Pullbacks and intersection theory for the Rost-Schmid groups

We describe how Rost's construction of pull back maps for $A^*(-, M, *)$ can be modified to give pull back maps for $H^*(-, \underline{K}^{MW}_*\{L\})$.

1st a remark: last time we had defined the five basic maps: f^*, f_* (for proper f), ∂^D_U for $D \hookrightarrow Y \hookleftarrow U = Y \setminus D$, $\{u_1, \dots, u_r\}\cdot$ and $\eta\cdot$. These all define maps of R-S complexes; for f^*, f_*, $\cdot\eta$ without change, for ∂^D_U with $-d_D$, for $[u_1, \dots, u_r]$ with $\epsilon^r d$ in target. As these changes in the differential are all by multiplication by a unit, this does not change the cohomology, so we get maps on $H^*(-, K^{MW}_*\{-\})$ in all cases. On H, f^*, f_* are functorial and commute with $[u_1 \dots u_r]\cdot$, $\eta\cdot$,

$f_* \circ \partial^{f^{-1}D}_{f^{-1}U} = \partial^D_U \circ f_*$; $f^* \circ \partial^D_U = \partial^{f^{-1}D}_{f^{-1}U} \circ f^*$ for f smooth

§1 Specialization to a principal divisor.

We consider the following situation. Let $i : D \hookrightarrow Y$ be the inclusion of a smooth effective Cartier divisor on a smooth finite type k-scheme Y. To simplify the discussion, we assume that D is a principal divisor, and choose a global generator t for $\mathcal{I}_D \subset \mathcal{O}_Y$. Let $j : U \hookrightarrow Y$ be the open complement, $U = Y \setminus D$ and let L be an invertible sheaf on Y, & let $N = N_D Y$ ($\cong \mathcal{O}_D$)

As before we have the additive decomposition

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$$C^n(Y, \underline{K}^{MW}_j\{L\}) = C^n(U, \underline{K}^{MW}_j\{L\}) \oplus C^n_D(Y, \underline{K}^{MW}\{L\})$$

$$C^n(Y, \underline{K}^{MW}_j\{L\}) = \bigoplus_{y \in Y^{(n)}} K^{MW}_{j-n}(k(y); \nu^Y_y \otimes L)$$

$$C^n(U, \underline{K}^{MW}_j\{L\}) = \bigoplus_{u \in U^{(n)}} K^{MW}_{j-n}(k(u), \nu^U_u \otimes L), \quad \nu^U_u = \nu^Y_u$$

$$C^n_D(Y, \underline{K}^{MW}\{L\}) = \bigoplus_{z \in D^{(n-1)}} K^{MW}_{j-n}(k(z), \nu^Y_z \otimes L)$$

where
$$\left.\begin{aligned} \nu^Y_y &= \det\left(\left(\frac{m_y}{m_y^2}\right)^\vee\right), & m_y &\subset \mathcal{O}_{Y,y} \\ \nu^U_u &= \det\left(\left(\frac{m_u}{m_u^2}\right)^\vee\right), & m_u &\subset \mathcal{O}_{U,u} = \mathcal{O}_{Y,u} \\ \nu^D_z &= \det\left(\left(\frac{m^D_z}{(m^D_z)^2}\right)^\vee\right), & m^D_z &\subset \mathcal{O}_{D,z} \end{aligned}\right\} \text{ maximal ideals}$$

With respect to this decomposition, the differential
$$d_Y^{\{L\}} : C^n(Y, \underline{K}^{MW}_j\{L\}) \to C^{n+1}(Y, \underline{K}^{MW}_j\{L\})$$
is written as
$$d_Y^{\{L\}} = \begin{pmatrix} d^n_U & \delta^n_{D,U} \\ 0 & d^n_{D,Y} \end{pmatrix}$$
with $\delta^n_{D,U} : C^n(U, \underline{K}^{MW}_j\{L\}) \to C^{n+1}_D(Y, \underline{K}^{MW}_j\{L\})$

We have an isomorphism for $z \in D$ $\quad m^Y_z \subset \mathcal{O}_{Y,z}$ (3)

$$\det\left(m^D_z / m^{D\,2}_z\right) \otimes N^\vee \xrightarrow{\sim} \det\left(m^Y_z / m^{Y\,2}_z\right) \qquad m^D_z \subset \mathcal{O}_{D,z} = \mathcal{O}_{Y,z}/(t)$$

$(\bar{t}_1, \dots, \bar{t}_{n-1}) = m^D_z$

$$d\bar{t}_1 \wedge \dots \wedge d\bar{t}_{n-1} \otimes dt \mapsto dt_1 \wedge \dots \wedge dt_{n-1} \wedge dt \qquad \overset{\int}{(t_1 \dots t_{n-1})} \subset m^Y_z$$

giving on dual $\quad \det\left(m^Y_z / m^{Y\,2}_z\right) \xrightarrow{\sim} \det\left(m^D_z / m^{D}_z\right)^\vee \otimes N$

This induces an iso of complexes

$$C^*_D(Y, K^{MW}_{-j}\{L\}) \to C^{*-1}(D, K^{MW}_{-j-1}\{N \otimes L\})$$

and we now consider $\delta^n_{D,U}$ as map

$$\delta^n_{D,U_j} : C^n(U, K^{MW}_j\{L\}) \to C^n(D, K^{MW}_{-j-1}\{N \otimes L\})$$

<u>Note</u> the δ^n satisfy

$$\delta^n_{D,U} \circ d^{n-1}_U = -d^{n-2}_D \circ \delta^{n-1}_{D,U}$$

This follows from $\quad d_Y^2 = 0$: $\quad 0 = \begin{pmatrix} d^n_U & 0 \\ \delta^n_{D,U} & d^{n-1}_D \end{pmatrix} \begin{pmatrix} d^{n-1}_U & 0 \\ \delta^{n-1} & d^{n-2}_D \end{pmatrix}$

so $\delta^\vee_{D,U}$ is a map of complexes if we use $\quad -d^n_D$

Next we have $[t] \in Z^0(\mathcal{U}, K^{MW}_{-1})$

$(= \ker d^0_{\mathcal{U}})$

Define the map

$$sp^n_t : C^n(\mathcal{U}, K^{MW}_{-j}\{L\}) \to C^n(D, K^{MW}_{-j}\{N \otimes L\})$$

global generator $\frac{\partial}{\partial t}$ for N $\cong$ $C^n(D, K^{MW}_{-j}\{L\})$

by $sp^n_t\left(\alpha_u \otimes \frac{\partial}{\partial t_1} \wedge \dots \wedge \frac{\partial}{\partial t_n} \otimes \lambda\right)$ for $u \in \mathcal{U}$

$$\mapsto (-1)^n \cdot \partial^{n+1}_{D,N}\left([t]\alpha_u \otimes \frac{\partial}{\partial t_1} \wedge \dots \wedge \frac{\partial}{\partial t_n} \otimes \lambda\right)$$

One checks sp^n_t defines a map of complexes

$$sp_t : C^*(U, \underline{K}_j^{MW}\{L\}) \to C^*(D, \underline{K}_j^{MW}\{L\})$$

§2 Smooth pull back

Let $p : Y \to X$ be a smooth morphism in Sm_k of relative d

We have the smooth pull back $p^* : C^*(X, \underline{K}_j^{MW}\{L\})$

$$\to C^*(Y, \underline{K}_j^{MW}\{p^*L\})$$

for $x \in X^{(n)}$, $\alpha \in K_{j-n}^{MW}(k(x))$, λ gen for L_x

since p^* induces an iso $m_x/m_x^2 \otimes_{k(x)} k(y) \xrightarrow[p^*]{\sim} m_y/m_y^2$

we have $p^*(\alpha \otimes \theta \otimes \lambda) = p^*(\alpha) \otimes p^*\theta \otimes p^*\lambda$

$p^* : K_{j-n}^{MW}(k(x)) \to K_{j-n}^{MW}(k(y))$ induced by $p^* : k(x) \hookrightarrow k(y)$

§3 Homotopy for a coordinated vector bundle $V \xrightarrow[p]{} X$

we have $r_V : C^*(V, \underline{K}_j^{MW}\{p^*L\}) \to C^*(X, \underline{K}_j^{MW}\{L\})$

with $r_V p^* = id$, $p^* r_V \underset{h_V}{\sim} id$

§ Deformation to the normal bundle and pullback for a closed immersion in Sm_k

As before let $i: Z \hookrightarrow X$ be a closed immersion in Sm_k, giving the deformation diagram

$$\begin{tikzcd}
N_Z X \arrow[r, hook, "i"'] \arrow[d] & D(X,Z) \arrow[d] & X \times \mathbb{A}^1 \setminus \{0\} \arrow[l, hook', "j"'] \arrow[d] \arrow[r, "\pi"] & X \\
0 \arrow[r, hook] & \mathbb{A}^1 & \mathbb{A}^1 \setminus \{0\} \arrow[l, hook']
\end{tikzcd}$$

$$Z \overset{s_0}{\hookrightarrow} N_Z X, \qquad Z \overset{i}{\hookrightarrow} X, \qquad \mathbb{A}^1 = \mathrm{Spec}\, k[t]$$

Note that $N_Z X \hookrightarrow D(X,Z)$ is a smooth principal Cartier divisor on the smooth k-scheme $D(X,Z)$, defined by t

so we have the case with

$$\begin{tikzcd}
C^*(X, \underline{K}_j^{MW}\{L\}) \arrow[r, "\pi^*"] \arrow[ddr, "i^!"'] & C^*(X \times (\mathbb{A}^1 \setminus 0), \underline{K}_j^{MW}\{p^* L\}) \arrow[d, "sp_t"] \\
 & C^*(N_Z X, \underline{K}_j^{MW}\{q^* L\}) \arrow[d, "r_N"] \\
 & C^*(Z, \underline{K}_j^{MW}\{i^* L\})
\end{tikzcd}$$

§ <u>General pullback</u>

Let $f: Y \to X$ be a morphism in Sm_k. Factor f

as $Y \underset{i_Y}{\hookrightarrow} Y \times X \underset{p_2}{\to} X$ and define

$$f^*: C^*(X, \underline{K}_j^{MW}\{L\}) \to C^*(Y, \underline{K}_j^{MW}\{p^* L\})$$

as $\quad f^* := i_Y^! \circ p_2^*$

(8)

Theorem 1) for $Z \xrightarrow{g} Y \xrightarrow{f} X$ we have

$$g^* f^* \sim (fg)^*$$

2) for $f : Y \to X$ smooth, the general pullback f^* is homotopic to the smooth pull back f^*

3) Given a transverse cartesian sq in Sm_k

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "g'"'] & X' \arrow[d, "g"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

with g proper, we have

$$f^* g_* \sim g'_* f'^*$$

§ External products

This is also essentially the same as for cycle modules. Given $x \in X^{(n)}$, $y \in Y^{(m)}$ for $X, Y \in Sm_k$, there are finitely many generic points $z_1, \dots, z_r$ of $\overline{\{x\}} \times_k \overline{\{y\}} \subset X \times_k Y$ and for each i we have $p_1(z_i) = x$, $p_2(z_i) = y$ giving us the pullback $p_{1,i}^* : k(x) \hookrightarrow k(z_i)$, $p_{2,i}^* : k(y) \hookrightarrow k(z_i)$ and the induced map

$$K_a^{MW}(k(x)) \otimes_{\mathbb{Z}} K_b^{MW}(k(y)) \to \bigoplus_i K_{a+b}^{MW}(k(z_i))$$

$$\alpha \otimes \beta \mapsto \{ p_{1,i}^*(\alpha)\, p_{2,i}^*(\beta) \}$$

similarly we have the isomorphism

$$(\nu_x^X \otimes \omega_{k(x)}) \otimes (\nu_y^Y \otimes \omega_{k(y)}) \xrightarrow{\sim} \nu_{z_i}^{X \times Y} \otimes \omega_{k(z_i)}$$

$$\frac{\partial}{\partial t_1} \wedge \dots \wedge \frac{\partial}{\partial t_n} \otimes \alpha \otimes \frac{\partial}{\partial s_1} \wedge \dots \wedge \frac{\partial}{\partial s_m} \otimes \beta \mapsto \frac{\partial}{\partial t_1} \wedge \dots \wedge \frac{\partial}{\partial t_n} \wedge \frac{\partial}{\partial s_1} \wedge \dots \wedge \frac{\partial}{\partial s_m} \otimes \alpha \wedge \beta$$

Taking same for x, y gives

$$p_1^* \otimes p_2^* : C^{n}(X, \underset{\sim}{K}^{MW}_{j}\{L\}) \otimes C^{m}(Y, K^{MW}_{l}\{M\})$$
$$\to C^{*}(X \times Y, \underset{\sim}{K}^{MW}_{j+l}\{p_1^* L \otimes p_2^* M\})$$

This is associative and

where R $\tau : X \times Y \to Y \times X$ the switch

we write induced map $d\tau : v_z^{X \times Y} \xrightarrow{\sim} v_{\tau(z)}^{Y \times X}$

$\downarrow$

in deg $\quad$ mult $(-1)^{n+m}$

$(-1)^{nm} \varepsilon^{(n-j)(m-l)}$ $\quad C^n \quad C^m$ graded commutative $K^{MW}_{n-j} \quad K^{MW}_{m-l}$

$\varepsilon^{(n-j)(m-l)}$

$$d_{X \times Y}(\underset{C^n}{\alpha} \boxtimes \underset{C^m}{\beta}) = d_X \alpha \boxtimes \beta + \varepsilon^{n-j}(-1)^n \alpha \boxtimes d_Y \beta$$
$$= (-1)^n \varepsilon^j \alpha \boxtimes d_Y \beta$$

so get a map of complexes if we use $\varepsilon^j d_Y$ instead of d_Y

$\Rightarrow$ we have well-defined external product

$$\boxtimes_{X,Y} : H^n(X, \underset{\sim}{K}^{MW}_{j}\{L\}) \otimes H^m(Y, \underset{\sim}{K}^{MW}_{l}\{L_Y\}) \to H^{n+m}(X \times Y, \underset{\sim}{K}^{MW}_{j+l}\{L \boxtimes L_Y\})$$

§ Products. For $X \in Sm_k$ we have products

$$H^n(X, \underline{K}_j^{MW}\{L_1\}) \otimes H^m(X, \underline{K}_l^{MW}\{L_2\}) \to$$
$$H^{n+m}(X, \underline{K}_{j+l}^{MW}\{L_1 \otimes L_2\})$$

by $\alpha \cdot \beta := \Delta_X^*(\alpha \boxtimes_{k,X} \beta)$

These satisfy

1) associativity, unital with $1 \in H^0(X, \underline{K}_0^{MW})$
 $1 \in K_0^{MW}(k(X))$
2) $f^*(\alpha\beta) = f^*(\alpha) \cdot f^*(\beta)$
3) $$\alpha \cdot \beta = (-1)^{nm} \varepsilon^{(n-j)(m-l)} \beta \cdot \alpha$$

for $\alpha \in H^n(X, \underline{K}_j^{MW}\{L\})$

$\beta \in H^m(X, \underline{K}_l^{MW}\{L'\})$

4) projection formula: for $f: Y \to X$ proper we have $f_*(f^*\alpha \cdot \beta) = \alpha \cdot f_*(\beta)$

$\alpha \in H^n(X, \underline{K}_j^{MW}\{L\})$, $\beta \in H^m(Y, \underline{K}_l^{MW}\{L'\})$