Lemarié--Rieusset Elementine
Talk 13 MSS25

Computations of the quadratic linking degree

Addendum to the last talk:

Def: Two disjoint irred. smooth finite-type closed F-subschemes (F: perfect field) of X (X: irred. smooth finite-type F-s.) of same dimension form an orientable link in X if there exist orientations of their normal sheaves in X.
Two orientable links (Z_1,Z_2) and (Z'_1,Z'_2) in X are coarsely equivalent if:
$\exists\, h\in \mathrm{Aut}(X)\quad Z'_1=h^*(Z_1)$ and $Z'_2=h^*(Z_2)$.

Ⅰ $\mathbb{A}^2_F\setminus\{0\}\amalg \mathbb{A}^2_F\setminus\{0\}\hookrightarrow \mathbb{A}^4_F\setminus\{0\}$

① The Hopf link

Let F be a perfect field. $\mathbb{A}^2_F=\mathrm{Spec}(F[u,v])$ and $\mathbb{A}^4_F=\mathrm{Spec}(F[x,y,\eta,t])$.

oriented link of type $(\mathbb{A}^2_F\setminus\{0\},\mathbb{A}^2_F\setminus\{0\},\mathbb{A}^4_F\setminus\{0\})$; oriented link; orientable link:

$Z_1=\{x=0,y=0\}$ in $\mathbb{A}^4_F\setminus\{0\}$ (i.e. the ideal sheaf of Z_1 is $\underbrace{(x,y)}_{\mathcal{Y}}$)

$Z_2=\{\eta=0,t=0\}$ in $\mathbb{A}^4_F\setminus\{0\}=:X$ $\quad (\mathcal{Y}/\mathcal{Y}^2)^\vee\wedge(\mathcal{Y}/\mathcal{Y}^2)^\vee \qquad x\in\mathcal{Y},\ \bar{x}\in \mathcal{Y}/\mathcal{Y}^2,\ \bar{x}^*\in(\mathcal{Y}/\mathcal{Y}^2)^\vee$

$\bar{o}_1$ is the $\simeq$-class of $o_1:\nu_{Z_1}:=\det(N_{Z_1}X)\xrightarrow{\sim}\mathcal{O}_{Z_1}\otimes_{\mathcal{O}_{Z_1}}\mathcal{O}_{Z_1}$ which $\bar{x}^*\wedge\bar{y}^*\mapsto 1\otimes 1$

$\bar{o}_2$ " $\quad o_2:\nu_{Z_2}:=\det(N_{Z_2}X)\xrightarrow{\sim}\mathcal{O}_{Z_2}\otimes_{\mathcal{O}_{Z_2}}\mathcal{O}_{Z_2}$ which $\bar{\eta}^*\wedge\bar{t}^*\mapsto 1\otimes 1$

"parametrisations":

$\varphi_1:\mathbb{A}^2_F\setminus\{0\}\hookrightarrow\mathbb{A}^4_F\setminus\{0\}$ closed immersion $\leftrightarrow$ morphism of F-algebras $F[x,y,\eta,t]\to F[u,v]$
which maps x,y,η,t to $0,0,u,v$ respectively (image: Z_1)

$\varphi_2:\mathbb{A}^2_F\setminus\{0\}\hookrightarrow\mathbb{A}^4_F\setminus\{0\}$ closed immersion $\leftrightarrow$ "
which maps x,y,η,t to $u,v,0,0$ respectively (image: Z_2)

Let $j_1\le 0$ and $j_2\le 0$.

$$[o_1]_{j_1}\in H^0(Z_1,\underline{K}^{MW}_{j_1}\{\nu_{Z_1}\})\ \underset{\bar{o}_1}{\mapsto}\ \eta^{-j_1}\in H^0(Z_1,\underline{K}^{MW}_{j_1})\quad(\text{only depends on }\bar{o}_1)$$

$$[o_1]_{j_1}=\eta^{-j_1}\otimes(\bar{x}^*\wedge\bar{y}^*)$$

$$\partial(\underbrace{S_{o_1,j_1}}_{\in H^1(X\setminus Z,\underline{K}^{MW}_{j_1+2})})=([o_1]_{j_1},0)\in H^0(Z:=Z_1\amalg Z_2,\underline{K}^{MW}_{j_1}\{\nu_Z\}),\quad \nu_Z=\det(N_Z X)$$

$$\begin{tikzcd}
C^1(X\setminus Z,\underline{K}^{MW}_{j_1+2}) \arrow[r,"\partial"] \arrow[d,"j^*"'] \arrow[dr, phantom, "\circlearrowleft"] & C^0(Z,\underline{K}^{MW}_{j_1}\{\nu_Z\})=K^{MW}_{j_1}(K(\xi_{Z_1}),\nu_Z|_{\xi_{Z_1}}) \\
C^1(X,\underline{K}^{MW}_{j_1+2}) \arrow[r,"d"'] & C^2(X,\underline{K}^{MW}_{j_1+2}) \arrow[u,"i^*"']
\end{tikzcd}$$

$(\nu_Z)_{\xi_{Z_1}}\otimes_{\mathcal{O}_{Z,\xi_{Z_1}}}K(\xi_{Z_1})$

$$\bigoplus_{x\in X^{(1)}}K^{MW}_{j_1+1}(K(x),\nu_x)=C^1(X,\underline{K}^{MW}_{j_1+2}),\qquad C^2(X,\underline{K}^{MW}_{j_1+2})=\bigoplus_{y\in X^{(2)}}K^{MW}_{j_1}(K(y),\nu_y)$$

$$d=\bigoplus_{x\in X^{(1)}}\bigoplus_{y\in\overline{\{x\}}^{(1)}}d^x_y$$

Note that $\{y=0\}$ is of codim. 1 in X (with ξ_y its gen. pt) and such that ξ_{Z_1} (generic point of Z_1) $\in\overline{\{y=0\}}^{(1)}$.

If $j_1\le -1$, let's try $\underbrace{\eta^{-j_1-1}\langle x\rangle}_{\in K^{MW}_{j_1+1}(F(x,\eta,t))}\otimes\bar{y}^*$, $\quad \bar{y}^*\in\left((y)/(y)^2\right)^\vee \qquad (\langle x\rangle=1+\eta[x])$

If $j_1=0$, let's try $\underbrace{[x]}_{\in K^{MW}_1(F(x,\eta,t))}\otimes\bar{y}^*$.

$d^{\xi_y}_{\xi_{Z_1}}$ is the twisted canonical residue morphism ass. to v_1 and $\det(\psi_1^*(\mathcal{N}_{\{y=0\}|X}))_{\xi_{Z_1}}$ with $\psi_1 : Z_1 \hookrightarrow \{y=0\}$ and:

$v_1 : K(\xi_y)^* = (\mathrm{Frac}(\mathcal{O}_{\{y=0\},\xi_{Z_1}}))^* \twoheadrightarrow \mathbb{Z}$ the (normalized) discrete valuation of $\mathcal{O}_{\{y=0\},\xi_{Z_1}}$ (so that $K(v_1) = K(\xi_{Z_1})$).

$\frac{a}{b} \mapsto v_1(a) - v_1(b)$

$v_1(a)$: the greatest $n \in \mathbb{N}_0$ such that $a \in \mathfrak{m}^n_{\{y=0\},\xi_{Z_1}}$

<u>Def:</u> Let $v : K^* \twoheadrightarrow \mathbb{Z}$ be a (norm.) discrete val. (of ring $\mathcal{O}_v$ and maximal ideal: $\mathfrak{m}_v$ and residue field $K(v)$).
Let L be a rank 1 $\mathcal{O}_v$-module. The twisted canonical residue morphism

$$\partial_{v,L} : K_*^{MW}(K, L \otimes_{\mathcal{O}_v} K) \to K_{*-1}^{MW}(K(v), (\mathfrak{m}_v/\mathfrak{m}_v^2)^\vee \otimes_{K(v)} (L \otimes_{\mathcal{O}_v} K(v)))$$

is the only morphism of graded groups such that:

$$\forall \alpha \in K_*^{MW}(K)\ \forall \ell \in L\setminus\{0\}\ \partial_{v,L}(\alpha \otimes (\ell \otimes 1)) = \partial_v^\pi(\alpha) \otimes (\overline{\pi}^* \otimes (\ell \otimes 1))$$

with π a uniformizing parameter for v (i.e. $v(\pi)=1$); this does not depend on π.

<u>Thm</u> (Thm A.10 in my article "The quadratic linking degree" / Thm 2.46 in my PhD thesis "Motivic knot theory"):
$\partial_v^\pi : K_*^{MW}(K) \to K_{*-1}^{MW}(K(v))$ is the only morphism of graded groups such that:

- $\forall n \leq 0\ \forall m \in \mathbb{Z}\ \forall u \in \mathcal{O}_v^*\ \ \partial_v^\pi(\langle \pi^m u \rangle \eta^{-n}) = \langle \overline{u} \rangle \eta^{-n+1} \chi^{odd}(m)$
- $\forall n \geq 1\ \forall m_1, \dots, m_n \in \mathbb{Z}\ \forall u_1, \dots, u_n \in \mathcal{O}_v^*\ \ \partial_v^\pi([\pi^{m_1} u_1, \dots, \pi^{m_n} u_n]) = \dots$

with $m_\varepsilon = \sum_{i=1}^{m} \langle (-1)^{i-1} \rangle$ if $m > 0$ and $m_\varepsilon = \varepsilon(-m)_\varepsilon$ otherwise, $\varepsilon := -\langle -1 \rangle$.

$\hookrightarrow \partial_v^\pi([\pi^m u]) = \langle \overline{u} \rangle m_\varepsilon \ (= \lceil \frac{m}{2} \rceil \langle \overline{u} \rangle + \lfloor \frac{m}{2} \rfloor \langle -\overline{u} \rangle$

(note that $\eta m_\varepsilon = \eta \chi^{odd}(m)$) using $\eta[a,b] = [ab] - [a] - [b]$

$\hookrightarrow \partial_v^\pi([\pi^{m_1} u_1, \pi^{m_2} u_2]) = -\lceil \frac{m_1}{2} \rceil [\overline{u_1}] - \lfloor \frac{m_1}{2} \rfloor [-\overline{u_1}] + \lceil \frac{m_2}{2} \rceil [-\overline{u_2}] + \frac{m_1 - m_2 + \chi^{odd}(m_1 - m_2)}{2} [\overline{u_1 u_2}] + \lfloor \frac{m_2}{2} \rfloor) [\overline{u_2}] + \frac{m_1 - m_2 - \chi^{odd}(m_1 - m_2)}{2} [-\overline{u_1 u_2}] + \dots$

$j_1 \leq -1$: $d^{\xi_y}_{\xi_{Z_1}}(\underbrace{\eta^{-j_1-1} \langle x \rangle \otimes \overline{y}^*}_{\text{hence } = S_{o_1, j_1}}) = \partial^x_{v_1}(\eta^{-j_1-1} \langle x \rangle) \otimes (\overline{x}^* \wedge \overline{y}^*) = \eta^{-j_1} \otimes (\overline{x}^* \wedge \overline{y}^*) = [o_1]_{j_1}$

$j_1 = 0$: $d^{\xi_y}_{\xi_{Z_1}}(\underbrace{[x] \otimes \overline{y}^*}_{\text{hence } = S_{o_1,0}}) = \partial^x_{v_1}([x]) \otimes (\overline{x}^* \wedge \overline{y}^*) = 1 \otimes (\overline{x}^* \wedge \overline{y}^*) = [o_1]_0$

Similarly, $[o_2]_{j_2} = \eta^{-j_2} \otimes (\overline{\eta}^* \wedge \overline{t}^*)$ and $S_{o_2, j_2} = \begin{cases} \eta^{-j_2-1} \langle \eta \rangle \otimes \overline{t}^* & \text{if } j_2 \leq -1 \\ [\eta] \otimes \overline{t}^* & \text{if } j_2 = 0 \end{cases}$

Formula for $\cdot$: $\langle f_1 \rangle \otimes \overline{g_1}^* \cdot \langle f_2 \rangle \otimes \overline{g_2}^* = \sum_{\substack{[x] \text{ irred.} \\ \text{comp of } D_1 \cap D_2}} (m_x)_\varepsilon \langle f_1 f_2 u_x \rangle \otimes (\overline{\pi}_x^* \otimes \overline{g_1}^*)$, $g_2 = u_x \pi_x^{m_x} \in \mathcal{O}_{X,x}/(g_1)$

We can use it for $(j_1, j_2) = (-1, -1)$.
Anyway we will get the same QLD and AQLD if $(j_1, j_2) \neq (0,0)$.
We would however like a formula for $(j_1, j_2) = (0,0)$, in which case $QLD \in GW(F) \oplus GW(F)$ and $AQLD \in GW(F)$ (and they map to QLD and AQLD for other (j_1, j_2) via $GW(F) \xrightarrow{\text{can}} W(F)$).

Conjectural formula for $\cdot$: $[f_1] \otimes \overline{g_1}^* \cdot [f_2] \otimes \overline{g_2}^* = \sum_{\substack{[x] \text{ irr.} \\ \text{comp of} \\ D_1 \cap D_2}} (m_x)_\varepsilon \langle u_x \rangle [f_1, f_2] \otimes (\overline{\pi}_x^* \otimes \overline{g_1}^*)$ with $g_2 = u_x \pi_x^{m_x} \in \mathcal{O}_{X,x}/(g_1)$

$$\mathcal{O}_{X|Z,\{y=0,t=0\}} = F[x,y,\eta,t]_{(y,t)}$$

$$\mathcal{O}_{X|Z,\{y=0,t=0\}}/(y) = F[x,y,\eta,t]_{(y,t)}/(y) \simeq F[x,\eta,t]_{(t)} \quad t \text{ is a unif. param.}: \begin{cases} \pi = t \\ m = 1 \\ u = 1 \end{cases}$$

Case $(j_1, j_2) = (-1,-1)$:

$$S_1 \cdot S_2 = (\langle x\rangle \otimes \bar{y}^*)\cdot(\langle \eta\rangle \otimes \bar{t}^*)$$
$$= \langle x\eta\rangle \otimes (\bar{t}^* \wedge \bar{y}^*)$$

$$QLC = \partial(S_1\cdot S_2) = d^{t,y}_{t,y,x}(\langle x\eta\rangle \otimes(\bar{t}^*\wedge\bar{y}^*)) \oplus d^{t,y}_{t,y,\eta}(\langle x\eta\rangle\otimes(\bar{t}^*\wedge\bar{y}^*))$$
$$= \partial^x_{v_x}(\langle x\eta\rangle)\otimes(\bar{x}^*\wedge\bar{t}^*\wedge\bar{y}^*) \oplus \partial^\eta_{v_\eta}(\langle x\eta\rangle)\otimes(\bar{\eta}^*\wedge\bar{t}^*\wedge\bar{y}^*)$$
$$= \eta\langle\eta\rangle\otimes(\bar{x}^*\wedge\bar{t}^*\wedge\bar{y}^*) \oplus \eta\langle x\rangle\otimes(\bar{\eta}^*\wedge\bar{t}^*\wedge\bar{y}^*)$$
$$= \underbrace{\eta\langle-\eta\rangle\otimes(\bar{t}^*}_{\varphi_1^* \mapsto \eta\langle -u\rangle\otimes\bar{v}^*}\wedge\underbrace{\bar{x}^*\wedge\bar{y}^*}_{\text{taken away by } \tilde{\sigma}_1}) \oplus \underbrace{\eta\langle x\rangle\otimes(\bar{y}^*}_{\varphi_2^*\mapsto\eta\langle u\rangle\otimes\bar{v}^*}\wedge\underbrace{\bar{\eta}^*\wedge\bar{t}^*}_{\text{taken away by } \tilde{\sigma}_2})$$

$$\xi: H^1_3(\mathbb{A}^2_{F_4}\setminus\{0\}, \underline{K}^{MW}_{0_2}) \xrightarrow{\partial} H^0(\{0\}, \underline{K}^{MW}_{-2}\{v_{\{0\}}\}) \xrightarrow[\bar{x}^*\wedge\bar{y}^*\wedge\bar{\eta}^*\wedge\bar{t}^*]{\overbrace{\bar{u}^*\wedge\bar{v}^*}} K^{MW}_{-2}(F) \xrightarrow{\eta^2\mapsto 1} W(F)$$
ξ':

$$\xi(\eta\underbrace{\langle u\rangle}_{\langle -u\rangle}\otimes\bar{v}^*) = \mathrm{can.}(\tilde{\sigma}(\eta^2\otimes(\bar{u}^*\wedge\bar{v}^*))) = \mathrm{can.}(\eta^2) = 1 \in W(F)$$
$\langle -1\rangle$ $\langle -1\rangle$ $\langle -1\rangle = -1 \in W(F)$

$QLD = (-1, 1) \in W(F)\oplus W(F)$

$AQLC = \eta\langle-\eta\rangle\otimes(\bar{t}^*\wedge\bar{x}^*\wedge\bar{y}^*) \in H^3(X, \underline{K}^{MW}_2)$

$AQLD = -1 \in W(F)$

(inv. for cable equiv.:
· rank mod 2 (1)
· Σ_k (0) $W(\mathbb{R}) \cong \mathbb{Z}$
· if $F = \mathbb{R}$, $|\cdot|$ (1))
↑ absolute value

≥ for AQLD, if the only group autom. of $W(F)$ are $x\langle a\rangle$ (e.g. $F = \mathbb{R}, \mathbb{C}$, a finite field of char. $\not\equiv 1 (4)$)
$a \in F^*$

Case $(j_1, j_2) = (0,0)$ (with the conj. • formula):

$$S_{a_1,0}\cdot S_{a_2,0} = ([x]\otimes\bar{y}^*)\cdot([\eta]\otimes\bar{t}^*)$$
$$= [x,\eta]\otimes(\bar{t}^*\wedge\bar{y}^*)$$

$$QLC = \partial(S_{a_1,0}\cdot S_{a_2,0}) = \partial^x_{v_x}(\underbrace{[x,\eta]}_{\in K_2^{MW}(F(x,\eta))})\otimes(\bar{x}^*\wedge\bar{t}^*\wedge\bar{y}^*) \oplus \partial^\eta_{v_\eta}(\underbrace{[x,\eta]}_{\in K_2^{MW}(F(x,\eta))})\otimes(\bar{\eta}^*\wedge\bar{t}^*\wedge\bar{y}^*)$$

$$\partial^x_{v_x}([x,\eta]) = -\overbrace{[1]}^{0} + [\eta] = [\eta]$$

$$\partial^\eta_{v_\eta}([x,\eta]) = [-1] - [-x] \quad (= \varepsilon[x])$$

$$QLC = \underbrace{\overbrace{\langle -1\rangle[-\eta]}^{([-\eta]-[-1])} \otimes(\bar{t}^*}_{\varphi_1^*\mapsto([-u]-[-1])\otimes\bar{v}^*}\wedge\underbrace{\bar{x}^*\wedge\bar{y}^*}_{\text{taken away by } \tilde{\sigma}_1}) \oplus \underbrace{([-1]-[-x])\otimes(\bar{y}^*}_{\varphi_2^*\mapsto([-1]-[-u])\otimes\bar{v}^*}\wedge\underbrace{\bar{\eta}^*\wedge\bar{t}^*}_{\text{taken away by } \tilde{\sigma}_2})$$

$$\xi: H^1_3(\mathbb{A}^2_{F_4}\setminus\{0\}, \underline{K}^{MW}_2) \xrightarrow{\partial} H^0(\{0\}, \underline{K}^{MW}_{0_4}\{v_{\{0\}}\}) \xrightarrow[\bar{x}^*\wedge\bar{y}^*\wedge\bar{\eta}^*\wedge\bar{t}^*]{\overbrace{\bar{u}^*\wedge\bar{v}^*}} K^{MW}_0(F) \xrightarrow{1\mapsto 1} GW(F)$$
ξ':

$$\xi(([-u]-[-1])\otimes\bar{v}^*) = \mathrm{can.}(\tilde{\sigma}(\langle -1\rangle\otimes(\bar{u}^*\wedge\bar{v}^*))) = \langle -1\rangle \in GW(F)$$

$QLD = (\langle -1\rangle, -\langle -1\rangle) \in GW(F)\oplus GW(F)$

$AQLC = ([-\eta]-[-1])\otimes(\bar{t}^*\wedge\bar{x}^*\wedge\bar{y}^*) \in H^3(X, \underline{K}^{MW}_4)$

$AQLD = \langle -1\rangle \in GW(F)$

(inv. for cable equiv.:
QLD: rank (1 and −1), $GW(\mathbb{R}) \cong \mathbb{Z}\oplus\mathbb{Z}$
Σ_k (0), if $F = \mathbb{R}$: pair $(\{0,1\}, \{0,-1\})$;
AQLD: abs. value of the rank (1), Σ_k (0),
if $F = \mathbb{R}$: gcd (1))

if the only group autom. of $GW(F)$ are $x \pm \langle a\rangle$ (e.g. $F = \mathbb{C}$, finite field of char. 2)
$a \in F^*$

$$\Sigma_k\Big(\sum_{i=1}^{n}\varepsilon_i\langle a_i\rangle\Big) = \sum_{1\le i_1<\dots<i_k\le n}\Big(\prod_{1\le l\le k}\varepsilon_{i_l}\Big)\Big\langle\prod_{1\le j\le k} a_{i_j}\Big\rangle \in GW(F) \ \checkmark$$

$k > 0$ even; $\in GW(F)$ $\quad (\varepsilon_i \in \{-1,1\}, a_i \in F^*)$ $\quad \Sigma_k: W(F) \to {}^{W(F)}/_{(1)} \big/ {}_{(\Sigma_2(\varepsilon_{-1}))} / \dots \ \checkmark$