§1 Higher Chow groups, motivic cohomology and Milnor K-theory

Recall: $K^M_*(F) = (F^\times)^{\otimes *} / \langle \{c \otimes 1-c \mid c \in F \setminus \{0,1\}\} \rangle$

Theorem 1.1 There is a natural isomorphism of graded rings

$$K^M_*(F) \cong \bigoplus_{n \geq 0} H^{n,n}(\operatorname{Spec} F, \mathbb{Z})$$

Definition Let $\mathcal{H}^{p,q}(A)$ be the Nisnevich sheafification of $\{U \mapsto H^{p,q}(U,A)\}$.

Corollary The assignment $F \mapsto K^M_*(F)$ extends to a sheaf of graded rings $\mathcal{K}^M_*$ on Sm_k s.t.

$$\mathcal{K}^M_* \xrightarrow{\sim} \bigoplus_{n \geq 0} \mathcal{H}^{n,n}(\mathbb{Z})$$

Remark Let $\mathcal{O}$ be a DVR with residue field f and quotient field F. There is a map $\partial : K^M_n(F) \to K^M_{n-1}(f)$ characterized by
- $\partial\{a_1, \dots, a_n\} = 0$ for $a_i \in \mathcal{O}^\times \subset F^\times$
- $\partial\{t, a_2, \dots, a_n\} = \{\bar{a}_2, \dots, \bar{a}_n\}$, $a_i \in \mathcal{O}^\times$

(t) = max ideal, $\bar{a}$ = image of a in $f^\times$

The original definition of $\mathcal{K}^M_*$ is

$$\mathcal{K}^M_{n/X} = \ker\left[\bigoplus_{x\in X^{(0)}} i_{x*} K^M_n(k(x)) \xrightarrow{\partial} \bigoplus_{x\in X^{(1)}} i_{x*} K^M_{n-1}(k(x))\right]$$

One can show that the two definitions agree.

The proof of Theorem 1 (Nesterenko-Suslin/Totaro) relies on an isomorphism

$$H^{p,q}(X,\mathbb{Z}) \cong \underset{\substack{\uparrow \\ \text{Bloch's higher Chow groups}}}{CH^q}(X, 2q-p)$$

Def: A face F of $\Delta^n = \operatorname{Spec}(k[t_0,\dots,t_n]/(\sum t_i - 1))$ is a closed subscheme of the form $V(t_{i_1},\dots,t_{i_r})$

- $z^q(X,n) \subset Z^q(X\times\Delta^n)$ is the subgroup freely generated by codim q integral closed $W \subset X\times\Delta^n$ s.t. $\operatorname{codim}_{X\times F} X\times F\cap W = q$ (or $\cap = \emptyset$) for all faces $F\subset\Delta^n$

- Let $\delta_i^n : \Delta^n \to \Delta^{n+1}$ be the inclusion $(t_0, \dots, t_n) \mapsto (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_n)$

Then $\delta_i^{n*} : z^q(X, n) \to z^q(X, n-1)$

is well-defined. Let

$$d_n = \sum (-1)^i \delta_i^{n*}$$

give the complex $(z^q(X, *), d)$

- $CH^q(X, p) := H_p(z^q(X, *))$

- For $X \in Sm_k$ let $z^q_X(*)_{Zar, Nis}$ = sheafification of $U \mapsto z^q(U, *)$

<u>Theorem</u> (Bloch)

The natural map $H_p(z^q(X, *)) \to \mathbb{H}^{-p}(X_?, z^q_X(-*))$

is an iso for $? = Zar, Nis$

<u>Theorem</u> (Friedlander-Suslin-Voevodsky)

There is a iso in $D^-_{Nis}(Sh^{Ab}(X))$

$$z^q_X(2q - *)_{Nis} \cong \mathbb{Z}(q)^*|X_{Nis}$$

inducing an iso $CH^q(X, 2q-p) \cong H^{p,q}(X, \mathbb{Z})$

Cor 1. $H^{2n+i}(X, \mathbb{Z}(n)) = 0$ for $i > 0$ 3. $H^p(X, \mathbb{Z}(q)) = 0$ $\forall p \in \mathbb{Z}, q < 0$

(Beilinson) 2. $\mathcal{H}^{n+i,n} = 0$ for $i > 0$

Pf 1. $H^{2n+i}(X, \mathbb{Z}(n)) \cong CH^n(X, -i) = 0$ for $i > 0$

2. $H^{n+i}(\mathrm{Spec}\, F, \mathbb{Z}(n)) = CH^n(\mathrm{Spec}\, F, n-i) = 0$

since $Z^n(F, n-i) \subset Z^n(\Delta_F^{n-i}) = 0$

But $\mathcal{H}^{n+i,n}$ is an NST

$\Rightarrow \mathcal{H}^{n+i,n}_{x \in X} \hookrightarrow H^{n+i,n}(F, \mathbb{Z}) = 0$

3. $H^p(X, \mathbb{Z}(q)) = \mathrm{Hom}_{DM_{gm}}(M(X), \mathbb{Z}(q)[p])$. Let $r = -q > 0$

Assume X sm

(doesn't use Cancel) $= \mathrm{Hom}(M(X) \otimes \mathbb{Z}(-q), \mathbb{Z}(0)[p])$

$\subseteq \bigcap_{j=1}^{r} \ker\left[\left(H^{p,0}(X \times (\mathbb{P}^1)^{\wedge r}, \mathbb{Z}\right) \xrightarrow{i_j^*} H^{p,0}(X \times (\mathbb{P}^1)^{r-1}, \mathbb{Z})\right)$

Recall $\mathbb{Z}(0)^{tr} \cong \mathbb{Z}$ $\Rightarrow H^{p,0}(X \times (\mathbb{P}^1)^{\wedge}, \mathbb{Z}) = \begin{cases} 0 & p \neq 0 \\ \mathbb{Z}^{\pi_0(X \times (\mathbb{P}^1)^r)} & p = 0 \end{cases}$ (X sm)

$\Rightarrow \bigcap \ker = 0$

Def $\mathbb{Z}(q)^{tr} = 0$ for $q < 0$ (recognizing (3))

Finally, we have the theorem of Nesterenko-Suslin/Totaro

<u>Theorem</u> (Nesterenko-Suslin/Totaro)

Sending $a_1, \dots, a_n \in (F^\times)^n$ to the point

$$\sum a_i \neq 1 \qquad \left(\frac{1}{1-\sum a_i}, \frac{-a_1}{1-\sum a_i}, \dots, \frac{-a_n}{1-\sum a_i}\right) \in \Delta^n_F$$

descends to a

$$K^M_n(F) \xrightarrow{\sim} CH^n(F,n)$$

This gives $K^M_n(F) \cong H^{n,n}(F,\mathbb{Z})$

which one checks is compatible with multiplication

§2 Lichtenbaum motivic cohomology (Details in Lecture 9)

Recall the complex of sheaves $\mathbb{Z}(q)^\bullet = C^{sing}_*(\mathbb{Z}^{tr}(q))^*$ on $Sm_{/k,\, Nis}$

with $\mathbb{H}^p(X_{Nis}, A\otimes\mathbb{Z}(q)) \cong H^{p,q}(X, A)$ A an ab. grp

Def $\mathbb{Z}_L(q) = \mathbb{Z}(q)_{et}$, $H^{p,q}_L(X,A) = \mathbb{H}^p(X_{et}, A\otimes\mathbb{Z}_L(q))$

"Lichtenbaum motivic cohomology" [we set $\mathbb{Z}(q) = \mathbb{Z}_L(q) = 0$ for $q<0$]

Remarks • we have the change of topology map

$$H^{p,q}(-,A) \to H^{p,q}_L(-,A)$$

• This is an iso for $A=\mathbb{Q}$ [true on fields by properties of H^*_{Gal}, then use injectivity for $\mathcal{F}\in HI(k)$]

• $H^{p,q}_L(-,\mathbb{Z}/n) \cong H^p_{et}(-,\mu_n^{\otimes q})$ for n prime to char k

$H^{p,q}_L(-,\mathbb{Q}/\mathbb{Z}_{(\ell)}) = H^p_{et}(-,\mathbb{Q}_\ell/\mathbb{Z}_\ell(q))$ for $\ell \neq$ char k

Def (H90) $n\in\mathbb{N}$, ℓ a prime. $H90(n,\ell)$ holds over k if

$\forall F\supset k$ we have $H^{n+1,n}_L(F,\mathbb{Z}_{(\ell)}) = 0$

Theorem 2.1 (see 2...) Suppose $H90(n, l)$ holds over k. Then $\forall F \supset k$,

1. $H^{n,n}(F, \mathbb{Q}/\mathbb{Z}_{(l)}) \to H_L^{n,n}(F, \mathbb{Q}/\mathbb{Z}_{(l)})$ is surjective

2. i) $H^{p,q}(F, \mathbb{Q}/\mathbb{Z}_{(l)}) \xrightarrow{\sim} H_L^{p,q}(F, \mathbb{Q}/\mathbb{Z}_{(l)})$, $\forall\, p \leq q \leq n$

   ii) $H^{p,q}(F, \mathbb{Z}_{(l)}) \xrightarrow{\sim} H_L^{p,q}(F, \mathbb{Z}_{(l)})$ for all $q \leq n$, $p \leq q+1$

3. $H^p(F, \frac{\mathbb{Z}}{l^m}(q)) \xrightarrow{\sim} H_{et}^p(F, \mu_{l^m}^{\otimes q})$ $\forall\, p \leq q \leq n$, $\forall\, m \geq 1$

4. The norm resi map

$$K_n^M(F)/l^m \to H^n_{Gal}(F, \mu_{l^m}^{\otimes n})$$

is an iso for all $m \geq 1$

Moreover, let $\alpha : Sm_{et} \to Sm_{k, Nis}$ be the change of topology map. Then $\mathbb{Z}_{(l)}(q) \to \tau_{\leq q+1} R\alpha_* \mathbb{Z}_{(l)}(q)$, $\frac{\mathbb{Z}}{l^m}(q) \to \tau_{\leq q} R\alpha_* \mu_{l^m}^{\otimes q}$ are q-iso. for all $q \leq n$, $m \geq 1$.

Pf (Sketch) (1) We have the commutative diagram with exact rows

$$\begin{array}{ccccccc}
\to & H^{n,n}(F, \mathbb{Q}) & \to & H^{n,n}(F, \mathbb{Q}/\mathbb{Z}_{(l)}) & \to & H^{n+1,n}(F, \mathbb{Z}_{(l)}) = 0 \\
 & \downarrow \wr & & \downarrow & & \downarrow \\
\to & H_L^{n,n}(F, \mathbb{Q}) & \to & H_L^{n,n}(F, \mathbb{Q}/\mathbb{Z}_{(l)}) & \to & H_L^{n+1,n}(F, \mathbb{Z}_{(l)})
\end{array}$$

(1) $\Downarrow$

(2) : in fact (1) $\Rightarrow$ (2) is a theorem of Suslin-Voevodsky in char 0, Gersten-Levine injectional

(3) : (2) $\Rightarrow$ (3) ; For (i), use the exact sequence

$$0 \to \mathbb{Z}/\ell^m \to \mathbb{Q}/\mathbb{Z}_{(\ell)} \xrightarrow{\times \ell^m} \mathbb{Q}/\mathbb{Z}_{(\ell)} \to 0$$

and $H^{p,q}(-, \mathbb{Z}/\ell^m) = H^p_{et}(-, \mu_{\ell^m}^{\otimes q})$

Similarly, use the exact sequence

$$0 \to \mathbb{Z}_{(\ell)} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}_{(\ell)} \to 0$$

and the iso $H^{p,q}(-, \mathbb{Q}) \cong H^{p,q}_L(-, \mathbb{Q})$ to show (ii)

(4) : (3) $\Rightarrow$ $H^{n,n}(F, \mathbb{Z}/\ell^m) \xrightarrow{\sim} H^n_{Gal}(F, \mu_{\ell^m}^{\otimes n})$

We have the Moore seq

$$0 \to H^{n,n}(F, \mathbb{Z})/\ell^m \to H^{n,n}(F, \mathbb{Z}/\ell^m) \to H^{n+1,n}(F, \mathbb{Z})_{\ell^m - 1} \to 0$$

But $H^{n+1,n}(F, \mathbb{Z}) = 0$ and $H^{n,n}(F, \mathbb{Z}) = K^M_n(F)$

Finally, by (3) we have for $q < n$, $p \leq q+1$ and for $x \in X \in Sm_k$ a generic point

$$\mathcal{H}^{p,q}(\mathbb{Z}_{(2)})_x = H^{p,q}(F, \mathbb{Z}_{(2)}) \xrightarrow{\sim} H_L^{p,q}(F, \mathbb{Z}_{(2)}) = \mathcal{H}_L^{p,q}(\mathbb{Z}_{(2)})_x$$

Both $\mathcal{H}^{p,q}(\mathbb{Z}_{(2)})$ & $\mathcal{H}_L^{p,q}(\mathbb{Z}_{(2)})$ are in $HI(k)$

so we have $\mathcal{H}^{p,q}(\mathbb{Z}_{(2)}) \xrightarrow{\sim} \mathcal{H}_L^{p,q}(\mathbb{Z}_{(2)})$ for $q \le n$, $p \le q+1$

$$\Rightarrow \quad \mathbb{Z}_{(2)}(q) \xrightarrow{\sim} \tau_{\le q+1} R\alpha_* \mathbb{Z}_{(2),L}(q)$$

The proof for the mod ℓ^m-complexes is the same

$+ \quad \mathbb{Z}/\ell^m{}_L(q) \cong \mu_{\ell^m}^{\otimes q}$.

Strategy (not difficult) to show $H_L^{n+1,n}(F, \mathbb{Z}_{(2)}) = 0$

If F has no odd degree extensions and $K_n^M(F)/2 = 0$

then $H_L^{n+1,n}(F, \mathbb{Z}_{(2)}) = 0$

lemma) (I) for $F \hookrightarrow F'$ odd degree extension $H^{n+1,n}(F, \mathbb{Z}_{(2)}) \hookrightarrow H^{n+1,n}(F', \mathbb{Z}_{(2)})$

so we can always replace a field F with its maximal odd extension

(II) For $\underline{a} = (a_1 \cdots a_n) \in (F^\times)^n$ we have the norm quadric $Q_{\underline{a}}$.

lemma) (hard) main point! $\{a_1 \cdots a_n\} \to 0$ in $K_n^M(F(Q_{\underline{a}}))/2$

$H_L^{n+1,n}(F, \mathbb{Z}_{(2)}) \hookrightarrow H_L^{n+1}(F(Q_{\underline{a}}), \mathbb{Z}_{(2)})$

By iterating (I) & (II) construct $F \hookrightarrow F_\infty$ with

$$H_L^{n+1,n}(F, \mathbb{Z}_{(2)}) \hookrightarrow H_L^{n+1,n}(F_\infty, \mathbb{Z}_{(2)}) = 0$$

§3 Pfister quadrics and norm quadrics

Recall for $a \in F$, $\langle\langle a\rangle\rangle = \langle 1\rangle + \langle -a\rangle$ (dim 2)

$\underline{a} = (a_1, \dots, a_n) \in F^{\times n}$, $\langle\langle a_1, \dots, a_n\rangle\rangle = \langle\langle a_1\rangle\rangle \cdot \dots \cdot \langle\langle a_n\rangle\rangle$

"n-fold Pfister form", dim $= 2^n$

$V(\langle\langle a_1, \dots a_n\rangle\rangle) = Q \subset \mathbb{P}^{2^n - 1}$ is a "Pfister quadric"

$Q_{\underline{a}} = V(\langle\langle a_1 \dots a_{n-1}\rangle\rangle + \langle -a_n\rangle) \subset \mathbb{P}^{2^{n-1}}$

is a "norm quadric"

Note for ψ a Pfister form $/F$ the set of "representable" $\{\psi(x) \neq 0 \mid x \in F^\times\} = D(\psi)$

is a subgroup of $F^\times$. So a solution $(x, 1)$ $\psi + \langle -b\rangle = 0$

is writing $b \in D(\psi)$ "b is a norm of ψ".

Ex $\psi = \langle\langle a\rangle\rangle$. $\psi(x,y) = x^2 - ay^2 = Nm_{F(\sqrt{a})/F}(x + y\sqrt{a})$

$\psi = \langle\langle a, b\rangle\rangle$ $\psi(x_0, x_1, x_2, x_3) = x_0^2 - ax_1^2 - bx_2^2 + abx_3^2$

$= Nm_{H_F(a,b)/F}(x_0 + x_1 i + x_2 j + x_3 k)$

$\psi = \langle\langle a, b, c\rangle\rangle$: Nm from octonions

§4 Statement of main theorems of Rost

Theorem 4.1 (Rost nilpotence theorem) Let Q be a smooth quadric over k (char $\neq 2$) and take $f \in \mathrm{End}_{\mathrm{Mot}_{CH}(k)}(m(Q))$. Suppose $f_{k^{sep}} = 0$.

Then f is nilpotent

[Cor $f_{k^{sep}}$ an iso $\Rightarrow$ f is an iso]

Theorem 4.2 (the Rost motive) Let $Q = Q_{\underline{a}}$ be a norm quadric ($Q_{\underline{a}} = V(\langle\langle a_1, \dots, a_{n-1}\rangle\rangle + \langle -a_n\rangle)$) over a field F. Let $\mathcal{L} = Q_{F^{sep}}$, $p \in Q(F^{sep})$.

Then there is an idempotent $r \in \mathrm{End}(m(Q))$ with $r_{F^{sep}} = p \times \mathcal{L} + \mathcal{L} \times p$

Moreover: Let $f : Q_{\underline{a}} \to \mathrm{Spec}\, F$ be the structure map, giving map $f_* : m(Q_{\underline{a}}) \to m(\mathrm{Spec}\, F) = \mathbb{Z}$

$f^* = f_*^{\vee} : \mathbb{Z}\{2^{n-1}-1\} \to m(Q_{\underline{a}})$ in $\mathrm{Mot}_{CH}(F)$

Let $M_{\underline{a}} = (m(Q_{\underline{a}}), \pi)$ and ψ^*, ψ_* the maps

$$(*) \qquad \mathbb{Z}\{2^{n-1}-1\} \xrightarrow{\psi^*} M_{\underline{a}} \xrightarrow{\psi_*} \mathbb{Z}$$

in $Mot_{CH}(F)$ induced by f^*, f_*. The

$$\begin{tikzcd}
\mathbb{Z}\{2^{n-1}-1\} \arrow[r, "f^*"] \arrow[dr, "\psi^*"'] & m(Q_{\underline{a}}) \arrow[r, "f_*"] \arrow[d, shift right] & m(\mathrm{Spec}\, k) = \mathbb{Z} \quad (**) \\
& M_{\underline{a}} \arrow[u, shift right] \arrow[ur, "\psi_*"'] &
\end{tikzcd}$$

Then if $L \supset F$ is a field such that $Q_{\underline{a}}(L) \neq \emptyset$ the sequence $(*)$ is split exact in $Mot_{CH}(L)$

Theorem 4.3 (Rost injectivity theorem) Let $Q = Q_{\underline{a}}$, $\underline{a} = (a_1, \dots, a_n)$, be a norm quadric over a field F. Then the pushforward map

$$p_* : H^{2^{n-1}-1}(Q_{\underline{a}}, \mathcal{K}^M_{2^{n-1}}) \to k^\times$$

is injective.

Remark We rephrase Remark 4.3 in terms of motivic cohomology via

Proposition Let $X \xrightarrow{p} \mathrm{Spec}\, F$ be smooth projective of dim n. Then there is a natural isomorphism

$$H^n(X, K^M_{n+1}) \cong H^{2n+1, n+1}(X, \mathbb{Z})$$

transforming the pushforward $p_*: H^n(X, K^M_{n+1}) \to F^\times$ to the Gysin pushforward $p_*: H^{2n+1,n+1}(X,\mathbb{Z}) \to H^{1,1}(F,\mathbb{Z}) \cong F^\times$

Pf Use the local-global spectral sequence

$$E_2^{p,q} = H^p(X_{Nis}, \mathcal{H}^{q,n+1}(\mathbb{Z})) \Rightarrow H^{p+q, n+1}(X, \mathbb{Z})$$

We have $E_2^{p,q} = 0$ if $q > n+1$ ($\mathcal{H}^{n+1+i, n+1} = 0 \ \forall i > 0$) or if $p > n = \dim X$. This gives the iso

$$H^n(X_{Nis}, \mathcal{H}^{n+1,n+1}(\mathbb{Z})) = E_2^{n,n+1} \cong H^{2n+1,n+1}(X,\mathbb{Z})$$

$$\wr\wr$$

$$H^n(X_{Nis}, K^M_{n+1})$$

In the next section we outline a proof of Theorem 4.2.

§5 Working in $Mot_{CH}(k)$ [Write $M\{i\}$ for $M\otimes \mathbb{L}^{\otimes i}$, $\mathbb{Z}\{i\}$ for $\mathbb{L}^{\otimes i}$]

__Recall__ $Hom_{Mot_{CH}(k)}(m(X), m(Y)) = CH_{\dim X}(X\times Y)$

__Lemma 5.1__
For $r \geq 0$, $Hom_{Mot_{CH}(k)}(\mathbb{Z}\{r\}, m(X)) \cong CH_r(X)$

and $Hom_{Mot_{CH}(k)}(m(X), \mathbb{Z}\{r\}) \cong CH^r(X)$

compatible with W_*, W^* for $W \in Cor_{CH}(X,Y)$

__Cor__ For $r \in \mathbb{Z}$, $Hom_{Mot_{CH}(k)}(m(X)\{r\}, m(Y))$
$= CH_{\dim X + r}(X\times Y)$

compatible with $W \in Cor_{CH}(Y,Z)$, $W' \in Cor_{CH}(Z,X)$

Pf This follows from Lemma using duality

__Cor__ $Hom(\mathbb{Z}\{i\}, \mathbb{Z}\{j\}) = \begin{cases} 0 & i \neq j \\ \mathbb{Z} & i = j \end{cases}$

Pf $Hom[\mathbb{Z}\{i\}, \mathbb{Z}\{j\}] = CH_{i-j}(\operatorname{Spec} k)$.

Proof of Lemma $\quad \mathbb{Z}\{r\} = \left((\mathbb{P}^1)^r, (\infty^r \times (\mathbb{P}^1)^r) \right)$

$$\Rightarrow \operatorname{Hom}_{Mot_{CH}(k)}\left(\mathbb{Z}\{r\}, m(X)\right)$$

$$= \left(\infty^r \times (\mathbb{P}^1)^r\right)^* \left(\operatorname{Hom}_{Mot_{CH}(k)}\left(m((\mathbb{P}^1)^r), m(X)\right)\right)$$

$$= \left(\infty^r \times (\mathbb{P}^1)^r\right)^* \left(CH_r\left((\mathbb{P}^1)^r \times X\right)\right)$$

Note that $\quad CH_n(\mathbb{P}^1 \times Y) \cong CH_n(Y) \oplus CH_{n-1}(Y)$

by $\quad (W, W') \mapsto \infty \times W + \mathbb{P}^1 \times W'$

so

$$CH_r\left((\mathbb{P}^1)^r \times X\right) = \bigoplus_{I \subset \{1, \dots, r\}} CH_{r-|I|}(X)$$

by $\quad a \in CH_{r-|I|}(X)_I \mapsto \left(\infty^{I^c} \times (\mathbb{P}^1)^I \times a\right)$

Moreover $\quad \left(\infty^r \times (\mathbb{P}^1)^r\right)^* \left(\infty^{I^c} \times \mathbb{P}^{1\,I} \times a\right) \cap (\mathbb{P}^1)^{\{1,\dots,r\}} \times X$

$$= p_{13*}\left((\mathbb{P}^1)^r \times (\infty^{I^c} \times \mathbb{P}^{1\,I} \times a) \cdot (\infty^r \times (\mathbb{P}^1)^r \times X) \right)$$

$$= \begin{cases} 0 & \text{if } I \neq \emptyset \\ \infty^r \times a, \ a \in CH_r(X), & \text{if } I = \emptyset \end{cases}$$

so $\quad \left(\infty^r \times (\mathbb{P}^1)^r\right)_* CH_r\left((\mathbb{P}^1)^r \times X\right) \cong CH_r(X).$

Similarly one computes

$$\mathrm{Hom}_{\mathrm{Mot}_{CH}(k)}(m(X), \mathbb{Z}\{r\}) = [\infty^{\wedge} \times (\mathbb{P}^1)^n]_* (CH_{\dim X}(X \times (\mathbb{P}^1)^r))$$
$$= \{ a \times (\mathbb{P}^1)^r \mid a \in CH_{\dim X - r}(X) \}$$
$$\cong CH^r(X).$$

Write $\varphi_b : \mathbb{Z}\{r\} \to m(X)$, $\varphi^a : m(X) \to \mathbb{Z}\{r\}$ for the maps corresp to $b \in CH_r(X)$, $a \in CH^r(X)$

Lemma 2 Take $X \in Sm_k^{pr}$, $a \in CH^r(X)$, $b \in CH_r(X)$

i. $a \times b \in CH_{\dim X}(X \times X)$ satisfies

$$(a \times b) \circ (a \times b) = \deg(a \cdot b) \cdot (a \times b) \quad (\deg : CH_0^{\dim X}(X) \to CH^0(k))$$

so $a \times b$ is an idempotent $\Leftrightarrow \deg(a \cdot b) = 1$.

ii. If $\deg(a \cdot b) = 1$, then the maps ψ^a, ψ_b defined by

$$\begin{tikzcd} \mathbb{Z}\{r\} \arrow[r, "\varphi_b"] \arrow[dr, "\psi_b"'] & m(X) \arrow[r, "\varphi^a"] \arrow[d, shift right] & \mathbb{Z}\{r\} \\ & m(X, a \times b) \arrow[u, shift right] \arrow[ur, "\psi^a"'] & \end{tikzcd}$$

are inverse isomorphisms

Pf An elementary computation

Example 1. Take $X \xrightarrow{f} \operatorname{Spec} k$, smooth, projective, irreducible, of dim n over k, take $a = [X] \in CH^0(X)$

$b = [X] \in CH_n(X)$. Then $\varphi^a = \varphi^{[X]} : m(X) \to \mathbb{Z}$ is p_* and $\varphi_b = \varphi_{[X]} : \mathbb{Z}\{n\} \to m(X)$ is $f^* := (f_*)^\vee$. Indeed $\varphi^{[X]}$ is given by $[X] \in CH_{\dim X}(X \times pt)$
$m(\operatorname{Spec} k)$

which is Γ_f, and $(f_*)^\vee$ is given by $[{}^t\Gamma_f] = [X]$ in $CH_{\dim X}(pt \times X)$, so $(p_*)^\vee = \varphi_{[X]}$

2. Suppose we have $p \in X(k)$, give idempotent $p \times X$ and $X \times p$. Note that

$$(p \times X) \circ (X \times p) = (X \times p) \circ (p \times X) = 0$$

so $p \times X + X \times p$ is also an idempotent

and $m(X, p \times X + X \times p) = m(X, p \times X) \oplus m(X, X \times p)$

Then we have commutative diagram

$$
\begin{tikzcd}
\mathbb{Z}\{\dim X\} \arrow[r, "f^*"] \arrow[dr, "\psi_p", "\sim"'] & m(X) \arrow[d, "\mathrm{proj}"] \\
 & m(X, p^*X)
\end{tikzcd}
\qquad
\begin{tikzcd}
m(X) \arrow[r, "f_*"] & \mathbb{Z} \\
m(X, X_*p) \arrow[u, leftrightarrow, "\mathrm{incl.}"] \arrow[ur, "\psi^p"', "\sim"] &
\end{tikzcd}
$$

and a split exact sequence

$$0 \to \mathbb{Z}\{\dim X\} \xrightarrow{\psi^*} m(X, p^*X + X_*p) \xrightarrow{\psi_*} \mathbb{Z} \to 0$$

where ψ^*, ψ_* are defined by

$$
\begin{tikzcd}
\mathbb{Z}\{\dim X\} \arrow[r, "f^*"] \arrow[dr, "\psi^*"'] & m(X) \arrow[r, "f_*"] \arrow[d, shift right, "\mathrm{proj}"'] & \mathbb{Z} \\
 & m(X, p^*X + X_*p) \arrow[u, shift right, "\mathrm{incl.}"'] \arrow[ur, "\psi_*"'] &
\end{tikzcd}
$$

§ Cellular varieties

$X \in Sch_k$ is *cellular* if $\exists$ filtration by closed subsets $\emptyset = X_{-1} \subset X_0 \subset \dots \subset X_n = X$, $\dim X_i = i$

s.t. $X_i \setminus X_{i-1} \cong \coprod_{j=1}^{m_i} \mathbb{A}^i$. Let $\overline{\mathbb{A}^i}^j$ be the closure of the j^{th} component

Prop Suppose X is smooth cellular. Then $M^{eff}(X)$ is Tate:

i) $$M^{eff}(X) \cong \bigoplus_{i=0}^{\dim X} \bigoplus_{j=1}^{m_i} \mathbb{Z}(i)[2i]$$

and $$CH_i(X) = \bigoplus_{j=1}^{m_i} \mathbb{Z} \cdot [\overline{\mathbb{A}^i}^j]$$

ii) if X is in addition projective then

$$m(X) = \bigoplus_{i=0}^{\dim X} \bigoplus_{j=0}^{m_i} \mathbb{Z}\{i\}$$

Cor Take X and Y, smooth, cellular

i. If X is cellular then $CH_*(X \times Y) \cong CH_*(X) \otimes CH_*(Y)$ by $W \otimes W' \mapsto W \times W'$

ii. if X and Y are cellular then so is $X \times Y$.

Ex 1. $M(\mathbb{P}^n) = \bigoplus_{i=0}^{n} \mathbb{Z}\{i\}$ $CH_i(\mathbb{P}^n) = [L_i]$ $i=0,\dots,n$

2. split quadric $Q^{2n+1} \subset \mathbb{P}^{2n+2}$ defined by $\sum_{i=0}^{n} X_i Y_i - Z^2 = 0$

or $Q^{2n} \subset \mathbb{P}^{2n+1}$ " $\sum_{i=0}^{n} X_i Y_i = 0$

$$m(Q^{2n+1}) = \bigoplus_{i=0}^{2n+1} \mathbb{Z}\{i\}, \quad m(Q^{2n}) = \bigoplus_{i=0}^{2n} \mathbb{Z}\{i\} \oplus \mathbb{Z}\{n\}$$

$$CH_i(Q^{2n+1}) = \begin{cases} \mathbb{Z}[L_i], & L_i \cong \mathbb{P}^i \text{ for } i=0,\dots,n \\ \mathbb{Z}[h^{2n+1-i}], & h: \text{hyperplane section}, \text{ for } i=n+1,\dots,2n+1 \end{cases}$$

and $h^{2n+1-i} = 2L_i \quad i=0,\dots,n$

$$CH_i(Q^{2n}) = \begin{cases} \mathbb{Z}[L_i] & i=0,\dots,n-1 \\ \mathbb{Z}[L_n] \oplus \mathbb{Z}[L_n'] & i=n \\ \mathbb{Z}[h^{2n-i}], & i=n+1,\dots,2n \end{cases}$$

with $h^{2n-i} = 2L_i$ for $i=0,\dots,n-1$, $h^n = L_n + L_n'$

for n odd: $L_n^{(2)} = 0 = L_n'^{(2)}$, $L_n \cdot L_n' = L_0$

for n even: $L_n^{(2)} = L_0 = L_n'^{(2)}$, $L_n \cdot L_n' = 0$

Finally, some facts about quadrics

- φ dim $n+2$ non-degenerate form$/_F$. $Q = V(\varphi)$ dimension n in $\mathbb{P}^{n+1}_F$. Call Q $\begin{pmatrix}\text{anisotropic}\\ \text{isotropic}\end{pmatrix}$ if $Q(F)\begin{pmatrix}=\emptyset\\ \neq\emptyset\end{pmatrix}$

- $Q(F) \neq \emptyset \iff \varphi \cong \mathbb{H} \perp \varphi'$

- Q is split $\iff$ $\varphi \cong \frac{n}{2}\cdot\mathbb{H}$ (n even)

  $\lambda\varphi \cong \frac{(n-1)}{2}\mathbb{H} + \langle -1\rangle$ (n odd)

  some $\lambda \in F^\times$

- If φ is a Pfister form then n is 2^k (even) and φ is hyperbolic $\iff$ Q is isotropic i.e. $\varphi \cong \left(\frac{n}{2}\right)\cdot\mathbb{H}$

- Springer Theorem: if $Q \xrightarrow{f} \operatorname{Spec} F$ is anisotropic then $f_*(CH_0(Q)) \subset CH_0(F) = \mathbb{Z}$ is $2\mathbb{Z}$

- Swan Theorem: $f_*: CH_0(Q) \to CH_0(F) = \mathbb{Z}$ is injective

- Cassels-Pfister subform Theorem (weak form) For $Q = V(\varphi)$, $Q' = V(\psi)$. Suppose Q' is anisotropic and $\psi_{F(Q)}$ is hyperbolic. Then $\dim Q \leq \dim Q'$

We can now sketch the proof of Theorem 9.2, relying on Rost's nilpotence theorem (Theorem 9.1)

Recall the set-up: we start with $\underline{a} = (a_1, \dots, a_n) \in F^{\times n}$

Let $\;= \langle\langle a_1, \dots, a_{n-1}\rangle\rangle \perp \langle -a_n \rangle$, $Q = Q_{\underline{a}} = V(\varphi)$

Let F^{sep} be the separable closure of F, take $p \in Q(F^{sep})$

Then we want to construct an idempotent r on $M(Q)$

s.t. $r_{sep} = p \times Q + Q \times p$. Call such an r a "Rost projector" for Q

$CH_0(Q_{F^{sep}}) = \mathbb{Z} \Rightarrow$ this is independent of the choice of p

<u>Lemma</u> If Q is isotropic over F, choose $q \in Q(F)$. Then we prove this later!!

$q \times Q + Q \times q$ is the unique Rost projector on Q

Pf $Q = V(\varphi)$, $q \in Q(F)$ gives the decomposition $\varphi = \mathbb{H} \perp \varphi'$

and $M(Q) = \mathbb{Z} \oplus M(Q')\{1\} \oplus \mathbb{Z}\{d\}$ $\quad d = \dim Q$

$\Rightarrow \operatorname{End}(M(Q)) = \mathbb{Z} \times \operatorname{End}(M(Q')) \times \mathbb{Z}$. If r is a Rost projector then $r = q \times Q + Q \times q + r'$ $\quad r' \in \operatorname{End}(X')$ $\quad r'_{\bar{F}} = 0$

By the nilpotence theorem $r'^N = 0$ some N. Since $r' \circ (q \times Q + Q \times q) = 0$ $= (q \times Q + Q \times q) \circ r'$ we have $r = r^N = (q \times Q + Q \times q)^N = q \times Q + Q \times q$

<u>Cor</u> Once we show the existence of a Rost projector r on $Q_{\underline{a}}$, we have $r_K = q \times Q_{\underline{a}K} + Q_{\underline{a}K} \times q$ for any $q \in Q(K)$, $K \supset F$.

By Ex 2, this verifies the last part of Th 4.2

That

$$0 \to \mathbb{Z}\{\alpha\} \xrightarrow{\varphi^*} M_a(Q_L) \xrightarrow{\varphi_*} \mathbb{Z} \to 0$$

is split exact if $Q(L) \neq \emptyset$

The construction of r is in 2 steps

<u>Step 1</u>. We show that $p \times Q_{F^{sep}} + Q_{F^{sep}} \times p \in CH_{\dim Q}(Q_{F^{sep}} \times Q_{F^{sep}})$ is defined over F: $\exists\ r_1 \in CH_{\dim Q}(Q \times Q)$ with

$$r_{1\,F^{sep}} = p \times Q_{F^{sep}} + Q_{F^{sep}} \times p$$

<u>Step 2</u> if $\alpha \in CH_{\dim Q}(Q \times Q)$ has $\alpha_{F^{sep}}$ an idempotent, then α^{l} is an idempotent for some $l > 0$

Then just take $r = r_1^{l}$ for this l.

Proof of (2) is very easy (using the nilpotence theorem)

$(\alpha^2 - \alpha)_{F^{sep}} = 0 \Rightarrow \exists\ L/F$ finite with

$(\alpha^2 - \alpha)_L = 0$. Rost nilpotence $\Rightarrow (\alpha^2 - \alpha)^N = 0$

for some $N > 0 \Rightarrow \{\alpha^i \mid i = 1, 2, \ldots\}$ is a finitely generated subgroup of $CH_d(Q \times Q)$, so there are only finitely many torsion elements in this group

Let $m = [L:F]$. Then $(\alpha^{\ell} - \alpha)_L = 0$

$\Rightarrow m(\alpha^{\ell} - \alpha) = 0 \Rightarrow \{\alpha^{\ell} - \alpha \mid \ell = 1, 2, \dots\}$ is finite

so $\exists$ $\ell' > \ell$ with $\alpha^{\ell} = \alpha^{\ell'}$ $\Rightarrow \alpha^{\ell+i} = \alpha^{\ell'+i}$ $\forall i$

so we have $\alpha^{\ell} = \alpha^{\ell'}$ for some $\ell > \ell'$ with

$\ell' - \ell \mid \ell = s(\ell' - \ell)$. Since $\alpha^{\ell} = \alpha^{\ell' + i(\ell' - \ell)}$ $\forall i$

we take $i = (s-1)$ $\alpha = \alpha^{\ell' + (s-1)(\ell' - \ell)}$ $\ell' + s(\ell'-\ell) - (\ell' - \ell)$

$= \alpha^{\ell' + \ell - \ell' + \ell} = \alpha^{2\ell}$

<u>Proof of (1)</u> The ideal to consider $\Psi = \langle\langle a_1, \dots, a_{n-1} \rangle\rangle$

and $Q' \subset Q$, $Q' = V(\Psi)$. Let $m = 2^{n-2} - 1$ so

$\dim Q' = 2m$, $\dim Q = 2m+1$, $Q' = Q \cap (X_{2m+2} = 0)$

Since $Q'(F(Q')) \neq \phi$ so Ψ is a Pfister form

$\Psi_{F(Q')} = (m+1)\mathbb{H}$ so then

(a) $Q_{F(Q')}$ is split

On the other hand $\dim Q' < \dim Q$ so by Cassels-Pfister $\psi_{k(Q)}$ is not hyperbolic $\underset{(\psi \text{ is Pfister})}{\Longrightarrow}$ $Q'_{k(Q)}$ is anisotropic

$\underset{\text{Springer}}{\Longrightarrow}$

(b) every 0-cycle on $Q'_{k(Q)}$ has even degree

We recall that a split quadric is cellular and we know generators and relations for CH_*.

With $\mathcal{Q}, \mathcal{Q}'$ the Q_{sep}, Q'_{sep} we consider

the cycle $p \times \mathcal{Q} + \mathcal{Q} \times p \in CH_{2n+1}(\mathcal{Q} \times \mathcal{Q})$

Since $\mathcal{Q}$ is cellular $CH_*(\mathcal{Q} \times \mathcal{Q}) = CH_*(\mathcal{Q}) \otimes_{\mathbb{Z}} CH_*(\mathcal{Q})$

is the free abelian group on classes $h^{2n+1-i} \otimes h^{2n+1-j}$

& the classes $h^i \otimes h^j$ are defined mod F. $\quad h^{2n+1-i} \otimes L_j = \frac{1}{2} h^{2n+1-i} \otimes h^{2n+1-j}$

Also the diagonal $\Delta_{\mathcal{Q}}$ is $\quad L_i \otimes h^{2n+1-j} = \frac{1}{2} h^{2n+1-i} \otimes h^{2n+1-j}$

$$\Delta_{\mathcal{Q}} = \frac{1}{2} \sum_{i=0}^{2n+1} h^i \otimes h^{2n+1-i} = L_0 \times h^0 + h^0 \times L_0 + \frac{1}{2} \sum_{i=1}^{2n} \sim \qquad L_0 \otimes L_j$$

So

$$p^*\mathcal{Q} + \mathcal{Q}^* p = \underset{\substack{\uparrow \\ \text{defined over } F}}{\lambda} - \frac{1}{2}\left(h\otimes h^{n+1} + h^{n+1}\otimes h\right)\underset{\substack{\uparrow \\ \text{defined over } F}}{\left(\sum_{i=1}^{n} h^{i-1}\otimes h^{n-i}\right)}$$

so need only show $\frac{1}{2}(\underset{\xi}{h\otimes h^{n+1}} + \underset{\xi'}{h^{n+1}\otimes h})$ is defined over F $\quad \pi = L_n$ on $\mathcal{Q}'$

Consider $\xi_{F^{sep}} = h\otimes L_n + L_n \otimes h = i_*(h\otimes \pi + L_n \otimes \mathcal{Q}')$

$$\begin{tikzcd}
CH^{n+2}(\mathcal{Q}\times\mathcal{Q}) & CH^{n+1}(\mathcal{Q}\times\mathcal{Q}') \arrow[l, "i_*"'] \arrow[r, "j^*"] & CH^{n+1}(\mathcal{Q}_{F(\mathcal{Q}')}) \ni L_{n\,F(\mathcal{Q}')} \\
 & CH^{n+1}(Q\times Q') \arrow[u, "\mathrm{res}_{F^{sep}/F}"] \arrow[r, "j^*"] & CH^{n+1}(Q_{F(Q')}) \arrow[u, "\mathrm{res}_{F^{sep}/F}"'] \arrow[u, bend right=60]
\end{tikzcd}$$

$\alpha \in CH^{n+1}(Q\times Q')$

by (c): splits $\Rightarrow L_{n\,F(Q')}$

so $\exists$ α with $j^*(\alpha) = L_n|_{F(Q')} = \frac{1}{2} h^{n+1}_{F(Q')}$

let $\tilde{\alpha} = \mathrm{res}(\alpha)$

Let η be the hyperplane class on $\mathcal{Q}'$

$\pi = L_n$ on $\mathcal{Q}'$

$CH^{n+1}(\overline{Q} \times \overline{Q}')$ is free on $h^i \otimes \eta^{n+1-i}$, $i = 0 \ldots n$

$\frac{1}{2}(\overline{Q} \otimes \eta^{n+1})$, $h \otimes \pi$

$\frac{1}{2}(h^{n+1} \otimes \overline{Q}')$

$L_n \otimes \overline{Q}'$

so write

$$\tilde{\alpha} = \sum_{i=0}^{n} a_i h^i \otimes \eta^{n+1-i} + \frac{b}{2}(h^{n+1} \otimes \overline{Q}') + \frac{a}{2}(\overline{Q} \otimes \eta^{n+1}) + c(h \otimes \pi)$$

and consider

$$\begin{tikzcd}
\tilde{\alpha} \in CH^{n+1}(\overline{Q} \times \overline{Q}') \arrow[r, "g^*"] & CH^{n+1}(\overline{Q}'_{F(\overline{Q})}) \\
\alpha \in CH^{n+1}(Q \times Q') \arrow[r, "g^*"] \arrow[u, "\mathrm{res}"] & CH^{n+1}(Q'_{F(Q)}) \arrow[u, "\mathrm{res}"]
\end{tikzcd}$$

Then $g^*(\tilde{\alpha}) = \frac{a}{2} \eta^{n+1}_{F(\overline{Q})} = \mathrm{res}(g^*(\alpha))$

$= a L_n$

$$\Rightarrow \deg_{F(Q)}(g^*(\alpha) \cdot \eta^n) = \deg_{F(\overline{Q})}(a L_n \cdot \eta^n) = a$$

But by (b) $Q'_{F(Q)}$ is anisotropic

$$\Rightarrow \quad a = 2a_0 \text{ and}$$

$$g^*(\tilde{\alpha}) = a_0 \, \eta^{n+1}_{F(Q)}$$

Also, since $g^*(\tilde{\alpha}) = \frac{1}{2} h^{n+1}_{F(Q')}$ we have $b = 1$

Now consider

$$(1d \times i)_*(\tilde{\alpha}) \in CH^{n+2}(Q \times Q) \qquad 1d \times 1_*(Q^2) = h$$

$$\left\{ \begin{array}{l} \sum^{n} a_i \, h^i \otimes h^{n+2-i} \\ + \; a_0 \, Q \otimes h^{n+2} \quad + \frac{1}{2} h^{n+1} \otimes h \quad + \frac{c}{2}(h \otimes h^{n+1}) \end{array} \right.$$

defined over F $\text{res}((1d \times i)_* \alpha)$

$$\Rightarrow \quad \frac{1}{2} h^{n+1} \otimes h + \frac{c}{2}(h \otimes h^{n+1}) \text{ is defined over } F$$

if $c = 2c_0 + 1$ then $\frac{c}{2}(h \otimes h^{n+1}) = c_0 (h \otimes h^{n+1}) + \frac{1}{2}(h \otimes h^{n+1})$

so $\frac{1}{2}(h^{n+1} \otimes h + h \otimes h^{n+1})$ is defined over F. If $c = 2c_0$ then

$\frac{1}{2}(h^{n+1} \otimes h)$ is defined over $F \Rightarrow \frac{1}{2}(h \otimes h^{n+1})$ is defined over F. $\square$

Theorem 4.1 (Rost nilpotence theorem) Let Q be a smooth quadric over k, $\mathrm{char}\, k \neq 2$, and consider $f \in \mathrm{End}_{\mathrm{Mot}_{CH}(k)}(m(Q))$. There is an integer N, depending only on $\dim Q$, such that, if $f_{k^{sep}} = 0$, then $f^N = 0$.

The proof relies on

Proposition Let B, X be smooth projective over k, $d = \dim_k B$, and consider $f \in \mathrm{End}_{\mathrm{Mot}_{CH}(k)}(m(X))$. Suppose $f_*(CH_r(X_b)) = 0$ $\forall\, b \in B$ and all r, $a \leq r \leq d+a$, some integer a. Then

$$f_*^{d+1}\left(\mathrm{Hom}_{\mathrm{Mot}_{CH}(k)}(m(B)\{a\}, m(X))\right) = 0$$

Idea of proof $\mathrm{Hom}(m(B)\{a\}, m(X)) = CH_{d+a}(B \times X)$

Take $\alpha \in CH_{d+a}(B \times X)$

Take b = generic point of B. We have the restriction map

$$CH_{d+a}(B \times X) \to CH_a(X_b) \ni f_*=0$$

Since $CH_a(X_b) = \operatorname{colim}_{U \subset B} CH_{a+b}(U \times X)$, $\exists\, U$ s.t.

$f_*(\alpha)|_{U \times X} = 0$. Let $B_1 = B \setminus U$ given localization, $\overset{i_1}{\subset} B$, $\dim B_1 = d-1$

$$CH_{d+a}(B_1 \times X) \xrightarrow{i_1 \times id} CH_{d+a}(B \times X) \to CH_{d+a}(U \times X) \to 0$$

so $f_*(\alpha) = (i_1 \times id)_*(\alpha_1 \in CH_{d+a}(B_1 \times X))$

(need an extension of f_* to the case of singular B)

Repeat: $f_*(\alpha_1) \to 0$ in $CH_{a+1}(b_1^j \times X)$ for each generic point b_1^j of B_1 $\Rightarrow$ $\exists\, U_1 \subset B_1$ containing each b_1^j s.t. $f_*(\alpha_1) \to 0$ in $CH_{a+d}(U_1 \times B_1)$

By induction we construct

$$\emptyset = B_{d+1} \subset B_d \subset B_{d-1} \subset \cdots \subset B_1 \subset B \qquad \dim B_i = d-i$$

$\alpha_i \in CH_{d+a}(B_i \times X)$ with $\alpha_i \to f_*^i(\alpha)$ in $CH_{d+a}(B \times X)$

$\Rightarrow \quad f_*^{d+1}(\alpha) = 0$

Let Q/k be a smooth quadric with a k-point p

Then writing $Q = V(q)$ q a quadratic poly, we find that $q(p) = 0 \Rightarrow q = q' \perp \mathbb{H}$

Let $Q' = V(q')$, $p = (0,1,0)$ $\mathbb{H}(x,y) = xy$

<u>Thm</u> $m(Q) = \mathbb{Z}\{d\} \oplus m(Q')\{1\} \oplus \mathbb{Z}$ $d = \dim Q$

Pf We work in $\mathrm{Mot}^{\mathrm{eff}}_{CH}(k) \hookrightarrow DM^{\mathrm{eff}}_{-}(k)$

and show $M^{\mathrm{eff}}(Q) = \mathbb{Z}(d)[2d] \oplus M^{\mathrm{eff}}(Q') \oplus \mathbb{Z}(0)$

we have written $q = q(x,y,z) = xy + q'(z)$

Let $Q_1 = V(x) \cap Q$. Then $Q \setminus Q_1 \xrightarrow{\sim} \mathbb{A}^d$ by

$$(x,y,z) \mapsto \frac{z}{x}$$

since $0 = xy + q'(z) \Rightarrow -\frac{y}{x} = \frac{1}{x^2} q'(z) = q'(z/x)$

Also $p \in Q_1$ and we map $\pi: Q_1 \setminus p \to Q'$ with fiber $\mathbb{P}^1 \setminus \{1\} \cong \mathbb{A}^1$

$$(0,y,z) \mapsto z$$

This gives us

(1) Gysin sequence (split)

$$M^{eff}(Q \setminus Q_1) \to M^{eff}(Q \setminus p) \to M^{eff}(Q_1 \setminus p)(1)[2] \to$$

$$\begin{array}{ccc} M^{eff}(\mathbb{A}^d) & \swarrow p_{Q \setminus p *} & \\ \wr\!\!\mid & & \\ \mathbb{Z}(0) = M^{eff}(pt) & & \end{array} \Rightarrow \begin{array}{c} M^{eff}(Q \setminus p) \\ \cong \\ \mathbb{Z}(0) \oplus M^{eff}(Q_1 \setminus p)(1)[2] \end{array}$$

(2) Gysin sequence (split)

$$M^{eff}(Q \setminus p) \to M^{eff}(Q) \to M^{eff}(p)(d)[2d]$$

$$p^* \qquad \mathbb{Z}(d)[2d]$$

$$\Rightarrow \quad M^{eff}(Q) \cong \mathbb{Z}(d)[2d] \oplus \mathbb{Z}(0) \oplus M^{eff}(Q_1 \setminus p)(1)[2]$$

(3) Map $Q_1 \setminus p \to Q'$ with fibre $\mathbb{P}^1 \setminus \{1\} \cong \mathbb{A}^1$

$(0, y, z) \to z \quad \Rightarrow M^{eff}(Q_1 \setminus p) \cong M^{eff}(Q')$

$(x, y, z) \in V(x) \cap V(q)$

$\Leftrightarrow x = 0,\ q'(z) = 0$

Thm (Rost nilpotence theorem) Let Q be a smooth quadric of dim d over k.

Then there is an integer $N(d)$ s.t. for $f \in \operatorname{End}(M(Q))$ (in $\mathrm{Mot}_{CH}(k)$) with $f_{\bar k} = 0$, $f^{N(d)} = 0$.

Pf (induction on d)

$d = 0 \Rightarrow Q^{(1)} = \operatorname{Spec} k \amalg \operatorname{Spec} k$ or $Q^{(2)} = \operatorname{Spec} k(\sqrt{a})$

(1) $\operatorname{End}(M(Q)) = \mathbb{Z} \oplus \mathbb{Z} = \operatorname{End}(M(Q_{\bar k})) \subseteq M_2(\mathbb{Z})$

(2) $\operatorname{End}(M(Q)) = CH_0(\operatorname{Spec}(k(\sqrt{a}) \otimes_k k(\sqrt{a})))$

$= CH_0(\operatorname{Spec}(k(\sqrt{a})) \amalg \operatorname{Spec} k(\sqrt{a}))$

$= \mathbb{Z} \oplus \mathbb{Z} \hookrightarrow M_2(\mathbb{Z})$ invariant under conj by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\operatorname{End}(M(Q_{\bar k})) = CH_0(\operatorname{Spec}(\bar k \otimes \bar k) \times \operatorname{Spec}(\bar k \times \bar k))$

$= M_2(\mathbb{Z})$

So take $N(0) = 1$

Case (a) $Q(k) \neq \emptyset$

For $d=1$, $Q' = \emptyset$ $\quad m(Q) = \mathbb{Z} \oplus \mathbb{Z}\{1\}$ $\Rightarrow$ $\mathrm{End}(m(Q))$

so take $N(1) = 1$ $\quad \mathbb{Z} \oplus \mathbb{Z} = \mathrm{End}(m(Q))_{k^{sep}}$

for $d \geq 2$, then $\quad m(Q) = \mathbb{Z} \oplus m(Q')\{1\} \oplus \mathbb{Z}\{d\}$

so $\mathrm{End}(m(Q)) = \mathbb{Z} \oplus \mathrm{End}(m(Q')) \oplus \mathbb{Z}$

$$\oplus \left[\begin{array}{c} \underset{\substack{\| \\ 0}}{CH_{-1}(Q')} \oplus \underset{\substack{\| \\ 0}}{CH^{d}(\mathrm{Spec}\,k)} \oplus \underset{\substack{\| \\ 0}}{CH^{-1}(Q')} \\ \oplus\, CH^{d-1}(Q') = 0 \end{array} \right]$$

$0 =$

$\oplus \underset{=0}{CH_{-d}(\mathrm{Spec}\,k)} \oplus \underset{=0}{CH_{d-1}(Q')}$ $\quad (\dim Q' = d-2)$

Now use induction:

$f = n \cdot \mathrm{id}_{\mathbb{Z}} + m \cdot \mathrm{id}_{\mathbb{Z}\{d\}} + f' \in \mathrm{End}(m(Q'))$

$f_{k^{sep}} = 0 \Rightarrow n = m = 0$

$f'_{k^{sep}} = 0$

$\Rightarrow f^{N(d-2)} = f'^{N(d-2)} = 0$

If $Q(k) = \emptyset$. $Q(k(x)) \neq \emptyset$ for all $x \in Q$

Take $f \in \mathrm{End}(Q \times Q) = CH_d(Q \times Q)$, with $f_{k^{sep}} = 0$

By case (a) $f_{k(x)}^{N(d-2)} = 0$ $\forall x \in Q$ (if $d<1$ then $f_{k(x)} = 0$)
$N(1) \leq 1$

By Prop 1 $\left(f^{N(d-2)}\right)^{d+1} = 0$

so can take $N(d) = (d+1)N(d-2)$

<u>Cor</u> Let $F \supset k$ be a field extension, Q as above, k perfect
$d = \dim Q$
$f \in \mathrm{End}(M(Q))$

1. If f_F is nilpotent then f is nilpotent
2. If f_F is an isomorphism, then f is an isomorphism

<u>Proof</u> If F is algebraic over k, $f_F = 0 \Rightarrow f_{\bar{k}} = 0$. If F is
(1) not algebraic $\exists$ finite type U over k s.t. f goes to
zero in $CH^d(Q \times Q \times U)$. But $U(\bar{k}) \neq \emptyset \Rightarrow f_{\bar{k}} = 0$

(2) Similar to (1), we may assume $F \subset \bar{k}$. Let $g = f_F^{-1}$
then $\exists$ K finite over k s.t. $g = (g_K)_F$ and $g_K = f_K^{-1}$

So, we may assume F is finite rank. Also Q splits over finite extension of k, so we may assume Q_F splits. Then $m(Q_F)$ is a sum of Lefschetz motives $\Rightarrow \operatorname{End}(m(Q_F)) = \begin{cases} \mathbb{Z}^{d+1} & d \text{ odd} \\ \mathbb{Z}^{d} \oplus M_2(\mathbb{Z}) & d \text{ even} \end{cases}$

f_F an iso $\Rightarrow$ characteristic poly of f_F is

$$\begin{cases} \prod_{1}^{d}(t\pm1)\ (t^2-mt\pm1) & d \text{ even} \\ \prod_{1}^{d+1}(t\pm1) & d \text{ odd} \end{cases}$$

This implies that $f_F^{-1} = P(f_F)$ for some $P \in \mathbb{Z}[T]$

let $g = P(f)$ and replace f with fg

Sufficient to show fg is an iso, ie. we may assume from the start that $f_F = id_{m(Q_F)}$

Then $(f - id_{m(Q)})_F = 0 \Rightarrow f - id_{m(Q)} = h$ is nilpotent and $f = id + h$ is thus an isomorphism.

Some words about Theorem 4.3

- There is a presentation of $H^n(X, K^M_{n+1})$ as a quotient of $\bigoplus_{x \in X_{(0)}} k(x)^\times$ modulo the subgroup generated by

$$i_{C*}\left(T\left(K_2^M(k(x))\right)\right) \quad k(x) = k(C) \quad C \subset X$$

an integral curve & $T : K_2(k(C)) \to \bigoplus_{x \in C^{(1)}} k(x)^\times$

the tame $\sum T_x$ symbol map

$$T_x\{f, g\} = (-1)^{(\mathrm{ord}_x f)(\mathrm{ord}_x g)} \left(\frac{g^{\mathrm{ord}_x f}}{f^{\mathrm{ord}_x g}}\right)(x)$$

One shows that the maps $Nm_{k(x)/k} : k(x)^\times \to k^\times$ descend to the map

$$p_* : H^n(X, K^M_{n+1}) \to k^\times$$

Now suppose $X = Q = V(q)$ for some quadratic over F form q on V. Let $n = \dim Q$

I. If Q is isotropic$_k$, it is not hard to show that for $p \in Q(k)$

$k(p)^\times \to H^n(Q, K^M_{n+1})$ is an iso and p_* is $id_{k(p)^\times}$

If Q is anisotropic Rost shows that (using k has no odd degree extension)

$H^n(Q, K^M_{n+1})$ is generated by $\bigoplus k(x)^\times$, $x \in Q_{(0)}$, $[k(x):k] = 2$

Let $\overline{H}^n$ = that subgrp

a. $\dim Q = 0$: trivial

b. $\dim Q = 1$ Then Q is the Severi-Brauer variety of a quaternion algebra $H(a,b)/k$, and

$H^1(Q, K_2) \cong K_1(H)$

$\Rightarrow \ker p_* : H^1(Q, K_2) \to k^\times = \ker Nrd : K_1(H) \to k^\times = SK_1(H)$

But (Wang's theorem) $SK_1(D) = 0$ for D a central division algebra of prime degree order

$\Rightarrow p_*$ is injective

Now for $x \in Q \Rightarrow k(x)$ is a splitting field of H

$\Rightarrow k(x) \supset F$: maximal subfield of H

$\Rightarrow k(x) \supset k(y)$, $y \in Q$, $[k(y):k] = 2$

Consider

$$\begin{tikzcd} Q & Q_{k(y)} \arrow[l, hook'] \arrow[l, bend right, "x"'] \\ k \arrow[u, dash] & k(y) \arrow[l, hook'] \arrow[u, dash] \end{tikzcd}$$

$\Rightarrow$ for $\alpha \in k(x)^*$,

$Nm_{k(x)/k}(\alpha) = Nm_{k(y)/k}(Nm_{k(x)/k(y)}(\alpha))$

$\Rightarrow [\alpha \in k(x)^\times, x] = [N_{k(x)/k(y)}\alpha, y]$ in $H^1(Q, K_2)$

c. $\dim Q = 2$. Let K/k be a quadratic extension and let $x \in Q_K$ have deg 2 $/K$ & $\alpha \in k(x)^\times$

Then $\exists$ plane sections $C, C' \subset Q$ and $y \in C_K$, $y' \in C'_K$ of deg 2 and $\beta \in k(y)^\times$, $\beta' \in k(y')^\times$ s.t.

$$N_{k(x)/K}\alpha = (N_{K(y)/K}\beta)\cdot(N_{K(y')/K}\beta')$$

Prop Let K/k be a quadratic extension, Q arbitrary. Then for $f: Q_K \to Q$ the projection, we have

$$f_*(\bar{H}^n(Q_K, K^M_{n+1})) \subset \bar{H}^n(Q, K^M_{n+1})$$

Pf for $\dim Q \leq 1$, this is clear. Given $x \in Q_K$ of $(K(x):K) = 2 \Rightarrow [k(f(x)):k] \leq 4$

so $f(x)$ is on a dim 2 quadric $Q' \subset Q$

so can assume $\dim Q = 2$

Then for $\alpha \in K(x)$ $\exists$ $y \in C$, $y' \in C' \subset Q$ s.t.

$\beta \in k(y)^*$, $\beta' \in k(x')^*$ with

$$Nm_{K(x)/k}(\alpha) = Nm_{K(y)/k}(\beta) \cdot Nm_{K(y')/k}(\beta')$$

Then $f: Q_K \to Q$

$f_C : C_K \to C$, $f_{C'} : C'_K \to C'$

$$f_{C*}(\{\beta, y\}) \in H^1(C, K_2) = \bar{H}^1(C, K_2)$$

$$f_{C'*}(\{\beta', y'\}) \in H^1(C, K_2) = \bar{H}^1(C', K_2)$$

$$\Rightarrow f_*(\{\alpha, x\}) \in \bar{H}^2(Q, K_3)$$

Thm Q smooth quadric of dim n/k

Then $\bar{H}^n(Q, K^M_{n+1}) = H^n(Q, K^M_{n+1})$

Pf We may assume k has no odd degree extension

let $K \supset k$ be a splitting field for Q. Then $K \supset k$ is a tower of deg 2 extension.

Also $\bar{H}^n(Q_K, K^M_{n+1}) = H^n(Q_K, K^M_{n+1})$ so going down the tower and using the Prop $\Rightarrow$ $\bar{H}^n(Q, K^M_{n+1}) = H^n(Q, K^M_{n+1})$

Now one needs to analyze the kernel of the surjection

$$(+)\quad k(x)^\times \to H^n(Q, K_{n+1}^n)$$

$x \in Q_{(0)}$

$[k(x):k]=2$

Suppose $Q = V(q)$ for a dim 4 form q

We have the spinor group $S\Gamma(q) \subset C_0(q)$ ← even Clifford alg

with $1 \to k^\times \to S\Gamma(q) \to SO(q) \to 1$

A point $x \in Q$ with $[k(x):k]=2 \iff q_x$: dim 2 subform of q

and $S\Gamma(q_0) \subseteq S\Gamma(q)$

$$\begin{array}{ccc} \cong & & \cup \\ k(x)^\times & & \\ \alpha & \mapsto & g_\alpha \end{array}$$

Root shows: Given (α,x), (β,y) as above $\exists!\, z \in Q$ with $\alpha\beta \in k(z)^\times$, $g_\alpha g_\beta = g_{\alpha\beta}$

Rost shows the kernel is exactly generated by elements of the form

R0) $[\alpha, x] - [\alpha', x']$ if $N_{k(x)/k}\,\alpha = N_{k(x')/k}\,\alpha'$

R1) Suppose $\{x,y\} \subset Q \cap \Pi$ some $\Pi \subset \mathbb{P}^{n+1}$, $\Pi \cong \mathbb{P}^3$

then $[\alpha,x] + [\beta,y] = [\alpha\beta, z]$ in $Q(q_0)$

Finally, Rost shows using this presentation of $H^n(Q, K_{n+1}^M)$

Theorem Let φ be a Pfister neighbor / k, $Q = V(\varphi)$ of dim

Then $p_*: H^n(Q, K_{n+1}^M) \to k^\times$ is injective