples**

1. $H^*(-, \mathbb{Z})$ has the additive formal group law $u + v$.
2. $K^*_{top}$ has the multiplicative formal group law $u + v - \beta uv$, $\beta =$ Bott element in $K^{-2}_{top}(pt)$.
3. $MU^*$?

There is a universal formal group law $F_{\mathbb{L}}$, with coefficient ring the *Lazard ring* $\mathbb{L}$. For a topological $\mathbb{C}$-oriented theory $E$, let

$$\phi_E : \mathbb{L} \to E^*(pt);\ \phi(F_{\mathbb{L}}) = F_E$$

be the ring homomorphism classifying $F_E$; for an oriented theory $A$ on $\mathbf{Sm}/k$, let

$$\phi_A : \mathbb{L} \to A^*(k);\ \phi(F_{\mathbb{L}}) = F_A.$$

be the ring homomorphism classifying $F_A$.

### Theorem (Quillen)

*(1) Complex cobordism $MU^*$ is the universal $\mathbb{C}$-oriented theory (on topological spaces).*

*(2) $\phi_{MU} : \mathbb{L} \to MU^*(pt)$ is an isomorphism, i.e., $F_{MU}$ is the universal group law.*

Let $\phi : \mathbb{L} = MU^*(pt) \to R$ classify a group law $F_R$ over $R$. If $\phi$ satisfies the "Landweber exactness" conditions, form the $\mathbb{C}$-oriented cohomology theory $MU \wedge_\phi R$, with

$$(MU \wedge_\phi R)^*(X) = MU^*(X) \otimes_{MU^*(pt)} R$$

and formal group law $F_R$.

**Theorem (Conner-Floyd)**

*$K^*_{top} = MU \wedge_\times \mathbb{Z}[\beta, \beta^{-1}]$; $K^*_{top}$ is the universal multiplicative oriented cohomology theory.*

# Algebraic cobordism

**Theorem (L.-Morel)**

*Let $k$ be a field of characteristic zero.*

*(1) There is a universal oriented cohomology theory $\Omega$ over $k$, called* algebraic cobordism.

*(2) The classifying map $\phi_\Omega : \mathbb{L} \to \Omega^*(k)$ is an isomorphism, so $F_\Omega$ is the universal formal group law.*

For an arbitrary formal group law $\phi : \mathbb{L} = \Omega^*(k) \to R$, $F_R := \phi(F_{\mathbb{L}})$, we have the oriented theory

$$X \mapsto \Omega^*(X) \otimes_{\Omega^*(k)} R := \Omega^*(X)_\phi.$$

$\Omega^*(X)_\phi$ is universal for theories whose group law factors through $\phi$.

Let

$$\Omega^*_\times := \Omega^* \otimes_{\mathbb{L}} \mathbb{Z}[\beta, \beta^{-1}]$$

$$\Omega^*_+ := \Omega^* \otimes_{\mathbb{L}} \mathbb{Z}.$$

We recover both $K_0$ and $\mathrm{CH}^*$ from $\Omega^*$.

**Theorem**

*The canonical map*

$$\Omega^*_\times \to K_0^{alg}[\beta, \beta^{-1}]$$

*is an isomorphism, i.e.,* $K_0^{alg}[\beta, \beta^{-1}]$ *is the universal multiplicative theory over* $k$.

**Theorem**

*The canonical map*

$$\Omega^*_+ \to \mathrm{CH}^*$$

*is an isomorphism, i.e.,* $\mathrm{CH}^*$ *is the universal additive theory over* $k$.

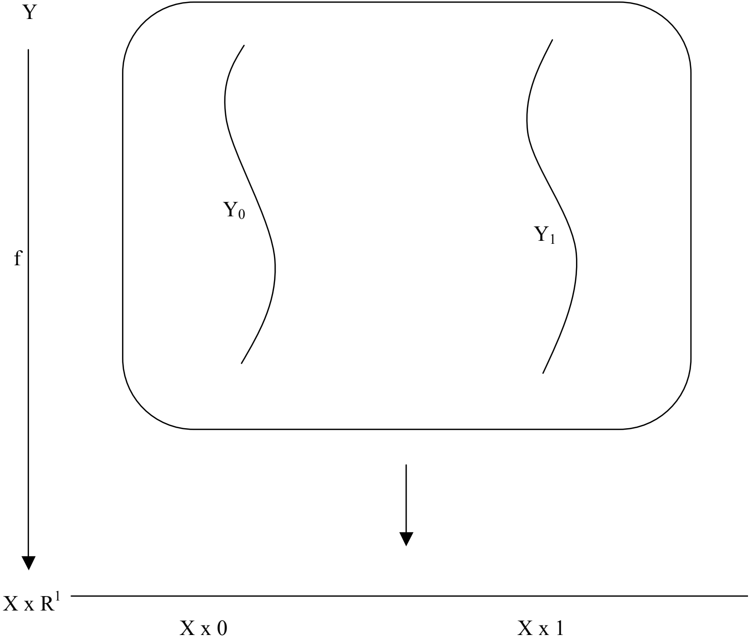
# The construction of algebraic cobordism

Let $X$ be a manifold. By classical transversality results in topology, $MU^n(X)$ has a presentation

$$MU^n(X) = \{f : Y \to X \mid f \text{ proper, } \mathbb{C} \text{ oriented, } n = \mathrm{codim} f\}/\sim$$

where $\sim$ is the cobordism relation: For $f : Y \to X \times \mathbb{R}^1$, transverse to $X \times \{0, 1\}$, $Y_i = f^{-1}(i)$, $i = 0, 1$,

$$[Y_0 \to X] \sim [Y_1 \to X]$$



Y
f
$Y_0$
$Y_1$
X x $R^1$
X x 0
X x 1

# Construction of algebraic cobordism

The original construction of $\Omega^*(X)$ was rather complicated, but necessary for proving all the main properties of $\Omega^*$.

Following a suggestion of Pandharipande, we now have a very simple presentation, with the same kind of generators as for complex cobordism. The relations are also similar, but we need to allow "double-point cobordisms".

**Definition** Take $X \in \mathbf{Sm}/k$. $\mathcal{M}^n(X)$ is the free abelian group on (iso classes of) projective morphisms $f : Y \to X$ with

1. $Y$ irreducible and smooth over $k$
2. $n = \dim_k X - \dim_k Y = \mathrm{codim} f$.

$\mathcal{M}^n(X)$ generates $\Omega^n(X)$.

The relations are given by double point cobordisms

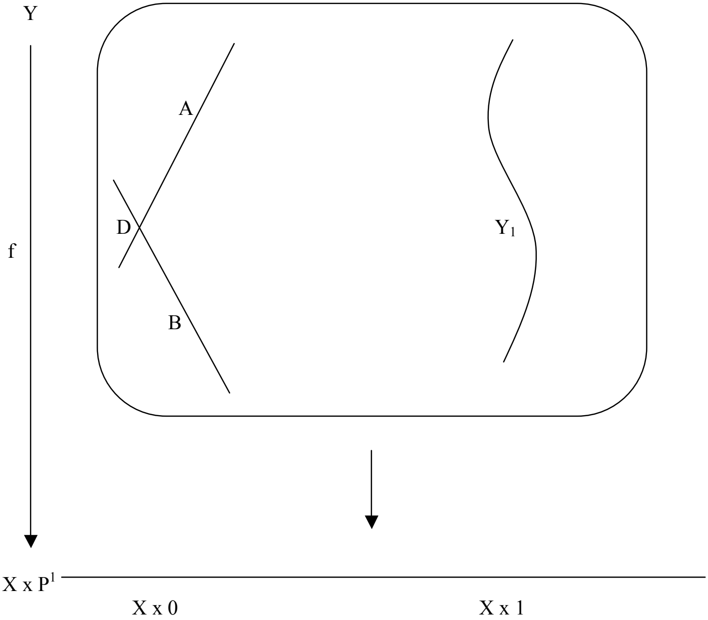
**Definition** A projective morphism $f : Y \to X \times \mathbb{P}^1$ in $\mathbf{Sm}/k$ is a *double-point cobordism* if $Y_1 := f^{-1}(X \times 1)$ is smooth and

$$Y_0 := f^{-1}(X \times 0) = A \cup B$$

where

1. $A$ and $B$ are smooth.
2. $A$ and $B$ intersect transversely on $Y$.

The codimension two smooth subscheme $D := A \cap B$ is called the *double-point locus* of the cobordism.


Y
A
D
B
f
Y1
X x P1
X x 0
X x 1

Let $f : Y \to X \times \mathbb{P}^1$ be a double-point cobordism, with

$$f^{-1}(X \times 0) = A \cup B;\ D := A \cap B.$$

Set $N_{D/A} :=$ the normal bundle of $D$ in $A$.

Set

$$\mathbb{P}(f) := \mathbb{P}(\mathcal{O}_D \oplus N_{D/A}),$$

a $\mathbb{P}^1$-bundle over $D$.

The definition of $\mathbb{P}(f)$ does not depend on the choice of $A$ or $B$:

$$\mathbb{P}_D(\mathcal{O}_D \oplus N_{D/A}) \cong \mathbb{P}_D(\mathcal{O}_D \oplus N_{D/B}).$$

Let $f : Y \to X \times \mathbb{P}^1$ be a double-point cobordism, $n = \mathrm{codim} f$.

Write $f^{-1}(X \times 0) = Y_0 = A \cup B$, $f^{-1}(X \times 1) = Y_1$, giving elements

$$[A \to X], [B \to X], [\mathbb{P}(f) \to X], [Y_1 \to X]$$

of $\mathcal{M}^n(X)$.

The element

$$R(f) := [Y_1 \to X] - [A \to X] - [B \to X] + [\mathbb{P}(f) \to X]$$

is the *double-point relation* associated to the double-point cobordism $f$.

**Definition** For $X \in \mathbf{Sm}/k$, $\Omega^*_{dp}(X)$ (double-point cobordism) is the quotient of $\mathcal{M}^*(X)$ by the subgroup of generated by relations $\{R(f)\}$ given by double-point cobordisms:

$$\Omega^*_{dp}(X) := \mathcal{M}^*(X)/<\{R(f)\}>$$

for all double-point cobordisms $f : Y \to X \times \mathbb{P}^1$.

In other words, we impose all double-point cobordism relations

$$[Y_1 \to X] = [A \to X] + [B \to X] - [\mathbb{P}(f) \to X]$$

We have the homomorphism

$$\phi : \mathcal{M}^*(X) \to \Omega^*(X)$$

sending $f : Y \to X$ to $f_*(1_Y) \in \Omega^*(X)$.

**Theorem (L.-Pandharipande)**

*The map $\phi$ descends to an isomorphism*

$$\phi : \Omega^*_{dp}(X) \to \Omega^*(X)$$

*for all $X \in \mathbf{Sm}/k$.*

# Donaldson-Thomas invariants

Let $X$ be a smooth projective 3-fold over $\mathbb{C}$.

$\mathrm{Hilb}(X, n)$ = the Hilbert scheme of length $n$ closed subschemes of $X$.

Maulik, Nekrasov, Okounkov and Pandharipande construct a "virtual fundamental class"

$$[\mathrm{Hilb}(X, n)]^{vir} \in \mathrm{CH}_0(\mathrm{Hilb}(X, n)).$$

This gives the partition function

$$Z(X, q) := 1 + \sum_{n \geq 1} \deg([\mathrm{Hilb}(X, n)]^{vir}) q^n$$

## Conjecture (MNOP)

*Let $M(q)$ be the MacMahon function:*

$$M(q) = \prod_{n\geq 1} \frac{1}{(1-q^n)^n} = 1 + q + 3q^2 + 6q^3 + 13q^4 + \dots .$$

*Then*

$$Z(X,q) = M(q)^{\deg(c_3(T_X \otimes K_X))}$$

*for all smooth projective threefolds $X$ over $\mathbb{C}$.*

*Note.* The MacMahon function has a combinatorial origin as the generating function for the number of 3-dimensional partitions of size $n$, i.e., expressions

$$n = \sum_{ij} \lambda_{ij}; \quad \lambda_{ij} \geq \lambda_{i+1,j} > 0,\ \lambda_{ij} \geq \lambda_{i,j+1} > 0.$$

MNOP verify:

**Proposition (Double point relation)**

*Let $\pi : Y \to \mathbb{P}^1$ be a double-point cobordism (over $\mathbb{C}$) of relative dimension 3. Write $\pi^{-1}(0) = A \cup B$, $\pi^{-1}(1) = Y_1$. Then*

$$Z(Y_1, q) = \frac{Z(A, q) \cdot Z(B, q)}{Z(\mathbb{P}(\pi), q)}$$

In other words, sending a smooth projective threefold $X$ to $Z(X, q)$ descends to a homomorphism

$$Z(-, q) : \Omega^{-3}(\mathbb{C}) \to (1 + \mathbb{Q}[[q]])^{\times}.$$

By general principles, the function

$$X \mapsto \deg(c_3(T_X \otimes K_X))$$

descends to a homomorphism $\Omega^{-3}(\mathbb{C}) \to \mathbb{Z}$.

Thus $X \mapsto M(q)^{\deg(c_3(T_X \otimes K_X))}$ descends to

$$M(q)^{?} : \Omega^{-3}(\mathbb{C}) \to (1 + \mathbb{Q}[[q]])^{\times}.$$

Next we have the result of MNOP:

**Proposition**

*The conjecture is true for* $X = \mathbb{CP}^3$, $\mathbb{CP}^1 \times \mathbb{CP}^2$, *and* $(\mathbb{CP}^1)^3$.

To finish, we use the well-known fact from topology:

**Proposition**

*The rational Lazard ring* $\mathbb{L}^* \otimes \mathbb{Q} = MU^{2*}(pt) \otimes \mathbb{Q}$ *is a polynomial ring over* $\mathbb{Q}$ *with generators the classes* $[\mathbb{CP}^n]$, $n = 0, 1, \ldots$, *with* $[\mathbb{CP}^n]$ *in degree* $* = -n$.

Thus $M(q)^?$ and $Z(-, q)$ agree on as $\mathbb{Q}$-basis of $\Omega^{-3}(\mathbb{C})_{\mathbb{Q}} = MU^{-6}(pt)_{\mathbb{Q}}$, hence are equal.

# Motivic homotopy theory

In stable homotopy theory, complex cobordism is represented by the Thom complex, the sequence of spaces

$$MU_{2n} := Th(E_n \to BU_n) := \mathbb{CP}(E_n \oplus 1)/\mathbb{CP}(E_n)$$

The algebraic version $MGL$ is the same, replacing $BU_n$ with the infinite Grassmann variety $Gr(n, \infty) = BGL_n$:

$$MGL_n := Th(E_n \to BGL_n) := \mathbb{P}(E_n \oplus \mathcal{O})/\mathbb{P}(E_n).$$

The "naive" theories $\mathrm{CH}^n$ and $K_0^{alg}$ are the $(2n, n)$ parts of the "refined" theories:

$$\mathrm{CH}^n(X) \cong H^{2n,n}(X, \mathbb{Z})$$

$$K_0(X) \cong K^{2n,n}(X)$$

The universality of $\Omega^*$ gives a natural map

$$\nu_n(X) : \Omega^n(X) \to MGL^{2n,n}(X).$$

**Theorem**

$\Omega^n(X) \cong MGL^{2n,n}(X)$ *for all* $n$*, all* $X \in \mathbf{Sm}/k$.

The proof relies on (unpublished) work of Hopkins-Morel.

# Other results and applications

The theory extends to arbitrary schemes as oriented Borel-Moore homology.

The homology version of algebraic cobordism, $\Omega_*$, is the universal theory and has a presentation with generators and relations just like $\Omega^*$.

The Connor-Floyd theorem extends:
$\Omega_*^+(X) \cong \mathrm{CH}_*(X)$
$\Omega_*^\times(X) \cong G_0(X)[\beta, \beta^{-1}]$ (S. Dai)

Application: construction of Brosnan's Steenrod operations on $\mathrm{CH}_*/p$ using formal group law tricks.

Markus Rost conjectured that certain mod $p$ characteristic classes $s(-)$ satisfy a degree formula:
Given a morphism $f : Y \to X$ of smooth varieties of the same dimension $d = p^n - 1$,

$$s(Y) \equiv \deg(f) \cdot s(X) \mod I(X)$$

where $I(X)$ is the ideal generated by field extension degrees $[k(x) : k]$, $x$ a closed point of $X$.

Properties of $\Omega^*$ yield a simple proof of the degree formula.

Applications: Incompressibility results (Merkurjev, et al), a piece of the proof of the Bloch-Kato conjecture.

An oriented cohomology theory $A^*$ gives an associated theory of $A$-motives (classical case: Chow motives).

Vishik has started a study of cobordism motives and proved an important nilpotence property. This helped in computations of the cobordism ring of quadrics.

Calmes-Petrov-Zainoulline have computed the ring structure for algebraic cobordism of flag varieties.

# Open problems

1. Give a "geometric" description of the rest of $MGL^{*,*}$

2. What kind of theory reflects the degeneration relations for positive degree D-T invariants?

3. Cobordism Gromov-Witten invariants: virtual fundamental class.

# Thank you!