Surjectivity of cycle maps

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Introduction

The complicated nature of the theory of cycles of codimension two and higher became apparent with Mumford's paper [M], which showed that $p_g = 0$ is a necessary condition for the representability of the group of zero-cycles on a smooth projective surface over $\mathbb{C}$. This was generalized by Roitman [R] when he showed that the vanishing of all the groups $H^0(\Omega^q)$, $q > 1$, is necessary for the representability of the group of zero-cycles on a smooth projective variety over $\mathbb{C}$. Bloch, Kas and Lieberman [BKL] investigated the zero-cycles on surfaces with $p_g = 0$, showing that the group of zero- cycles was in fact representable, at least if the surface is not of general type; Bloch [Bl] has conjectured that $p_g = 0$ is sufficient for the representability of the zero-cycles on a smooth projective surface. The case of surfaces of general type is still an open problem, although there has been some progress, most recently by Voisin [V].

Bloch's proof in [Bl] of Mumford's infinite dimensionality theorem views the diagonal in $X \times X$ as a family of zero-cycles on X, parametrized by X, and goes on to consider the consequences of the generic triviality of this family. This may be the first appearance of this point of view. Coombes and Srinivas used this idea in [CS] to get a decomposability result for $H^1(\mathcal{K}_2)$ of a surface. Bloch and Srinivas [BS] push this approach further, making a study of the cycle groups on a smooth variety X which relies on a partial decomposition of the diagonal in $X \times X$. They have applied this method to give some examples for which certain cycle groups are representable. This approach was recently used by Paranjape [P] in his discussion of the cycle groups of subvarieties of projective space of small degree and small codimension. Schoen [S] has also applied this method to give generalizations of the Mumford-Roitman criterion for non-representability to the Chow groups of cycles of positive dimension. Jannsen [J] used the ideas of Bloch and Srinivas in his discussion of smooth projective varieties X for which the rational topological cycle maps

$$\mathrm{CH}^p(X) \otimes \mathbb{Q} \to H^{2p}_{\mathcal{B}}(X, \mathbb{Q})$$

are injective. For such a variety, Jannsen shows that the diagonal in $X \times X$ decomposes in $\mathrm{CH}^*(X \times X)_{\mathbb{Q}}$ into a sum of product cycles

$$\Delta = A_0 \times B^0 + A_1 \times B^1 + \ldots + A_d \times B^d$$

where A_i is a dimension i cycle, B^i is a codimension i cycle, and $d = \dim(X)$. One consequence of this decomposition is that the total cycle map

$$\bigoplus_{p=0}^{d} \mathrm{CH}^p(X) \otimes \mathbb{Q} \to \bigoplus_{q=0}^{2d} H^q_{\mathcal{B}}(X, \mathbb{Q})$$

is an isomorphism; in particular, X has no odd cohomology.

In this paper, we prove an analog of Jannsen's result, considering the cycle map to rational Deligne cohomology rather than Betti cohomology. Assuming injectivity of the Deligne cycle maps, we arrive at a decomposition of the diagonal into a sum of codimension one cycles on products of the form $\Gamma_{i+1} \times D^i$, with $\dim(\Gamma_{i+1}) = i + 1$, $\mathrm{cod}(D^i) = i$ (see Theorem 1.2 for a more precise statement). The consequences of this decomposition are a surjectivity statement for certain cycle maps to Deligne cohomology and some other related maps (Theorem 2.5), a vanishing result for certain Hodge numbers (Theorem 3.2), and a decomposability result for the K-cohomology (Theorem 4.1). If we assume that *all* the rational cycle class maps for a smooth projective variety X are injective, then

(1) all the rational Hodge cycles on X are algebraic (Corollary 2.6)

(2) the Abel-Jacobi maps

$$cl^n \colon \mathrm{CH}^n(X)_{alg} \to J^n(X)$$

are all surjective (Corollary 3.3)

(3) the Hodge numbers $h^{p,q}(X)$ all vanish for $|p - q| > 1$.

(4) the maps

$$\mathrm{CH}^p(X) \otimes \mathbb{C}^\times \to H^p(X, \mathcal{K}_{p+1})$$

are all surjective.

The results on the Hodge numbers are a direct generalization of the results of Mumford-Roitman mentioned above. This points the way to some possible generalizations of Bloch's conjecture to a conjecture on the representability of cycle groups of higher dimension (see Questions 1 and 2 in §3). What is novel about the situation is that it involves *all* the groups of cycles of dimension 0 to s rather than the cycles of a single dimension s. Schoen has raised similar questions in his paper [S], from a slightly different point of view, replacing the injectivity assumption with an assumption that the generalized Hodge conjecture holds, and that the group of dimension s cycles is representable; we haven't attempted to reconcile these two points of view.

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§1. Decomposition of the diagonal

In this section, we show how the injectivity of the cycle map to Deligne cohomology leads to a decomposition of the diagonal. If X is a smooth projective variety, we let $\mathcal{Z}^n(X)$ denote the group of codimension n cycles on X, $\mathrm{CH}^n(X)$ the group of cycles modulo rational equivalence. We let $\mathcal{Z}_n(X)$ and $\mathrm{CH}_n(X)$ denote the group of dimension n cycles and cycle classes. If X is defined over $\mathbb{C}$, we have the cycle class map

$$cl^n\colon \mathcal{Z}^n(X) \to H^{2n}_{\mathcal{D}}(X, \mathbb{Z}(n)).$$

This map passes to rational equivalence, giving the map

$$cl^n\colon \mathrm{CH}^n(X) \to H^{2n}_{\mathcal{D}}(X, \mathbb{Z}(n)).$$

We refer to an element of $\mathcal{Z}^n(X)_{\mathbb{Q}}$ as a $\mathbb{Q}$-cycle. We also denote by cl^n the maps induced by cl^n after extending the coefficient ring. For the basic properties of Deligne cohomology and the cycle map, we refer the reader to [B].

Let $Hg^n(X)$ denote the group of codimension n *Hodge cycles* on X:

$$Hg^n(X) := \{x \in H^{2n}(X, \mathbb{Z}(n)) \mid x \otimes 1 \in F^n H^{2n}(X, \mathbb{C})\}.$$

We have the exact sequence describing $H^{2n}_{\mathcal{D}}(X, \mathbb{Z}(n))$ as an extension:

$$0 \to \frac{H^{2n-1}(X, \mathbb{C})}{H^{2n-1}(X, \mathbb{Z}(n)) + F^n H^{2n-1}(X, \mathbb{C})} \to H^{2n}_{\mathcal{D}}(X, \mathbb{Z}(n)) \to Hg^n(X) \to 0.$$

The n^{th} intermediate Jacobian, $J^n(X)$, is the complex torus on the left-hand side of the above sequence.

Lemma 1.1. *Let X be a smooth projective variety over $\mathbb{C}$ of dimension d. Suppose the $\mathbb{Q}$-cycle class map*

$$cl^n \colon CH^n(X)_{\mathbb{Q}} \to H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n))$$

is injective. Let D be a pure codimension $i = d - n$ closed subset of X, and let γ be a codimension d $\mathbb{Q}$-cycle on $X \times X$, supported on $X \times D$. Then there are closed subsets D' and Γ of X, codimension d $\mathbb{Q}$-cycles $\gamma_?$ and $\gamma^?$ on $X \times X$ such that

(1) D' has pure codimension $i + 1$ and Γ has pure dimension $i + 1$.

(2) $\gamma_?$ is supported on $\Gamma \times D$ and $\gamma^?$ is supported on $X \times D'$.

(3) $\gamma = \gamma_? + \gamma^?$ in $CH^d(X \times X)_{\mathbb{Q}}$.

Proof. If D has irreducible components $D_1, \ldots, D_s$, we can write γ as a sum

$$\gamma = \gamma^1 + \ldots + \gamma^s$$

with γ^j supported on $X \times D_j$. Thus we may assume that D is irreducible. Write γ as a sum, $\gamma = \gamma' + \gamma''$, such that each irreducible component of the support of γ' dominates D, and no irreducible component of the support of γ'' dominates D. Since γ'' is supported on $X \times p_2(supp(\gamma''))$, and $p_2(supp(\gamma''))$ has codimension at least $i+1$ on X, we may assume that $\gamma = \gamma'$. We may then find a smooth projective variety $\tilde{D}$, mapping birationally to D by $p\colon \tilde{D} \to D$, and a $\mathbb{Q}$-cycle $\tilde{\gamma}$ on $X \times \tilde{D}$ such that

(i) for each $y \in \tilde{D}$, $X \times y$ and $\tilde{\gamma}$ intersect properly on $X \times \tilde{D}$.

(ii) $(id_X \times p)_*(\tilde{\gamma}) = \gamma$.

Indeed, for a resolution of singularities $r\colon E \to D$, and a subvariety Z of $X \times D$, there is a subvariety W of $X \times E$ which is generically finite over Z. Thus each cycle γ as above can be lifted to a $\mathbb{Q}$-cycle γ_E on $X \times E$. Having done this, we may further blow-up E via $\tilde{D} \to E$ so that each component of γ_E has proper transform to $X \times \tilde{D}$ which is flat over $\tilde{D}$, giving us the desired resolution $\tilde{D}$ and $\mathbb{Q}$-cycle $\tilde{\gamma}$.

For a point $y \in \tilde{D}$, let γ_y be the $\mathbb{Q}$- cycle $p_{X*}((X \times y) \cdot \tilde{\gamma})$. Each γ_y has codimension n on X. Fix a point $0 \in \tilde{D}$. Since $\tilde{D}$ is connected, the cycles γ_0 and γ_y are homologous on X, for each y in $\tilde{D}$. Thus $cl^n(\gamma_y - \gamma_0)$ is in $J^n(X)_{\mathbb{Q}}$, for each $y \in \tilde{D}$. Let $cl\colon \tilde{D} \to J^n(X)_{\mathbb{Q}}$ be the map

$$cl(y) = cl^n(\gamma_y - \gamma_0).$$

In similar fashion, we have the map $ch\colon \tilde{D} \to \mathrm{CH}^n(X)_{\mathbb{Q}}$ defined by

$$ch(y) = \gamma_y - \gamma_0 \quad \text{mod rational equivalence.}$$

Both ch and cl extend by linearity to maps

$$\begin{aligned} ch\colon \mathrm{CH}_0(\tilde{D}) &\to \mathrm{CH}^n(X)_{\mathbb{Q}} \\ cl\colon \mathrm{CH}_0(\tilde{D}) &\to J^n(X)_{\mathbb{Q}}. \end{aligned}$$

The map cl factors further through the Albanese map

$$\alpha_{\tilde{D}}\colon CH_0(\tilde{D}) \to \mathrm{Alb}(\tilde{D}).$$

Clearly we have $cl^n \circ ch = cl$; since the map cl^n is injective by hypothesis, this implies that ch factors through $\mathrm{Alb}(\tilde{D})$ as well.

Take an embedding of $\tilde{D}$ in a $\mathbb{P}^N$, and let C be a smooth linear section of $\tilde{D}$ of dimension one; we assume that C contains 0. By the weak Lefschetz theorem, the map $\mathrm{Alb}(C) \to \mathrm{Alb}(\tilde{D})$ is surjective; in particular, this implies that, for each $y \in \tilde{D}$, there is a $\mathbb{Q}$-zero cycle a_y, supported on C, such that $cl(y) = cl(a_y)$. As the map ch factors through $\mathrm{Alb}(\tilde{D})$, we have $ch(y) = ch(a_y)$.

Take y to be a geometric generic point of $\tilde{D}$ over $\mathbb{C}$, so $\mathbb{C}(y) = \mathbb{C}(\tilde{D}) = \mathbb{C}(D)$. The zero-cycle a_y is defined over some finitely generated field extension of $\mathbb{C}(\tilde{D})$; by specializing a_y and changing notation, we may assume that the zero-cycle a_y is defined over a finite extension L of $\mathbb{C}(\tilde{D})$, of degree say M. Let b_y be the zero cycle $\frac{1}{M} \cdot Nm_{L/\mathbb{C}(\tilde{D})}(a_y)$. Then b_y is defined over $\mathbb{C}(\tilde{D})$, b_y is supported on C and $ch(y) = ch(b_y)$. In particular, there is a unique $\mathbb{Q}$-cycle $\tilde{\gamma}_?$ on $X \times \tilde{D}$ such that

(iii) $p_{X*}((X \times y) \cdot \tilde{\gamma}_?) = p_{X*}((X \times b_y) \cdot \tilde{\gamma})$, for y a geometric generic point of $\tilde{D}$ over $\mathbb{C}$.

(iv) each irreducible component of $\mathrm{supp}(\tilde{\gamma}_?)$ dominates $\tilde{D}$.

Let $S = p_X(\mathrm{supp}(\tilde{\gamma}) \cap X \times C)$. Since the fibers of $\mathrm{supp}(\tilde{\gamma})$ over $\tilde{D}$ all have dimension i, S has dimension at most $i+1$. By (iii) and (iv), $\tilde{\gamma}_?$ is supported on $S \times \tilde{D}$. Since $ch(y) = ch(b_y)$, (iii), together with the localization sequence for the Chow groups, implies there is a codimension one closed subset $\tilde{D}'$ of $\tilde{D}$, and a cycle $\tilde{\gamma}^? \in CH^{d-i}(X \times \tilde{D})$, supported on $X \times \tilde{D}'$, such that

(v) $\tilde{\gamma} = \tilde{\gamma}_? + \gamma_0 \times \tilde{D} + \tilde{\gamma}^?$ in $\mathrm{CH}^{d-i}(X \times \tilde{D})_{\mathbb{Q}}$.

Let Γ be a pure dimension $i+1$ closed subset of X containing S and $\mathrm{supp}(\gamma_0)$, let D' be a pure codimension $i+1$ closed subset of X containing $p(\tilde{D}')$. Take $\gamma_? = (id_X \times p)_*(\tilde{\gamma}_? + \gamma_0 \times \tilde{D})$, $\gamma^? = (id_X \times p)_*(\tilde{\gamma}^?)$. Since $(id_X \times p)_*(\tilde{\gamma}) = \gamma$, we have

$$\gamma = \gamma_? + \gamma^? \text{ in } \mathrm{CH}^d(X \times X)_{\mathbb{Q}}$$
$$\gamma^? \text{ is supported on } X \times D'$$
$$\gamma_? \text{ is supported on } \Gamma \times D,$$

as desired. $\square$

Theorem 1.2. *Let X be a smooth projective variety over $\mathbb{C}$ of dimension d, and let Δ be the class of the diagonal in $CH^d(X \times X)_{\mathbb{Q}}$. Suppose the $\mathbb{Q}$-cycle class maps*

$$cl^n: CH^n(X)_{\mathbb{Q}} \to H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n))$$

are injective for $n = d, d-1, \ldots, d-s$, for some integer s, $0 \leq s \leq d-2$. Then there are closed subsets $X = D^0, D^1, \ldots, D^{s+1}$, $\Gamma_1, \ldots, \Gamma_{s+1}$, and cycles $\gamma_1, \ldots, \gamma_s, \gamma^{s+1} \in CH^d(X \times X)_{\mathbb{Q}}$ such that

(1) D^i has pure codimension i, Γ_i has pure dimension i.

(2) γ_i is supported on $\Gamma_{i+1} \times D^i$, for $i = 0, \ldots, s$.

(3) γ^{s+1} is supported on $X \times D^{s+1}$.

(4) $\Delta = \gamma_0 + \ldots + \gamma_s + \gamma^{s+1}$ in $CH^d(X \times X)_{\mathbb{Q}}$.

Proof. We first apply Lemma 1.1 to the cycle Δ on $X \times X$, with $n = d$, $i = 0$ and $D = X$. This gives us the $\mathbb{Q}$-cycles γ_0 and γ^1, a codimension one closed subset D^1 and a dimension one closed subset Γ_1 with γ_0 supported on $\Gamma_1 \times X$, γ^1 supported on $X \times D^1$ and with $\Delta = \gamma_1 + \gamma^1$ in $\mathrm{CH}^d(X \times X)_{\mathbb{Q}}$. This proves the case $s = 0$. The general case follows by induction on s, applying Lemma 1.1 to the cycle γ^{s+1} supported on $X \times D^{s+1}$. $\square$

Note. We have systematically indexed our cycle groups by *codimension* rather than dimension for notational convenience. However, it seems instructive to view the hypotheses of Theorem 1.2 as requiring the injectivity of the rational cycle maps for cycles of *dimension* 0 to s.

§2. Surjectivity

In this section, we use the decomposition of the diagonal given in §1 to study the surjectivity of the cycle map.

Let X be a smooth projective variety over $\mathbb{C}$ of dimension d. Let γ be in $\mathrm{CH}^d(X \times X)_{\mathbb{Q}}$, supported on a product $\Gamma \times D$, with $\Gamma \subset X$ of pure dimension j, $D \subset X$ of pure codimension i. Let $p: \tilde{\Gamma} \to \Gamma$, $q: \tilde{D} \to D$ be birational maps, with $\tilde{\Gamma}$ and $\tilde{D}$ smooth and projective. If Z is a subvariety of $\Gamma \times D$, then there is a subvariety W of $\tilde{\Gamma} \times \tilde{D}$, with $(p \times q)(W) = Z$, and with W generically finite over Z. In particular, there is a cycle $\tilde{\gamma} \in \mathrm{CH}^{j-i}(\tilde{\Gamma} \times \tilde{D})_{\mathbb{Q}}$ with $(p \times q)_*(\tilde{\gamma}) = \gamma$.

The cycle γ determines the homomorphisms

$$\gamma_*: H^a_{\mathcal{D}}(X, \mathbb{Q}(b)) \to H^a_{\mathcal{D}}(X, \mathbb{Q}(b))$$

by

$$\gamma_*(\eta) = p_{2*}(p_1^*(\eta) \cup cl^d(\gamma)), \quad \text{for } \eta \in H^a_{\mathcal{D}}(X, \mathbb{Q}(b)).$$

Let $f\colon \tilde{\Gamma} \to X$, $g\colon \tilde{D} \to X$ be the obvious maps, and let $p_{\tilde{D}}\colon \tilde{\Gamma} \times \tilde{D} \to \tilde{D}$, $p_{\tilde{\Gamma}}\colon \tilde{\Gamma} \times \tilde{D} \to \tilde{\Gamma}$ denote the projections. The cycle $\tilde{\gamma}$ determines homomorphisms $\tilde{\gamma}_*\colon H^a_{\mathcal{D}}(\tilde{\Gamma}, \mathbb{Q}(b)) \to H^{a-2i}_{\mathcal{D}}(\tilde{D}, \mathbb{Q}(b-i))$ by

$$\tilde{\gamma}_*(\eta) = p_{\tilde{D}*}(p_{\tilde{\Gamma}}^*(\eta) \cup cl^{j-i}(\gamma)), \quad \text{for } \eta \in H^a_{\mathcal{D}}(\Gamma, \mathbb{Q}(b)).$$

Lemma 2.1. *Let $\eta \in H^a_{\mathcal{D}}(X, \mathbb{Q}(b))$. Then*

$$\gamma_*(\eta) = f_*(\tilde{\gamma}_*(g^*(\eta))).$$

Proof. We have

$$\begin{aligned}
\gamma_*(\eta) &= p_{2*}(p_1^*(\eta) \cup cl^d(\gamma)) \\
&= p_{2*}(p_1^*(\eta) \cup cl^d((g \times f)_*(\tilde{\gamma}))) \\
&= p_{2*}(p_1^*(\eta) \cup (g \times f)_*(cl^{j-i}(\tilde{\gamma}))) \\
&= p_{2*}((g \times f)_*((g \times f)^*(p_1^*(\eta)) \cup cl^{j-i}(\tilde{\gamma}))) \quad \text{(projection formula)} \\
&= f_*(p_{\tilde{D}*}(p_{\tilde{\Gamma}}^*(g^*(\eta)) \cup cl^{j-i}(\tilde{\gamma}))) \\
&= f_*(\tilde{\gamma}_*(g^*(\eta))).
\end{aligned}$$

$\square$

The Deligne cohomology groups $H^0_{\mathcal{D}}$ and $H^1_{\mathcal{D}}$ of a point $*$ are easily computed; we give here a partial computation:

For $k \geq 0$, we have

$$H^0_{\mathcal{D}}(*, \mathbb{Q}(-k)) = \mathbb{Q}(-k)$$
$$H^1_{\mathcal{D}}(*, \mathbb{Q}(1+k)) = \mathbb{C}/\mathbb{Q}(k)$$

Let $p_X\colon X \to *$ be the projection to a point. Using the cycle class map cl^n, we obtain the maps

$$cl^n_{0,-k}\colon \mathrm{CH}^n(X) \otimes H^0_{\mathcal{D}}(*, \mathbb{Q}(-k)) \to H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n-k))$$
$$cl^n_{1,k}\colon \mathrm{CH}^n(X) \otimes H^1_{\mathcal{D}}(*, \mathbb{Q}(1+k)) \to H^{2n+1}_{\mathcal{D}}(X, \mathbb{Q}(n+1+k)),$$

defined by

$$cl^n_{0,-k}(\eta \otimes \alpha) = cl^n(\eta) \cup p_X^*(\alpha)$$
$$cl^n_{1,k}(\eta \otimes \beta) = cl^n(\eta) \cup p_X^*(\beta),$$

for $\alpha \in H^0_{\mathcal{D}}(*, \mathbb{Q}(-k))$, $\beta \in H^1_{\mathcal{D}}(X, \mathbb{Q}(1+k))$ and $\eta \in \mathrm{CH}^n(X)$.

Lemma 2.2. *Let Y be a smooth irreducible projective variety over $\mathbb{C}$ of dimension d_Y. Then, for $k \geq 0$, we have*

$$H^0_{\mathcal{D}}(Y, \mathbb{Q}(-k)) = \mathbb{Q}(-k)$$
$$H^1_{\mathcal{D}}(Y, \mathbb{Q}(1+k)) = \mathbb{C}/\mathbb{Q}(1+k).$$

The map

$$cl^{d_Y}_{0,0}\colon CH^{d_Y}(Y) \otimes H^0_{\mathcal{D}}(*, \mathbb{Q}(0)) \to H^{2d_Y}_{\mathcal{D}}(Y, \mathbb{Q}(d_Y))$$

is surjective. If $\iota : * \to Y$ is a point of Y, the maps

$$\iota_*\colon H^0_{\mathcal{D}}(*, \mathbb{Q}(-k)) \to H^{2d_Y}_{\mathcal{D}}(Y, \mathbb{Q}(d_Y - k)), \quad k > 0$$

and

$$\iota_*\colon H^1_{\mathcal{D}}(*, \mathbb{Q}(1+k)) \to H^{2d_Y+1}_{\mathcal{D}}(Y, \mathbb{Q}(d_Y+1+k)), \quad k \geq 0$$

are isomorphisms.

Proof. The computation of $H^0_{\mathcal{D}}$ and $H^1_{\mathcal{D}}$ follow directly from the isomorphism

$$H^0_{\mathcal{D}}(Y, \mathbb{Q}(-k)) \to H^0(Y, \mathbb{Q}(-k)) \cap F^{-k}H^0(Y, \mathbb{C})$$

and the short exact sequence

$$0 \to \frac{H^0(Y,\mathbb{C})}{H^0(Y,\mathbb{Q}(1+k)) + F^{1+k}H^0(Y,\mathbb{C})}$$

$$\to H^1_{\mathcal{D}}(Y,\mathbb{Q}(1+k)) \to H^1(Y,\mathbb{Q}(1+k)) \cap F^{1+k}H^1(Y,\mathbb{C}) \to 0,$$

together with the identities (for $k \geq 0$)

$$\begin{aligned} F^{-k}H^0(Y,\mathbb{C}) &= H^0(Y,\mathbb{C}) \\ F^{1+k}H^0(Y,\mathbb{C}) &= 0 \\ F^{1+k}H^1(Y,\mathbb{C}) &= 0. \end{aligned}$$

For the surjectivity statement, we have the exact sequence

$$0 \to \frac{H^{2d_Y-1}(Y,\mathbb{C})}{H^{2d_Y-1}(Y,\mathbb{Z}(d_Y-k)) + F^{d_Y-k}H^{2d_Y-1}(Y,\mathbb{C})} \to H^{2d_Y}_{\mathcal{D}}(Y,\mathbb{Z}(d_Y-k))$$

$$\to H^{2d_Y}(Y,\mathbb{Z}(d_Y-k)) \cap F^{d_Y-k}H^{2d_Y}(Y,\mathbb{C}) \to 0.$$

For $k=0$, this is just the exact sequence

$$0 \to \mathrm{Alb}(Y) \to H^{2d_Y}_{\mathcal{D}}(Y,\mathbb{Z}(d_Y)) \to H^{2d_Y}(Y,\mathbb{Z}(d_Y)) \to 0;$$

and the cycle class map cl^{d_Y} breaks up into degree map to $H^{2d_Y}(Y,\mathbb{Z}(d_Y)) = \mathbb{Z}$ and the Albanese map $\alpha\colon \mathrm{CH}_0(Y)_0 \to \mathrm{Alb}(Y)$. As both these maps are surjective, $cl^{d_Y}_{0,0}$ is surjective as well. For $k<0$, we have

$$H^{2d_Y}_{\mathcal{D}}(X,\mathbb{Q}(d_Y-k)) = H^{2d_Y}(Y,\mathbb{Q}(d_Y-k)).$$

As this latter group is isomorphic to $\mathbb{Q}(-k)$, generated by the class of a point, the map ι_* is an isomorphism as claimed. The computation of the group $H^{2d_Y+1}_{\mathcal{D}}(X,\mathbb{Q}(d_Y+1+k))$ is similar. $\square$

Lemma 2.3. *Let X be a smooth projective variety over $\mathbb{C}$ of dimension d, let Γ be a closed subset of pure dimension $i+1$, D a closed subset of pure codimension i, and let $\gamma \in CH^d(X \times X)_{\mathbb{Q}}$ be a $\mathbb{Q}$-cycle supported on $\Gamma \times D$. Then, for all $n, k \geq 0$, $\gamma_*(H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n-k)))$ is contained in the image of $cl^n_{0,-k}$, and $\gamma_*(H^{2n+1}_{\mathcal{D}}(X, \mathbb{Q}(n+1+k)))$ is contained in the image of $cl^n_{1,k}$.*

Proof. As in the paragraph preceeding Lemma 2.1, we let $p\colon \tilde{\Gamma} \to \Gamma$, $q\colon \tilde{D} \to D$ be birational maps, with $\tilde{\Gamma}$ and $\tilde{D}$ smooth and projective. Let $g\colon \tilde{\Gamma} \to X$, $f\colon \tilde{D} \to X$ be the obvious maps, and let $\tilde{\gamma} \in \mathrm{CH}^1(\Gamma \times \tilde{D})_{\mathbb{Q}}$ be a $\mathbb{Q}$-cycle with $(g \times f)_*(\tilde{\gamma}) = \gamma$. By Lemma 2.1, we have

$$\gamma_*(\eta) = g_*(\tilde{\gamma}_*(f^*(\eta)))$$

for $\eta \in H^a_{\mathcal{D}}(X, \mathbb{Q}(b)))$. Also, the homomorphism $\tilde{\gamma}_* \circ g^*$ maps $H^a_{\mathcal{D}}(X, \mathbb{Q}(b)))$ to $H^{a-2i}_{\mathcal{D}}(\tilde{D}, \mathbb{Q}(b-i)))$, and g^* maps $H^a_{\mathcal{D}}(X, \mathbb{Q}(b)))$ to $H^a_{\mathcal{D}}(\tilde{\Gamma}, \mathbb{Q}(b)))$. Since $H^a_{\mathcal{D}}(\tilde{\Gamma}, \mathbb{Q}(b))) = 0$ for $a > 2i+3$, and $H^{a-2i}_{\mathcal{D}}(\tilde{D}, \mathbb{Q}(b-i))) = 0$ for $a < 2i$, we need only consider four cases:

(1) $a = 2n = 2i$, $b = n-k$;
(2) $a = 2n+1 = 2i+1$, $b = n+1+k$;
(3) $a = 2n = 2i+2$, $b = n-k$;
(4) $a = 2n+1 = 2i+3$, $b = n+1+k$.

For cases (1) and (2), it follows from Lemma 2.2 that $f_*(H^0_{\mathcal{D}}(\tilde{D}, \mathbb{Q}(-k))$ is in the image of $cl^i_{0,-k}$, and that $f_*(H^1_{\mathcal{D}}(\tilde{D}, \mathbb{Q}(1+k))$ is in the image of $cl^i_{1,k}$. For case (3) , it follows from Lemma 2.2 that $H^{2i+2}_{\mathcal{D}}(\tilde{\Gamma}, \mathbb{Q}(i+1-k))$ is generated by $cl^{i+1}_{0,-k}(\mathrm{CH}^{i+1}(\tilde{\Gamma}) \otimes H^0_{\mathcal{D}}(*, \mathbb{Q}(-k))$, i.e., by the classes of points of any dense Zariski open subset of $\tilde{\Gamma}$. If x is a point of $\tilde{\Gamma}$, let $\tilde{\gamma}_x$ be the divisor $p_{\tilde{D}*}(\tilde{\gamma} \cdot x \times \tilde{D})$, when the intersection $\tilde{\gamma} \cap x \times \tilde{D}$ has codimension one on $\tilde{\Gamma} \times D$. Then $\tilde{\gamma}_*(x)$ is the class in $H^2_{\mathcal{D}}(\tilde{D}, \mathbb{Q}(1)))$ of $\tilde{\gamma}_x$, when the latter is defined; using the projection formula, we see that

$$\tilde{\gamma}_*(H^{2i+2}_{\mathcal{D}}(\tilde{\Gamma}, \mathbb{Q}(i+1-k))) \subset cl^1_{0,-k}(\mathrm{CH}^1(\tilde{D}) \otimes H^0_{\mathcal{D}}(*, \mathbb{Q}(-k))).$$

Following $\tilde{\gamma}_*$ by f_*, and using the compatibility of cycle classes with proper pushforward, we see that

$$\gamma_*(H^a_{\mathcal{D}}(X, \mathbb{Q}(b))) \subset cl^{i+1}_{0,-k}(\mathrm{CH}^{i+1}(X) \otimes H^0_{\mathcal{D}}(*, \mathbb{Q}(-k))).$$

Case (4) is similar, and is left to the reader. $\square$

Lemma 2.4. *Let X be a smooth projective variety over $\mathbb{C}$ of dimension d, let D be a closed subset of pure codimension $s+1$, and let $\gamma \in CH^d(X \times X)_{\mathbb{Q}}$ be a $\mathbb{Q}$-cycle supported on $X \times D$. Then*

(i) $\gamma_*(H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n-k))) = \gamma_*(H^{2n+1}_{\mathcal{D}}(X, \mathbb{Q}(n+1+k))) = 0$, *for $n < s+1$, and for all $k \geq 0$.*

(ii) $\gamma_*(H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n-k)))$ *is contained in the image of $cl^n_{0,-k}$, and* $\gamma_*(H^{2n+1}_{\mathcal{D}}(X, \mathbb{Q}(n+1+k)))$ *is contained in the image of $cl^n_{1,k}$, for $n = s+1$, and for all $k \geq 0$.*

(iii) $\gamma_*(H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n)))$ *is contained in the image of $cl^n_{0,0}$, for $n = s+2$.*

Proof. The proofs of (i) and (ii) are similar to the argument in the proof of the preceeding lemma, and are left to the reader. For (iii), let $\tilde{D} \to D$ be a resolution of singularities, and let $f\colon \tilde{D} \to X$ be the obvious map. Arguing as in the preceeding lemma, we see that $\gamma_*(H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n)))$ is contained in $f_*(H^2_{\mathcal{D}}(\tilde{D}, \mathbb{Q}(1)))$. Since the cycle class map $cl^1\colon \mathrm{CH}^1(\tilde{D}) \to H^2_{\mathcal{D}}(\tilde{D}, \mathbb{Z}(1)))$ is an isomorphism, we find that $\gamma_*(H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n)))$ is contained $f_*(\mathrm{CH}^1(\tilde{D}))$, proving (iii). □

Theorem 2.5. *Let X be a smooth projective variety over $\mathbb{C}$ of dimension d. Suppose there is an integer s, with $0 \leq s \leq d-2$, such that the $\mathbb{Q}$-cycle class maps*

$$cl^n\colon CH^n(X)_{\mathbb{Q}} \to H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n))$$

are injective for $n = d, d-1, \ldots, d-s$. Then the maps

$$cl^n_{0,-k}\colon CH^n(X) \otimes H^0_{\mathcal{D}}(*, \mathbb{Q}(-k)) \to H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n-k))$$

and

$$cl^n_{1,k}\colon CH^n(X) \otimes H^1_{\mathcal{D}}(*, \mathbb{Q}(1+k)) \to H^{2n+1}_{\mathcal{D}}(X, \mathbb{Q}(n+1+k))$$

are surjective for $n = 0, \ldots, s+1$ and for all $k \geq 0$. The map

$$cl^n_{0,0}\colon CH^n(X) \otimes \mathbb{Q} \to H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n))$$

is surjective for $n = s+2$. In particular, if the $\mathbb{Q}$-cycle class maps cl^n are injective for all $n \geq 0$, then the maps $cl^n_{0,-k}$ and $cl^n_{1,k}$ are surjective for all $n \geq 0$ and for all $k \geq 0$.

Proof. This follows from Theorem 1.2, and Lemmas 2.3 and 2.4, noting the the map Δ_* is the identity. □

Corollary 2.6. *Let X be a smooth projective variety over $\mathbb{C}$ of dimension d. Suppose the $\mathbb{Q}$-cycle class maps*

$$cl^n\colon CH^n(X)_{\mathbb{Q}} \to H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n))$$

are injective for all n. Then the group $Hg^n(X) \otimes \mathbb{Q}$ of rational Hodge cycles of X is generated by the classes of algebraic cycles for all n.

Proof. The surjectivity of the rational cycle class map

$$\mathrm{CH}^n(X)_{\mathbb{Q}} \to Hg^n(X) \otimes \mathbb{Q}$$

follows directly from Theorem 2.5. □

Remark. We will show in the next section that the injectivity of the cycle maps implies that the intermediate Jacobians of X are generated by the classes of algebraic cycles which are algebraically equivalent to zero.

§3. Hodge numbers and the failure of injectivity of the cycle map

We proceed to examine some consequences of Theorem 1.2 for the Hodge numbers of a smooth projective variety, and derive a criterion for ensuring that the cycle class maps are *not* injective. This can be viewed as a generalization of the theorems of Mumford-Roitman ([M], [R]) on the non-representability of the group of zero cycles on smooth projective varieties with non-trivial holomorphic p-forms for $p > 1$. What is novel in this setting is that it is not clear *which* cycle group is contributing to the lack of injectivity, although there is an obvious question one can pose (see Question 1 below).

For a smooth projective variety X over $\mathbb{C}$, we let $H^{p,q}(X)$ denote (p, q)-component in the Hodge decomposition of $H^*(X, \mathbb{C})$, and let $h^{p,q}(X) = \dim_{\mathbb{C}}(H^{p,q}(X))$. Let $cl^{n,n}(\gamma)$ denote the cohomology class in $H^{n,n}(X)$ of $\gamma \in \mathrm{CH}^n(X)_{\mathbb{Q}}$. If Y and Z are smooth projective varieties over $\mathbb{C}$, with Z of dimension a, and if γ is in $\mathrm{CH}^b(Y \times Z)$, we have the homomorphism

$$\gamma_*\colon H^{p,q}(Y) \to H^{p+b-a,q+b-a}(Z)$$

defined by $\gamma_*(\eta) = p_{2*}(p_1^*(\eta) \cup cl^{b,b}(\gamma))$.

Lemma 3.1. *Let X, D and Γ be smooth projective varieties over $\mathbb{C}$, with maps $f\colon D \to X$, $g\colon \Gamma \to X$. Let $\tilde{\gamma}$ be in $CH^b(\Gamma \times D)$, and let $\gamma = (g \times f)_*(\tilde{\gamma})$. Then $\gamma_* = f_* \circ \tilde{\gamma}_* \circ g^*$.*

Proof. The proof is the same as the proof of Lemma 2.1. □

Let $CH^n(X)_{hom}$ denote the group of cycles homologous to zero, modulo rational equivalence, and let $CH^n(X)_{alg}$ denote the group of cycles algebraically equivalent to zero, modulo rational equivalence.

Theorem 3.2. *Let X be a smooth projective variety over $\mathbb{C}$ of dimension d. Suppose there is an integer s, $0 \le s \le d-2$ such that the $\mathbb{Q}$-cycle class maps*

$$cl^n\colon CH^n(X)_{\mathbb{Q}} \to H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n))$$

are injective for $n = d, d-1, \ldots, d-s$. Then the Hodge numbers $h^{p,q}(X)$ vanish if

(i) $p + q \le 2s + 2$ and $|p - q| > 1$,

or if

(ii) $p + q > 2s + 2$ and $p < s + 1$.

In particular, if the $\mathbb{Q}$-cycle class maps cl^n are injective for all $n \ge 0$, then the Hodge numbers $h^{p,q}(X)$ vanish if $|p - q| > 1$. In addition, the cycle class map cl^n induce a surjection

$$cl^n\colon CH^n(X)_{alg} \to J^n(X)$$

for $n \le s + 2$.

Proof. For (i), first suppose $p + q = 2n$ is even. By Theorem 2.5, the map

$$cl^n_{0,-k}\colon CH^n(X) \otimes H^0_{\mathcal{D}}(*, \mathbb{Q}(-k)) \to H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n-k))$$

is surjective for all $k \ge 0$. On the other hand, for $k = n$, we have

$$H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n-k)) = H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(0)) = H^{2n}(X, \mathbb{Q}),$$

and the map $cl^n_{0,-n}$ is the usual topological cycle class map to singular cohomology (after twisting by $\mathbb{Q}(-n)$). Since the topological cycle class map lands in $H^{n,n}(X)$, the surjectivity of $cl^n_{0,-n}$ forces the vanishing of the Hodge numbers $h^{p,q}(X)$ if $p \neq q$. This proves (i) for $p+q$ even.

For $p+q = 2n-1$ odd, consider the groups $CH^n(X)_{hom}$ and $CH^n(X)_{alg}$. As the difference of two cycles belonging to the same connected component of a family of cycles on X goes to zero in the quotient group

$$CH^n(X)_{hom}/CH^n(X)_{alg},$$

this latter group is generated by the connected components of the union of the Chow varieties of degree t cycles of codimension n on X, for varying t. In particular, $CH^n(X)_{hom}/CH^n(X)_{alg}$ is a countably generated group. On the other hand, $cl^n(CH^n(X)_{alg})$ is an abelian subvariety A of $J^n(X)$, with tangent space $T_0(A)$ contained in the the subspace $H^{n-1,n}(X)$ of $T_0(J^n(X))$. By Theorem 2.5, the restriction of cl^n to $CH^n(X)_{hom}$ gives a surjective map

$$CH^n(X)_{hom} \otimes \mathbb{Q} \to J^n(X) \otimes \mathbb{Q}.$$

Thus, the complex torus $J^n(X)/A$ is a countably generated group, which is impossible unless $J^n(X) = A$. But, as

$$T_0(J^n(X)) = H^{0,n}(X) \oplus H^{1,n-1}(X) \oplus \ldots \oplus H^{n-1,n}(X),$$

the Hodge numbers $h^{p,q}(X)$ vanish if $|p-q| > 1$, completing the proof of (i). The same argument, using the surjectivity of

$$cl^n \colon \mathrm{CH}^n(X)_{\mathbb{Q}} \to H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n))$$

for $n \leq s+2$, as given by Theorem 2.5, shows that

$$cl^n \colon CH^n(X)_{alg} \to J^n(X)$$

is surjective for $n \leq s+2$.

For (ii), we use the decomposition

$$\Delta = \gamma_0 + \ldots + \gamma_s + \gamma^{s+1}$$

of the diagonal Δ given by Theorem 1.2, with γ_i supported on $\Gamma_{i+1} \times D^i$. Take resolutions of singularities $\tilde{D}^i \to D^i$, $\tilde{\Gamma}_i \to \Gamma_i$, and let $g^i\colon \tilde{\Gamma}_i \to X$, $f^i\colon \tilde{D}^i \to X$ be the obvious maps. Take $\mathbb{Q}$-cycles $\tilde{\gamma}_i$ on $\tilde{\Gamma}_i \times D^{i-1}$ with $(g_i \times f^{i-1})_*(\tilde{\gamma}_i) = \gamma_i$. We note that $g_i^*(H^{p,q}(X)) = 0$ if $p+q > 2i$, for dimensional reasons. Applying Lemma 3.1, we see that $\Delta_* = \gamma_*^{s+1}$ as endomorphisms of $H^{p,q}(X)$, for $p+q > 2s+2$. Let $D = D^{s+1}$, let $\tilde{D} \to D$ be a resolution of singularities of D, and let $f\colon \tilde{D} \to X$ be the obvious map. Take a $\mathbb{Q}$-cycle $\tilde{\gamma}$ on $X \times \tilde{D}$ such that $\gamma^{s+1} = (id_X \times f)_*(\tilde{\gamma})$; applying Lemma 3.1 again, we see that

$$H^{p,q}(X) = \Delta_*(H^{p,q}(X)) = \gamma_*^{s+1}(H^{p,q}(X)) \subset f_*(H^{p-s-1,q-s-1}(\tilde{D})),$$

the second equality being valid for $p+q > 2s+2$. In particular, we have $H^{p,q}(X) = 0$ if $p+q > 2s+2$ and $p < s+1$, proving (ii). □

Corollary 3.3. *Let X be a smooth projective variety over $\mathbb{C}$ of dimension d. Suppose that the $\mathbb{Q}$-cycle class maps*

$$cl^n\colon CH^n(X)_{\mathbb{Q}} \to H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n))$$

are injective for all n. Then the Hodge numbers $h^{p,q}(X)$ vanish if $|p-q| > 1$, and the cycle class maps

$$cl^n\colon CH^n(X)_{alg} \to J^n(X)$$

are surjective for all n.

Proof. This follows directly from Theorem 3.2. □

If we adjoin the identities $h^{p,q}(X) = h^{q,p}(X) = h^{d-p,d-q}(X)$ to the information supplied by Theorem 3.2, we obtain a nice picture of the Hodge diamond of X, assuming that the $\mathbb{Q}$-cycle maps cl^n are injective for $n =$

$d, d-1, \ldots, d-s$. Here the stars represent all the coordinates (p, q) where it is possible that $h^{p,q}(X) \neq 0$; in this example $d = 20$, $s = 5$.

Theorem 3.2, taken in the light of Bloch's conjecture that the zero-cycles on a smooth projective surface with $p_g = 0$ should be detected by the Albanese map, leads to the following:

Question 1. Let X be a smooth projective variety over $\mathbb{C}$ of dimension d. Suppose there is an integer $s \geq 0$ such that the Hodge numbers $h^{p,q}(X)$ vanish if

(i) $p+q \leq 2s+2$ and $|p-q| > 1$,

and if

(ii) $p+q > 2s+2$ and $p < s+1$.

Then are the cycle class maps

$$cl^p\colon \mathrm{CH}^p(X) \to H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p))$$

injective for $p = d, d-1, \ldots, d-s$? If not, are at least the $\mathbb{Q}$-cycle class maps

$$cl^p\colon \mathrm{CH}^p(X) \otimes \mathbb{Q} \to H^{2p}_{\mathcal{D}}(X, \mathbb{Q}(p))$$

injective for $p = d, d-1, \ldots, d-s$?

In light of the proof of Theorem 3.2, it might be better to replace (ii) with

(ii)′ There are smooth projective varieties $Y_1, \ldots, Y_s$ of dimension $d_X - s - 1$ and morphisms $Y_i \to X$ inducing a surjection of $\mathbb{Q}$-Hodge structures

$$\oplus_i H^*(Y_i, \mathbb{C}) \otimes \mathbb{Q}(-s-1) \to \oplus_{n=2s+2}^{2d_X} H^n(X, \mathbb{C}),$$

or even

(ii)″ For each $n > 2s+2$, there is a pure $\mathbb{Q}$- motive (i.e. a compatible collection of Galois representations, together with Hodge and Betti realizations, in the sense of Deligne [D] and Jannsen [J2]) M_n of weight $n-2s-2$ and an isomorphism of $\mathbb{Q}$-motives $M_n \otimes \mathbb{Q}(-s-1) \to H^n(X)$. □

As far as we know, the integral question is unsettled even for torsion cycles, except for zero-cycles (Roitman [R2], Bloch [Bl]) and for codimension two cycles (Murre [M]).

In any case, the contrapositive of Theorem 3.2 gives a criterion for the *failure* of the injectivity of the cycle map.

Corollary 3.4. *Let X be a smooth projective variety over $\mathbb{C}$ of dimension d. Suppose there is an integer s, $0 \leq s \leq d-2$, such that some Hodge number $h^{p,q}(X)$ is non-zero, with*

(i) $p+q \leq 2s+2$ *and* $|p-q| > 1$,

or with

(ii) $p+q > 2s+2$ *and* $p < s+1$.

Then there is an integer n, $d-s \leq n \leq d$ such that the $\mathbb{Q}$-cycle class map

$$cl^n \colon CH^n(X)_{\mathbb{Q}} \to H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n))$$

is not injective. □

Nori [N] has given examples of projective varieties with $\mathrm{CH}^n(X)_h \otimes \mathbb{Q} \neq 0$, but with $J^n(X) = 0$ as *generic* complete intersections of sufficiently high degree in certain smooth quadrics. It would be interesting to check the Hodge numbers of these varieties, to see if similar non-injectivity results could be obtained by applying Corollary 3.4. With reference to Question 1, one could ask if the minimal s satisfying the conditions of Corollary 3.4 points to precisely the cycle group of highest codimension for which the cycle class map fails to be injective, i.e.,

Question 2. Let X be a smooth projective variety over $\mathbb{C}$ of dimension d. Let s be the minimal integer such that some Hodge number $h^{p,q}(X)$ is non-zero, with

(i) $p+q \leq 2s+2$ and $|p-q| > 1$,

or with

(ii) $p+q > 2s+2$ and $p < s+1$

(supposing such an s exists). Then does the $\mathbb{Q}$-cycle class map

$$cl^n \colon \mathrm{CH}^n(X)_{\mathbb{Q}} \to H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n))$$

have a non-trivial kernel for $n = d-s$? □

§4. Relations with K-theory

The injectivity of the cycle maps, and the ensuing decomposition of the diagonal given by Theorem 1.2, have consequences for higher K-theory, most notably K_1, although one can say something about the other K-groups as well. This leads to a generalization of a result of Coombes and Srinivas [CS], who showed that the map

$$\mathrm{CH}^1(X) \otimes K_1(\mathbb{C}) \to H^1(X, \mathcal{K}_2)$$

is surjective, assuming that the group of zero-cycles modulo rational equivalence on X is representable.

Using the Gersten resolution (see [Q]) of the K-sheaves $\mathcal{K}_p$ on a smooth variety X over a field k, one arrives at the exact sequence

$$0 \to H^0(X, \mathcal{K}_p) \to K_p(k(X)) \to \oplus_{x \in X^{(1)}} K_{p-1}(k(x)),$$

where $X^{(p)}$ is the set of codimension p points of X. In particular, the map $H^0(X, \mathcal{K}_p) \to K_p(k(X))$ is injective; thus, if $p: Y \to X$ is a proper birational map of smooth varieties, the maps

$$p_*: H^0(Y, \mathcal{K}_p) \to H^0(X, \mathcal{K}_p); \quad p^*: H^0(X, \mathcal{K}_p) \to H^0(Y, \mathcal{K}_p)$$

are inverse isomorphisms. If we require X to be smooth and projective, the group $H^0(X, \mathcal{K}_p)$ is thus a birational invariant (assuming resolution of singularities for varieties over k). In particular, we may define the group $K_p(X)^{gen}$ for X an arbitrary projective variety over $\mathbb{C}$ by setting $K_p(X)^{gen} = H^0(\tilde{X}, \mathcal{K}_p)$, where $\tilde{X} \to X$ is a resolution of singularities. We have

$$\begin{aligned} K_0(X)^{gen} &= \mathbb{Z}; \\ K_1(X)^{gen} &= \mathbb{C}^\times, \end{aligned}$$

for X an arbitrary projective variety over $\mathbb{C}$. The groups $K_p(X)^{gen}$ for $p > 1$ are more mysterious, and in general contain $K_p(\mathbb{C})$ as a proper summand.

The cup product in K-theory gives rise to the natural maps

$$\begin{aligned} K_0(X) \otimes K_q(\mathbb{C}) &\to K_q(X) \\ H^p(X, \mathcal{K}_p) \otimes K_q(X)^{gen} &\to H^p(X, \mathcal{K}_{p+q}), \end{aligned}$$

we call the image of these maps the *decomposable* part of $K_q(X)$ or of $H^p(X, \mathcal{K}_{p+q})$, respectively. There is a possibly larger subgroup of $H^p(X, \mathcal{K}_{p+q})$, which we now describe.

Let $\mathcal{Z}^p(X, q)$ be the group

$$\mathcal{Z}^p(X, q) = \bigoplus_{x \in X^{(p)}} K_q(\bar{x})^{gen},$$

where $\bar{x}$ is the closure of x in X. Via the Gersten resolution for $\mathcal{K}_{p+q}$, we have the natural map

$$\mathcal{Z}^p(X, q) \to H^p(X, \mathcal{K}_{p+q}).$$

We call the image of this map the *geometrically decomposable* part of $H^p(X, \mathcal{K}_{p+q})$. For $q = 0, 1$, the decomposable part and geometrically decomposable part of $H^p(X, \mathcal{K}_{p+q})$ agree; in general, the geometrically decomposable part contains the decomposable part. We extend the definition of the decomposable and geometrically decomposable parts to the rational versions $K_q(X)_{\mathbb{Q}}$ and $H^p(X, \mathcal{K}_{p+q})_{\mathbb{Q}}$ in the obvious way.

Theorem 4.1. *Let X be a smooth projective variety over $\mathbb{C}$ of dimension d. Suppose the $\mathbb{Q}$-cycle class maps*

$$cl^n \colon CH^n(X)_{\mathbb{Q}} \to H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n))$$

are injective for $n = d, d-1, \ldots, d-s$, for some integer s, $0 \le s \le d-2$. Then the groups $H^p(X, \mathcal{K}_{p+q})_{\mathbb{Q}}$ are geometrically decomposable for $0 \le p \le s+1$. In particular, the map

$$CH^p(X) \otimes \mathbb{C}^\times \otimes \mathbb{Q} \to H^p(X, \mathcal{K}_{p+1})_{\mathbb{Q}}$$

is surjective for $0 \leq p \leq s+1$.

Proof. The bi-graded ring $\oplus_{p,q} H^p(X, \mathcal{K}_q)_{\mathbb{Q}}$ satisfies the Bloch-Ogus axioms [BO] for a twisted duality theory; in particular, if γ is a codimension d cycle on $X \times X$, γ gives rise to the endomorphism $\gamma_*\colon H^p(X, \mathcal{K}_{p+q})_{\mathbb{Q}} \to H^p(X, \mathcal{K}_{p+q})_{\mathbb{Q}}$, and the obvious analog of Lemmas 2.1 and 3.1 hold. We apply Theorem 1.2, retaining the notation of that theorem. The vanishing of $H^p(Y, \mathcal{K}_{p+q})$ for $p > \dim(Y)$ and for $p < 0$, together with the decomposition of the diagonal

$$\Delta = \gamma_0 + \ldots + \gamma_s + \gamma^{s+1}$$

implies that, on $H^p(X, \mathcal{K}_{p+q})$,

$$\Delta_* = \begin{cases} \gamma_{p-1*} + \gamma_{p*}; & \text{if } 0 \leq p \leq s \\ \gamma_{s*} + \gamma_*^{s+1}; & \text{if } p = s+1 \end{cases}$$

For Y smooth of dimension d_Y, the map

$$\mathrm{CH}^{d_Y}(Y) \otimes K_q(\mathbb{C}) \to H^{d_Y}(Y, \mathcal{K}_{d_Y+q})$$

is surjective; arguing as in the proof of Lemma 2.3, we see that the image $\gamma_{p-1*}(H^p(X, \mathcal{K}_{p+q}))$ is in the decomposable part of $H^p(X, \mathcal{K}_{p+q})$. Similarly, the argument of Lemma 2.3 shows that $\gamma_{p*}(H^p(X, \mathcal{K}_{p+q}))$ is in the geometrically decomposable part of $H^p(X, \mathcal{K}_{p+q})$. Finally, arguing as in the proof of Lemma 2.4, we see that $\gamma_*^{s+1}(H^p(X, \mathcal{K}_{p+q}))$ is in the geometrically decomposable part of $H^p(X, \mathcal{K}_{p+q})$. This proves the theorem. □

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