TALK 2: MILNOR K-THEORY AND THE ROST COMPLEX

References:

[Mil] Milnor "Algebraic K-theory and quadratic forms", 1970

[Ros] Rost "Chow groups with coefficients", 1996

In this talk we introduce some fundamental tools that we need in the seminar.

Program: §1. Milnor K-theory
§2. Cycle premodules
§3. Cycle modules
§4. The 4 basic maps

§1. Milnor K-theory

[Mil, §1]

Let F be a field. We denote $F^\times := F \setminus \{0\}$.

Def. The Milnor K-theory of F is the $\mathbb{Z}$-graded ring

$$K_*^M(F) := \underbrace{\bigoplus_{n\geq 0} (F^\times)^{\otimes_{\mathbb{Z}} n}}_{\text{tensor ring of } F^\times\text{. It is a } \mathbb{Z}\text{-graded ring.}} \Big/ \underbrace{(a\otimes b \mid a,b\in F^\times,\ a+b=1)}_{\text{ideal generated by homogeneous elements} \leadsto \text{ the quotient is a } \mathbb{Z}\text{-graded ring.}}.$$

Explicitly, the grading is given by

$$K_*^M(F) = \bigoplus_{n\in\mathbb{Z}} K_n^M(F), \quad K_n^M(F) = 0 \quad \text{for } n<0$$

$$K_n^M(F) = (F^\times)^{\otimes_{\mathbb{Z}} n} \Big/ \underbrace{\langle a_1\otimes \dots \otimes a_n \mid a_1,\dots,a_n \in F^\times,\ a_i + a_{i+1} = 1 \text{ for some } i\rangle}_{\text{subgroup generated}} \quad \text{for } n\geq 0$$

In particular,

$$K_0^M(F) = \mathbb{Z}$$

$$K_1^M(F) = F^\times$$

$$K_2^M(F) = F^\times \otimes_{\mathbb{Z}} F^\times \Big/ \underbrace{\langle a\otimes(1-a) \mid a\neq 0,1\rangle}_{\text{"Steinberg relation"}}$$

<u>Notation</u> : We use the additive notation for the operation on $K^M_1(F)$.

To make it precise, we introduce the group isomorphism

$$F^\times \xrightarrow{\cong} K^M_1(F)$$
$$a \longmapsto \{a\}$$. Notice that $\{1\}=0$, $\{a^{-1}\}=-\{a\}$

So, we have that, $\forall a,b\in F^\times$,

$$\{ab\} = \{a\}+\{b\}.$$

We also denote

$$\{a_1,\dots,a_n\} := \{a_1\}\cdot\ldots\cdot\{a_n\}.$$ in $K^M_n(F)$

Notice that $\{ab,c\} = \{a,c\}+\{b,c\}$.

So, Milnor K-theory is the associative ring with unit

- generated by symbols $\{a\}$, for $a\in F^\times$. is generated by elements in deg 1 !
- with relations generated by
  - $\{ab\} = \{a\}+\{b\}$, for $a,b\in F^\times$
  - $\{a,1-a\}=0$, for $a\in F^\times$, $a\neq 1$.

Each $K^M_n(F)$ satisfies the universal property given by the combination of the universal properties of the tensor ring and quotient:

$\forall$ G group, $\forall$ $\underbrace{F^\times\times\ldots\times F^\times}_{n\text{-times}} \to G$ n-linear function

s.t. $(a_1,\dots,a_n)\longmapsto 0$ if $a_i+a_{i+1}=1$ for some i,

$\exists!$ $K^M_n(F)\to G$ group hom s.t.

$$\begin{tikzcd} F^\times\times\ldots\times F^\times \arrow[rr] \arrow[dr] & & G \\ & K^M_n(F) \arrow[ur] & \end{tikzcd}$$ commutes.

$(a_1,\dots,a_n) \mapsto \{a_1,\dots,a_n\}$

Some properties of Milnor K-theory directly following from the definition:

- $\{a,-a\}=0$, $\forall a\in F^\times$.
- $K^M_*(F)$ is anticommutative, i.e. $\forall x\in K^M_m(F)$, $y\in K^M_n(F)$

  Notice that $\{a,b\}=-\{b,a\}$. $\quad xy = (-1)^{mn}\,yx$ in $K^M_{m+n}(F)$

  $\{a,a\}$
  $\shortparallel$
- $\{a\}^2 = \{a,-1\}$ $\quad\forall a\in F^\times$
- $\{a_1,\dots,a_n\}=0$ $\quad\forall a_1,\dots,a_n\in F^\times$ s.t. $a_1+\ldots+a_n=0$ or 1

[Mil, §2]

An important construction on Milnor K-theory is the residue map. We consider the setting in which F is endowed with a discrete valuation v.

Recall: v a discrete valuation on F is a function $v: F \longrightarrow \mathbb{Z} \cup \{\infty\}$ s.t.

- $v(ab) = v(a) + v(b)$
- $v(a+b) \geq \min\{v(a), v(b)\}$
- $v(a) = \infty \iff a = 0$

We define $\mathcal{O} := \{a \in F \mid v(a) \geq 0\}$, called ring of integers.
$\cup$
$\mathfrak{m} := \{a \in F \mid v(a) > 0\}$

$\mathcal{O}$ is a DVR (= PID with exactly one maximal ideal) with maximal ideal $\mathfrak{m}$. A generator $\pi \in \mathfrak{m}$ is called a uniformizer. It is unique up to a unit $u \in \mathcal{O}^\times$. They are the elements of F with valuation 1. It holds that $F = \mathrm{Frac}(\mathcal{O})$.

The quotient $k(v) := \mathcal{O}/\mathfrak{m}$ is called the residue class field.

Let $\mathcal{O}$ be the ring of integers, $\mathfrak{m} \subset \mathcal{O}$ its maximal ideal, $U := \mathcal{O}^\times = \mathcal{O} \setminus \mathfrak{m}$ the units, $k(v) = \mathcal{O}/\mathfrak{m}$ the residue class field.

We denote

$$\begin{aligned} U \subset \mathcal{O} &\twoheadrightarrow \mathcal{O}/\mathfrak{m} = k(v) \\ u &\longmapsto \bar{u} \end{aligned}$$

Prop: [Mil, Lemma 2.1]

There exists a unique group hom $\forall n \geq 1$

$$\partial : K^M_n(F) \longrightarrow K^M_{n-1}(k(v)) \quad \text{RESIDUE MAP}$$

s.t. $\{\pi, u_2, \dots, u_n\} \longmapsto \{\bar{u}_2, \dots, \bar{u}_n\}$ — for $n=1$, this means that $\{\pi\} \longmapsto 1 \in \mathbb{Z} \cong K^M_0(F)$

when π is any uniformizer and $u_i \in U$.
↳ the definition of ∂ does not depend on a choice of π!

Proof: (Sketch)

Uniqueness: $K^M_n(F)$ is generated (as a group) by elements of the kind

$$\{\pi\}^r \cdot \{u_{r+1}\} \cdots \{u_n\} \qquad \text{for } r = 0, 1, \ u_i \in U.$$

Indeed it is generated by $\{a_1, \dots, a_n\}$ with $a_i \in F^\times$.
Since $F = \mathrm{Frac}(U)$, then $a_i = \pi^{e_i} u_i$ for some $e_i \in \mathbb{Z}$, $u_i \in U$.
Using linearity, $\{\pi, \pi\} = \{\pi, -1\}$ and anticommutativity, we see that we can take as generators elements of the above kind

ex: $\{\pi^2 u, \pi^{-1}\} = \{\pi^2, \pi^{-1}\} + \{u, \pi^{-1}\}$

$\{u, \pi^{-1}\} = -\{\pi^{-1}, u\} = \{\pi, u\}$

$\{\pi^2, \pi^{-1}\} = \{\pi, \pi^{-1}\} + \{\pi, \pi^{-1}\} =$
$= -2\{\pi, \pi\} = -2\{\pi, -1\}$

For $r=1$, the image is unique by the statement of the prop.
For $r=0$, the image is zero:
$\{u_1, \dots, u_n\} = \{u_1 \pi \cdot \pi^{-1}, \dots, u_n\} = \{u_1 \pi, \dots, u_n\} - \{\pi, \dots, u_n\} = 0$

are 2 uniformizers! we are assuming that ∂ doesn't depend on them.

Existence: It is proved constructing explicitely an n-bilinear function

$$\underbrace{F^\times \times \dots \times F^\times}_{n \text{ times}} \longrightarrow K^n_{n-1}(F^\times)$$

$$\underset{e_i \in \mathbb{Z} \quad u_i \in U}{(\pi^{e_1} u_1, \dots, \pi^{e_n} u_n)} \longmapsto \varphi \quad \text{which does \underline{not} depend on } \pi!$$

$$\text{s.t. } (\pi, u_2, \dots, u_n) \longmapsto \{\bar{u}_2, \dots, \bar{u}_n\}$$

Then, it is proved that whenever two consecutive $\pi^{e_i} u_i$ add to 1, then $\varphi = 0$.
By the universal property of $K^n_n(F)$, this defines the wanted group hom.

■

It works to explicit these maps for $n = 1, 2$:

- For $n=1$, it is the valuation map

$$\begin{array}{ccc} K^n_1(F) & \xrightarrow{\partial} & K^n_0(k(v)) \\ \uparrow\wr & & \| \\ F^\times & \xrightarrow{v} & \mathbb{Z} \end{array}$$

$a = \pi^i u \rightsquigarrow \{a\} = \{\pi^i u\} = i\{\pi\} + \{u\} \overset{\partial}{\longmapsto} i \cdot 1 + 0 = i = v(a)$

- For $n=2$, it is s.t.

$$K^n_2(F) \xrightarrow{\partial} K^n_1(k(v))$$

$$\{\pi^i u_1, \pi^j u_2\} \longmapsto \left\{ (-1)^{ij} \frac{\bar{u}_2^{\,i}}{\bar{u}_1^{\,j}} \right\}$$ is usually called "Tame Symbol"

$\{\pi^i u_1, \pi^j u_2\} = \{\pi^i, \pi^j\} + \{\pi^i, u_2\} + \{u_1, \pi^j\} + \{u_1, u_2\} = ij\{\pi, -1\} + i\{\pi, u_2\} - j\{\pi, u_1\} + \{u_1, u_2\}$

$ij\{-1\} + i\{\bar{u}_2\} - j\{\bar{u}_1\} + 0 = \left\{ (-1)^{ij} \frac{\bar{u}_2^{\,i}}{\bar{u}_1^{\,j}} \right\}$

Putting the residue maps all together we get an hom of graded groups of deg -1

$$\partial : K^M_*(F) \longrightarrow K^M_*(k(v)).$$

From the residue map, we immediately define another group hom

$$s^\pi : K^M_n(F) \longrightarrow K^M_n(k(v)) \qquad \text{SPECIALIZATION MAP}$$
$$x \longmapsto \partial(\{-\pi\}\cdot x)$$

This map is less "natural" than the residue map since it depends on the choice of a uniformiser. Given π and $\pi' = u\pi$ two different uniformisers, then the relation is:

$$s^{\pi'}(x) = s^\pi(x) - \{\bar{u}\}\,\partial(x)$$

$$s^{\pi'}(x) = \partial(\{-\pi'\}\cdot x) = \partial(\{-u\pi\}\cdot x) = \partial(\{-\pi\}\cdot x + \{u\}\cdot x) =$$
$$= \partial(\{-\pi\}\cdot x) + \partial(\{u\}\cdot x) \underset{\uparrow}{=} s^\pi(x) - \{\bar{u}\}\,\partial(x)$$

$$\{u\}\{\pi, u_2, \dots, u_n\} = \{u, \pi, u_2, \dots, u_n\} = -\{\pi, u, u_2, \dots, u_n\} \overset{\partial}{\longmapsto} -\{\bar{u}, \bar{u}_2, \dots, \bar{u}_n\}$$
$$= -\{\bar{u}\}\{\bar{u}_2, \dots, \bar{u}_n\}$$

Anyway it has some nice properties that ∂ doesn't have.
For example, it respects the product in $K^M_*(F)$: $\forall x \in K^M_m(F)$, $y \in K^M_n(F)$

$$s^\pi(xy) = s^\pi(x)\, s^\pi(y). \qquad \text{in } K^M_{m+n}(F)$$

So, putting the specialization maps all together, we get an hom of graded rings of deg 0

$$s^\pi : K^M_*(F) \longrightarrow K^M_*(k(v)).$$

It is s.t. $\{\pi, u_2, \dots, u_n\} \longmapsto 0$

$$\{u_1, \dots, u_n\} \longmapsto \{\bar{u}_1, \dots, \bar{u}_n\}$$

- $\{-\pi\}\{\pi, u_2, \dots, u_n\} = \{-\pi, \pi\}\{u_2, \dots, u_n\} \overset{\partial}{\longmapsto} 0$

  $\hookrightarrow = \{-1, \pi\} + \{\pi, \pi\} = -\{\pi, -1\} + \{\pi, -1\} = 0$

- $\{-\pi\}\{u_1, \dots, u_n\} = \{\underset{\substack{\xi \\ \text{is a uniformizer!}}}{-\pi}, u_1, \dots, u_n\} \overset{\partial}{\longmapsto} \{\bar{u}_1, \dots, \bar{u}_n\}$

[Ros, §1]

There are also a couple of other group hom on Milnor K-theory that we will need for the notion of cycle modules.
They express functoriality in the change of the field F.

- Given $\varphi: F \hookrightarrow E$ a field extension, we have an hom of graded rings of deg 0

$$\varphi_*: K^M_*(F) \longrightarrow K^M_*(E) \qquad \text{PUSHFORWARD MAP}$$

$$\text{s.t.} \quad \{a_1, \dots, a_n\} \longmapsto \{\varphi(a_1), \dots, \varphi(a_n)\}.$$

It would be better to call it a PULLBACK MAP (and denote it φ^*)
since it would be better to think geometrically φ as the morphism $\mathrm{Spec}E \longrightarrow \mathrm{Spec}F$.
But we follow Rost's notation to avoid future confusion.

We easily see that is well defined and gives rise to an hom of graded rings.

- Given $\varphi: F \hookrightarrow E$ a finite field extension, we have an hom of graded groups of deg 0:

$$\varphi^*: K^M_*(E) \longrightarrow K^M_*(F). \qquad \text{PULLBACK (or NORM) MAP}$$

The existence and characterizations are theorems of Bass, Tate and Kato.
It worths to explicit these maps for $n=0,1$:

- For $n=0$, it is the multiplication by the degree of the field extension

$$\begin{array}{ccc} K^M_0(E) & \xrightarrow{\varphi^*} & K^M_0(F) \\ \| & & \| \\ \mathbb{Z} & \xrightarrow{\cdot [E:F]} & \mathbb{Z} \end{array}$$

- For $n=1$, it is the norm map of the field extension E/F:

$$\begin{array}{ccc} K^M_1(E) & \xrightarrow{\varphi^*} & K^M_1(F) \\ \wr\wr & & \wr\wr \\ E^\times & \xrightarrow{Nr_{E/F}} & F^\times \end{array}$$

Recall: $Nr_{E/F}(a) := \det(m_a)$
where $m_a: E \longrightarrow E$, $b \longmapsto ab$
is an F-linear map

§2. CYCLE PREMODULES [Ros, §1]

Cycle modules are the coefficients that we use to construct Rost complex.
We first define cycle premodules, which are quite natural notion of modules over Milnor K-theory. They are defined by a list of data and axioms.
Then, cycle modules are a particular kind of cycle premodules which satisfy particular axioms.
Milnor K-theory is an example of a cycle module and is the coefficient from which we obtain classical Chow groups.

We first describe the setting:

$X = \bigcup_{\text{finite}} \operatorname{Spec} A_i[S_i^{-1}]$, $S_i \subset A_i$ mult. system, where $X' = \bigcup \operatorname{Spec} A_i$ is

- Fix k a field. By a scheme$/k$ we mean a localization of a separated k-scheme of finite type$/k$. This implies that all X schemes$/k$ we consider are noetherian and $\forall x \in X$ the residue field $K(x)$ is a f.g. field extension of k.
  (e.g. schemes of finite type$/k$)
  We fix a base scheme B, which is a scheme$/k$.

[We can think: $B = \operatorname{Spec} K$, K/k f.g. extension, and schemes are schemes of finite type$/K$]

- A field over B is a finitely generated field extension F/k with a morphism of k-schemes $\operatorname{Spec} F \to B$. [In case $B = \operatorname{Spec} K$, $\mathcal{F}(B)$ is the class of F/K f.g. field extensions]
  We denote by $\mathcal{F}(B)$ the class of fields over B.
  A field extension over B is a field extension $\varphi : F \hookrightarrow E$ with $F, E \in \mathcal{F}(B)$ such that
$$\begin{array}{ccc} \operatorname{Spec} E & \longrightarrow & \operatorname{Spec} F \\ & \searrow \; B \; \swarrow & \end{array}$$
  commutes.

- A valuation on a field over B $F \in \mathcal{F}(B)$ is a non trivial discrete valuation v on F, with a morphism of k-schemes $\operatorname{Spec} \mathcal{O} \to B$, and
  i.e. $v(F^\times) \neq \{0\}$
  $\mathcal{O}$ is the localization of an integral domain of finite type over k at a prime ideal of codim 1 (or equivalently, F/k and $k(v)/k$ are finitely generated field extensions and $\operatorname{tr.deg}_k(F) = \operatorname{tr.deg}_k(k(v)) + 1$).
  (f.g. as a k-alg)

this is exactly the situation we have in §3., coming from geometry:
$\mathcal{O} = \mathcal{O}_{X,x}$, $x \in X^{(1)}$, for X normal scheme$/B$ is a DVR
$\rightsquigarrow \operatorname{Frac}(\mathcal{O}) = K(X)$ is a field endowed with a valuation over B.

in Rost's article are considered also the $\mathbb{Z}/2$-graded abelian groups, but its just a technicality that we won't need.

Def: A CYCLE PREMODULE over B is a function
$$M : \mathcal{F}(B) \longrightarrow Ab^{\mathbb{Z}} = \text{class of } \mathbb{Z}\text{-graded abelian groups}$$
$$F \longmapsto M(F) = \bigoplus_{n\in\mathbb{Z}} M_n(F)$$
with the following data and axioms:

(D1) For any $\varphi : F \hookrightarrow E$ field extension over B there exists an hom of graded groups of deg 0:
$$\varphi_* : M(F) \longrightarrow M(E) \qquad \text{RESTRICTION MAP}$$

(D2) For any $\varphi : F \hookrightarrow E$ finite field extension over B there exists an hom of graded groups of deg 0:
$$\varphi^* : M(E) \longrightarrow M(F) \qquad \text{CORESTRICTION MAP}$$

(D3) For any $F \in \mathcal{F}(B)$, $M(F)$ is a graded left $K^M_*(F)$-module

(D4) For any v discrete valuation on F over B, there exists an hom of graded groups of deg -1:
$$\partial : M(F) \longrightarrow M(k(v)) \qquad \text{RESIDUE MAP}$$
Chosen a uniformizer π, we define the hom of graded groups of deg 0
$$s^\pi : M(F) \longrightarrow M(k(v)) \qquad \text{SPECIALIZATION MAP}$$
$$\xi \longmapsto \partial(\{-\pi\}\cdot\xi)$$
↳ scalar multiplication of (D3)

(R1) (Relations btw φ_* and φ^*)

"functoriality of φ_*, φ^*"
(a) For any $F \overset{\varphi}{\hookrightarrow} E \overset{\psi}{\hookrightarrow} L$ field extensions over B,
$$(\psi\varphi)_* = \psi_*\varphi_* .$$
(b) For any $F \overset{\varphi}{\hookrightarrow} E \overset{\psi}{\hookrightarrow} L$ finite field extensions over B,
$$(\psi\varphi)^* = \varphi^*\psi^* .$$

"base change formula"
(c) For any $F \overset{\varphi}{\hookrightarrow} E$, $F \overset{\psi}{\hookrightarrow} L$ field extensions over B, φ finite,
let $R := L\otimes_F E$, $\mathfrak{p} \in \operatorname{Spec} R$, $l_{\mathfrak{p}} := \text{lenght}(R_{\mathfrak{p}})$ = largest chain of ideals of $R_{\mathfrak{p}}$ = largest chain of ideals of R contained in $\mathfrak{p}$
$$\psi_{\mathfrak{p}} : E \longrightarrow R \longrightarrow R/\mathfrak{p} \quad , \quad \varphi_{\mathfrak{p}} : L \longrightarrow R \longrightarrow R/\mathfrak{p} .$$
Then
$$\psi_*\varphi^* = \sum_{\mathfrak{p}\in\operatorname{Spec}R} l_{\mathfrak{p}} \cdot \varphi_{\mathfrak{p}}^* \psi_{\mathfrak{p}*} .$$

Notice that any $\mathfrak{p} \in \operatorname{Spec} R$ is a maximal ideal, hence $R/\mathfrak{p}$ is a field over B with $\operatorname{Spec} R \longrightarrow \operatorname{Spec} E \longrightarrow B$.
Indeed, by Sharp-Grothendieck formula, $\dim(R) = \min(\text{tr.}\deg_F(E), \text{tr.}\deg_F(L)) = 0$ ← the E/F is finite, hence algebraic

(R2) (Relations btw φ_*, φ^* and scalar multiplication)

"φ_* is K^n_*-linear" } (a) For any $\varphi : F \hookrightarrow E$ field extension over B, $x \in K^n_*(F)$, $y \in K^n_*(E)$, $\xi \in M(F)$, $\mu \in M(E)$,

$$\varphi_*(x \cdot \xi) = \varphi_*(x) \cdot \varphi_*(\xi).$$

"Projection formulas" } (b,c) For any $\varphi : F \hookrightarrow E$ finite field extension over B, $x \in K^n_*(F)$, $y \in K^n_*(E)$, $\xi \in M(F)$, $\mu \in M(E)$,

$$\varphi^*(\varphi_*(x) \cdot \mu) = x \cdot \varphi^*(\mu)$$

$$\varphi^*(y \cdot \varphi_*(\xi)) = \varphi^*(y) \cdot \xi.$$

(R3) (Relations btw φ_*, φ^*, scalar multiplication and ∂)

(a) For any $\varphi : F \hookrightarrow E$ field extension over B, w a discrete valuation on E over B which restricts to $v := w|_F$ a discrete valuation on F over B with ramification index e, let $\bar{\varphi} : k(v) \hookrightarrow k(w)$ be the field extension over B induced on residue class fields. Then

$$\partial_v \varphi_* = e \cdot \bar{\varphi}_* \partial_w.$$

Recall: we have

$$\begin{array}{ccccc} m_w \cap F = m_v & \subset & \mathcal{O}_v & \subset & F \\ \cap & & \cap & & \cap \\ m_w & \subset & \mathcal{O}_w & \subset & E \end{array}$$

and $m_v \mathcal{O}_w = m_w^e$

$e =:$ ramification index

(b) For any $\varphi : F \hookrightarrow E$ finite field extension over B, v a discrete valuation on F over B, w discrete valuations on E over B which extend v ($v = w|_F$), let $\bar{\varphi}_w : k(v) \hookrightarrow k(w)$ be the field extensions over B induced on residue class fields. Then

$$\partial_v \varphi^* = \sum_w \bar{\varphi}_w^* \partial_w.$$

(c) For any $\varphi : F \hookrightarrow E$ field extension over B, w a discrete valuation on E over B, if $w|_F$ is trivial, then $\partial_w \varphi_* = 0$.

(d) For any $\varphi : F \hookrightarrow E$ field extension over B, w a valuation on E over B s.t. $w|_F$ is trivial, we have an induced field extension $\tilde{\varphi} : E \to k(v)$. Then, for any choise of a uniformizer π,

$$s^\pi \varphi_* = \tilde{\varphi}_*.$$

This is why s^π is called specialization map! }

(e) For any v discrete valuation on F over B, $u \in U$, $\xi \in M(F)$,

$$\partial(\{u\} \cdot \xi) = -\{\bar{u}\} \cdot \partial(\xi).$$

The first example of cycle premodule (and the only one important for us) is Milnor K-theory itself.

Thm: [Ros, Thm 1.4]

Milnor K-theory $K^M_*: \mathcal{F}(B) \longrightarrow Ab^{\mathbb{Z}}$

$$F \longmapsto K^M_*(F) = \bigoplus_{n\in\mathbb{Z}} K^M_n(F)$$

with (D1) = pushforward map

(D2) = pullback (or norm) map

(D3) = ring multiplication

(D4) = residue map

is a cycle premodule.

Other examples are:

- Quillen K-theory of fields: $F \longmapsto K_*(F)$
- Galois cohomology:
- de Rham cohomology: $F \longmapsto H^n_{dR}(F/B)$

Milnor and Quillen K-theory of fields are absolute theories, we don't need the base scheme B, we can think B = Spec k.

these are relative theories, we really need a base scheme B!

$H^n_{dR}(F/B)$ = sheaf cohomology of the sheaf complex $\Omega^{\cdot}_{\operatorname{Spec} F/B}$

The notion of cycle premodules comes with a natural def of morphisms btw them.

Def: A **morphism of cycle premodules** $/B$ $\alpha: M \longrightarrow N$ is a collection of group hom for each $F \in \mathcal{F}(B)$

$$\alpha_F: M(F) \longrightarrow N(F)$$

s.t. they satisfy the obvious compatibilities with data (D1)–(D4).

(1) $\varphi_* \alpha_F = \alpha_E \varphi_*$

(2) $\varphi^* \alpha_F = \alpha_E \varphi^*$

(3) $\{a\}\cdot \alpha_E = \alpha_E(\{a\}\cdot \_)$

(4) $\partial \alpha_E = \alpha_{k(v)} \partial$

§ 3. CYCLE MODULES [Ros, §2]

Cycle modules are cycle premodules with 2 additional axioms that allow to define the Rost complex.

First we fix some notations: (we also keep the conventions from §2.)

- M will always denote a cycle premodule over B.
- Let X be a scheme $/B$. For any point $x \in X$, the residue field $K(x)$ is a field over B. We denote $M(x) := M(K(x))$.
- Let X be a scheme $/B$. A point $x \in X$ has:
  - dimension p, if $\dim \overline{\{x\}} = p$. We denote $X_{(p)} :=$ set of points of X of dim p.
  - codimension p, if $\dim \mathcal{O}_{X,x} = p$. " " $X^{(p)} :=$ " " " " " " codim p.
- If X is irreducible, we denote by ξ_X its generic point. it is s.t. $\overline{\{\xi_X\}} = X$.

  Recall that if X is normal, ($\mathcal{O}_{X,x}$ is normal ring $\forall x \in X$ & X is irreducible) then any point $x \in X$ of codimension 1, $x \in X^{(1)}$, $\mathcal{O}_{X,x}$ is a DVR over B (b/c it is a local noetherian normal ring of dim 1).

  So, $\mathrm{Frac}(\mathcal{O}_{X,x}) = K(\xi_X)$ is a field with a discrete valuation over B, whose residue field class is $K(x)$.

  We denote the residue map for $K(\xi_X)$ with

$$\partial_x : M(\xi_X) \longrightarrow M(x)$$

- For any X scheme $/B$, given two points $x, y \in X$, let $Z := \overline{\{x\}}$. (We consider the reduced scheme structure. Z is an irreducible scheme $/B$ with generic point x. So, Z is an integral scheme.)

  We define the group hom

$$\partial^x_y : M(x) \longrightarrow M(y)$$

s.t.

$$\partial^x_y := \begin{cases} 0 & \text{if } y \notin Z^{(1)} \\ \partial_y & \text{if } y \in Z^{(1)} \text{ and } Z \text{ is normal} \\ \sum_{z \in \pi^{-1}(y)} \pi_z^* \partial_z & \text{if } y \in Z^{(1)} \text{ and } Z \text{ is not normal,} \end{cases}$$

(this case is in fact included in the next ($\pi = id$))

where $\pi : \tilde{Z} \to Z$ is a normalization (makes sense b/c Z is integral!)

and $\forall z \in \pi^{-1}(y)$, $\pi_z : K(y) \hookrightarrow K(z)$ is the induced field extension

Notice that the source and the target coincide b/c the residue field at $x \in Z \subseteq X$ is the same computed in Z or X!

Notice that this is a finite sum b/c $\pi^{-1}(y)$ is finite.

Indeed, π is s.t. $\forall U \cong \mathrm{Spec} A \overset{\circ}{\subset} Z$ affine open subset, $\pi^{-1}(U) \cong \mathrm{Spec} \bar{A}$, where $\bar{A}$ is the integral closure of A. The ring hom $A \hookrightarrow \bar{A}$ is finite because it is integral and finitely generated, hence fibers are finite.

b/c $k \to A \subset \bar{A} \hookrightarrow \mathrm{Frac}(A) = K(x)$ is a f.g. field extension

Def: A cycle premodule M over B is a CYCLE MODULE if it satisfies the following properties

(FD) (FINITE SUPPORT OF DIVISOR)

enables to define differentials d in Rost complex

For any X normal scheme$/B$, $\mu \in M(\xi_X)$,

$$\partial_x(\mu) \neq 0 \quad \text{for finitely many } x \in X^{(1)}.$$

Notice that μ is fixed! This is different to ask that $\partial_x = 0$ for finitely many $x \in X^{(1)}$.

Notice that (FD) implies that, for any X scheme$/B$, $x, y \in X$, $\mu \in M(x)$

$$\partial^x_y(\mu) \neq 0 \quad \text{for finitely many } y \in X.$$

(C) (CLOSEDNESS)

guarantees that $d^2 = 0$ in Rost complex

$\mathcal{O}_{X,x}$ is an integral domain $\forall x \in X$ & X is irreducible

has just 1 closed point

For any X integral, local scheme$/B$ of dim 2, the composition

$$M(\xi_X) \xrightarrow[\textcircled{1}]{\sum_{x \in X^{(1)}} \partial^{\xi_X}_x} \bigoplus_{x \in X^{(1)}} M(x) \xrightarrow[\textcircled{2}]{\sum_{x \in X^{(1)}} \partial^x_{x_0}} M(x_0)$$

① this lands in the $\oplus$ b/c of the observation following (FD)

② this is defined by the univ. prop. of the $\oplus$

is 0, where $x_0 \in X$ is the only closed point of X.

We also have a notion of morphism, s.t. cycle modules form a full subcategory of cycle premodules.

Def: A MORPHISM OF CYCLE MODULES is a morphism btw the underlying cycle premodules.

From these two additional axioms can be deduced the following properties.

Prop: [Ros, Prop 2.2]

gives homotopy property on Rost groups

For any M cycle module over B, F field over B,

(H) (HOMOTOPY PROPERTY FOR $\mathbb{A}^1$)

Consider $\mathbb{A}^1_F = \mathrm{Spec}(F[t])$. It is an irreducible scheme$/B$ with generic point $\xi = (0)$, s.t. $K(\xi) = F(t)$. $= \mathrm{Frac}(F[t])$

→ more precisely, the observation after. We need it to say that d lands in the $\oplus$

By axiom (FD), we have the morphism

$$d := \sum_{x \in \mathbb{A}^1_{F(0)}} \partial_x^{\xi} : M(F(t)) \to \bigoplus_{x \in \mathbb{A}^1_{F(0)}} M(x)$$

Denote by $\varphi : F \hookrightarrow F(t)$ the field extension.

Then the following sequence

$$0 \to M(F) \xrightarrow{\varphi_*} M(F(t)) \xrightarrow{d} \bigoplus_{x \in \mathbb{A}^1_{F(0)} = \mathbb{A}^{1\,(1)}_F} M(x) \to 0$$

is exact.

we need it to define pushforward map

(RC) (RECIPROCITY FOR CURVES)

$X \to \mathrm{Spec} F$ is proper dim 1 & irreducible

Let X be a proper curve over F.

→ same considerations as above

By axiom (FD), we have the morphism

$$d := \sum_{x \in X_{(0)}} \partial_x^{\xi_X} : M(\xi_X) \to \bigoplus_{x \in X_{(0)}} M(x)$$

For any $x \in X_{(0)}$, denote by $\varphi_x : F \hookrightarrow K(x)$ the field extension.

Then the following composition

$$M(\xi_X) \xrightarrow{d} \bigoplus_{x \in X_{(0)} = X^{(1)}} M(x) \xrightarrow{\sum_{x \in X_{(0)}} \varphi_x^*} M(F)$$

is 0.

[Ros, Rmk 2.7]

The homotopy property gives a motivation of why Milnor K-theory is a good choice as being a good ring of scalars for cycle modules. More precisely, it is reasonable to ask that $M(F)$ is a graded $T^*F^\times$ - module. ($=$ tensor ring of $F^\times$) We see why it's also good to ask why it passes to the Steinberg relations.

For any $\varphi: F \hookrightarrow E$ field extension over B, $a \in E^\times$ $a \neq 1$, we consider

$$\eta : M(F) \longrightarrow M(E)$$
$$\xi \longmapsto \{a, 1-a\} \cdot \varphi_*(\xi)$$

We want to deduce that assuming some reasonable properties on M (such as, reasonable specialization maps and homotopy property) then $\eta = 0$ for $\varphi = id_F$. Then, $M(F)$ passes to the Steinberg relations and so it is a graded module on Milnor K-theory.

The way we prove this is by reducing to the case $E = F(t)$.

In the case $E = F(t)$, we use homotopy invariance to deduce $\eta = 0$.

Again, the first example is Milnor K-theory.

Thm: Milnor K-theory K^M_* is a cycle module.

[Ros, Rmk 2.4]

Proof: (Very sketchy)

(FD) is a known fact of classical algebra

(H) was proved by Milnor in [Mil]

(RC) for $X = \mathbb{P}^1$ is intrinsic in the def of norm map

(C) was proved by Kato using (RC) by passing through completions.

§ 4. THE FOUR BASIC MAPS

The idea is that, given X, Y schemes $/B$, M a cycle module $/B$, we would like to define a family of reasonable maps

$X \bullet\!\!\rightarrow Y$ called "generalized correspondences"

which are group hom → not graded! i.e. they don't have to respect the grading given by p.

$$C_*(X;M) \longrightarrow C_*(Y;M)$$

where $C_*(X;M) = \bigoplus_{p\geq 0} C_p(X;M)$ is the Rost complex associated to X and M.

These maps are a combination of sums and compositions of the "4 basic maps" we are going to define.

The reasonable map will also have some additional conditions that say that intuitively say they are "morphism of complexes" (indeed the condition is a (anti)commutativity with the differentials of Rost complex, depending on the "sign" of the map)

We are going to see all the definitions. The first thing to define is Rost complex. But first we need some notation.

Notation: Given M, N cycle modules $/B$, X, Y schemes $/B$, $X' \subset X$ and $Y' \subset Y$ subsets, for a group hom

$$\omega : \bigoplus_{x\in X'} M(x) \longrightarrow \bigoplus_{y\in Y'} N(y)$$

we denote by $\omega^x_y : M(x) \longrightarrow N(y)$ its components, for $x \in X'$, $y \in Y'$.

These are sufficient to describe ω b/c by universal property of $\oplus$ and Π they induce a group hom

$$\omega : \bigoplus_{x\in X'} M(x) \longrightarrow \prod_{y \in Y'} M(y).$$

The fact that ω lands in fact in the $\oplus$ tells that the group hom ω^x_y have the property:

recalls the condition given in (FD)! ⟵ Fixed $\xi \in M(x)$, $\omega^x_y(\xi) \neq 0$ for finitely many $y \in Y'$ ⊛

|| So if we want to give ω via ω^x_y, we need to check this property!

Def: Given M a cycle module $/B$, X a scheme $/B$, for any integer $p \geq 0$, we define

$$C_p(X;M) := \bigoplus_{x \in X_{(p)}} M(x)$$

indexed by dimension!

$$\downarrow d \qquad \text{s.t. } d^x_y := \partial^x_y$$

$$C_{p-1}(X;M) := \bigoplus_{y \in X_{(p-1)}} M(y)$$

recall was defined with codimension 1 points!

Notice that the definition makes sense b/c by axiom (FD), ∂^x_y satisfy property ⊛

$C_*(X;M) := \bigoplus_{p\geq 0} C_p(X;M)$ with the group endomorphism d is called the CYCLE (or ROST) COMPLEX of X WITH COEFFICIENTS IN M.

This is in fact a complex:

Lemma: $d \circ d = 0$, i.e. $(C_*(X;M), d)$ is a chain complex.

[Ros, lemma 3.3]

Proof:

Consider $d \circ d : C_{p+1}(X;M) \longrightarrow C_p(X;M) \longrightarrow C_{p-1}(X;M)$

$$\begin{array}{ccc} \| & \| & \| \\ \bigoplus_{x \in X_{(p+1)}} M(x) & \bigoplus_{y \in X_{(p)}} M(y) & \bigoplus_{z \in X_{(p-1)}} M(z) \end{array}$$

We prove that $(d\circ d)^x_z = 0 \quad \forall x \in X_{(p+1)},\ z \in X_{(p-1)}$.

$$(d\circ d)^x_z : M(x) \xrightarrow{\sum_y \partial^x_y} \bigoplus_{y\in X_{(p)}} M(y) \xrightarrow{\sum_y \partial^y_z} M(z)$$

If $z \notin \overline{\{x\}}$, then, for any $y \in X_{(p)}$ we have the following cases:

(i) $y \in \overline{\{x\}}$. Then $z \notin \overline{\{y\}}$ $\to$ otherwise $z \in \overline{\{y\}} \subset \overline{\{x\}}$

so $\partial^y_z = 0$, thus $(d\circ d)^x_z = 0$.

(ii) $y \notin \overline{\{x\}}$. Then $\partial^x_y = 0$, thus $(d \circ d)^x_z = 0$.

Otherwise, consider the case $z \in \overline{\{x\}}$. Let $Y := \operatorname{Spec} \mathcal{O}_{\overline{\{x\}},z}$.

we consider on $\overline{\{x\}}$ the reduced scheme structure, so $\overline{\{x\}}$ is an integral scheme.

$$\| \\ \{ y \in \overline{\{x\}} \mid z \in \overline{\{y\}} \}$$

Let X be a scheme.

Indeed, for any $x \in U \subseteq X$ (with $x = \mathfrak{p} \in \operatorname{Spec} A$), $\overline{\{x\}} \subset U$ and

$\operatorname{Spec} \mathcal{O}_{X,x} = \operatorname{Spec} A_{\mathfrak{p}} = \{ \mathfrak{q} \in \operatorname{Spec} A \mid \mathfrak{p} \subset \mathfrak{q} \iff \mathfrak{q} \in V(\mathfrak{p}) = \overline{\{x\}}^U = \overline{\{x\}} \} = \{ y \in U \mid y \in \overline{\{x\}} \}$

for every $Z, Z' \subseteq X$ closed irreducible,
any maximal chain of closed irreducible
btw Z and Z' are all of the same length.

$\parallel$

The conditions we considered on schemes imply that X is catenary, so:

$\dim Y = \dim \mathcal{O}_{\overline{\{x\}}, z} = 2$

$X_{(p)} \cap Y = Y_{(1)} = Y^{(1)}$

$\uparrow$ b/c Y is equidimensional

Recall: X is catenary $\Leftrightarrow$ $\mathcal{O}_{X,x}$ is catenary $\forall x \in X$

$\Downarrow$

Z is catenary $\forall Z \subseteq X$ closed

Notice that Y is integral, b/c $\overline{\{x\}}$ is, with generic point x.
is local, with closed point z.

Then, we can apply axiom (C) to Y: the composition

$$M(x) \xrightarrow{\sum_y \partial^x_y} \bigoplus_{y \in Y^{(1)}} M(y) \xrightarrow{\sum_y \partial^y_z} M(z)$$

is 0.

The components of this composition are exactly the $(d \circ d)^x_z$ in the case we are considering, hence they are all $= 0$.

$\blacksquare$

A remarkable fact is that a morphism of cycle modules $/X$

$$\alpha : M \longrightarrow N$$

induces a group hom btw the corresponding Rost complexes:

$$\alpha_{\#} : C_*(X; M) \longrightarrow C_*(X; N)$$

given by the $\oplus$ of

$$\alpha_{\#} : \underset{\bigoplus_{x \in X_{(p)}} M(x)}{C_p(X; M)} \longrightarrow \underset{\bigoplus_{y \in X_{(p)}} N(y)}{C_p(X; N)}$$

s.t. $(\alpha_{\#})^x_y := \begin{cases} \alpha_{k(x)} & \text{if } x = y \\ 0 & \text{else} \end{cases}$ $\alpha_{k(x)} : M(k(x)) \longrightarrow N(k(x))$, $M(k(x)) = M(x)$, $N(k(x)) = N(x)$

Notice that $\circledast$ is satisfied b/c $\forall x \in X_{(p)}\ \forall \xi \in M(x)$
$(\alpha_{\#})^x_y(\xi) \neq 0$ only for $y = x \in X_{(p)}$

This is just a functoriality of the Rost complex, is not one of the 4 basic maps, that we now define.

From now on, M is always a cycle module $/B$.

Def. Given $g: X \to Y$ a morphism of schemes $/B$, we define the group hom

$$g_*: C_*(X;\Pi) \to C_*(Y;\Pi) \quad \text{PUSHFORWARD MAP}$$

given by the $\oplus$ of

$$g_*: C_p(X;\Pi) \to C_p(Y;\Pi)$$
$$\begin{matrix} \| & & \| \\ \bigoplus_{x \in X_{(p)}} \Pi(x) & & \bigoplus_{y \in Y_{(p)}} \Pi(y) \end{matrix}$$

this is always true when $f: X \to Y$ is finite! Rost doesn't ask that the pushforward maps should be defined just for finite morph., but this finiteness condition is present here

$$\text{s.t. } (g_*)^x_y = \begin{cases} \varphi^* & \text{if } y = f(x) \text{ and } \varphi: k(y) \hookrightarrow k(x), \text{ the field extension } /B \text{ induced by } g, \text{ is finite} \\ 0 & \text{else} \end{cases}$$

Notice that $\circledast$ is satisfied b/c $\forall x \in X_{(p)}$ $\forall \xi \in \Pi(x)$ $(f_*)^x_y(\xi) \neq 0$ only for $y = f(x) \in Y_{(p)}$, in case $k(x)/k(y)$ is finite

Def. Given $f: Y \to X$ a morphism of schemes $/B$ of constant relative dim s, (fibers are $\emptyset$ or equidimensional of dim s)
we define the group hom,

$$f^*: C_*(X;\Pi) \to C_*(Y;\Pi) \quad \text{PULLBACK MAP}$$

given by the $\oplus$ of

$$f^*: C_p(X;\Pi) \to C_{p+s}(Y;\Pi)$$

which are defined as follows.

We do a more general construction that will be useful in future talks.

Given any $f: Y \to X$ morph. of schemes $/B$, let

$$s(f) := \max\{\dim(y) - \dim(f(y)) \mid y \in Y\}.$$

(Notice that for f of constant relative dim s, $s(f) = s$).

Let $\mathcal{A}$ be a coherent sheaf $/Y$.

For any $x \in X$, let $Y_x = Y \times_X \operatorname{Spec} k(x)$ denote the fiber.

For any $y \in Y_x^{(0)}$ we define the integer

$$[\mathcal{A}, f]^x_y := \operatorname{length}_R(\tilde{\mathcal{A}})$$

where $R := \mathcal{O}_{Y_x, y}$ and $\tilde{\mathcal{A}}$ is the f.g. R-module given by the pullback of $\mathcal{A}$ along $\operatorname{Spec} R = \operatorname{Spec} \mathcal{O}_{Y_x,y} \to Y_x \to Y$.

is an artinian ring b/c has dim 0 $\rightsquigarrow$ $\tilde{\mathcal{A}}$ f.g. R-mod has finite length

For any integer $s \geq s(g)$, we define the group hom
$$[A, f, s] : C_p(X; M) \longrightarrow C_{p+s}(Y; M)$$
s.t.
$$[A, f, s]^x_y := \begin{cases} [A, g]^x_y \cdot \varphi_* & \text{if } f(y) = x \\ 0 & \text{else} \end{cases}$$
, where $\varphi : k(x) \hookrightarrow k(y)$ is the field extension/B induced by f

Notice that ⊛ is satisfied b/c $\forall x \in X$ $\forall \xi \in M(x)$
$$[A, f, s]^x_y(\xi) \neq 0 \quad \text{only for } f(y) = x$$

We define f^* above as
$$f^* := [\mathcal{O}_Y, f, s].$$

Def: Given X a scheme /B, $a_1, \dots, a_n \in \mathcal{O}(X)^\times$ global units,
we define the group hom
$$\{a_1, \dots, a_n\} : C_*(X; M) \longrightarrow C_*(X; M)$$ MULTIPLICATION BY UNITS
given by the ⊕ of
$$\{a_1, \dots, a_n\} : \underset{\underset{x \in X_{(p)}}{\bigoplus} M(x)}{\underset{\shortparallel}{C_p(X; M)}} \longrightarrow \underset{\underset{y \in X_{(p-1)}}{\bigoplus} M(y)}{\underset{\shortparallel}{C_p(X; M)}}$$
s.t.
$$(\{a_1, \dots, a_n\})^x_y = \begin{cases} \underset{\underset{K^M_n(k(x))}{\cup}}{\{a_1(x), \dots, a_n(x)\}} \cdot \_ & \text{for } x = y \\ 0 & \text{else} \end{cases}$$
, where $\mathcal{O}(X)^\times \longrightarrow k(x)^\times$, $a_i \longmapsto a_i(x)$

Notice that ⊛ is satisfied b/c $\forall x \in X$ $\forall \xi \in M(x)$
$$(\{a_1, \dots, a_n\})^x_y(\xi) \neq 0 \quad \text{only for } y = x$$

Def: Given X a scheme /B, $i : Y \hookrightarrow X$ closed immersion, let
$j : U := X \setminus i(Y) \hookrightarrow X$ be the complementary open immersion.
Y and U are also schemes /B.
We define the group hom
$$\partial^U_Y : C_*(U; M) \longrightarrow C_*(Y; M)$$ BOUNDARY MAPS

given by the $\oplus$ of

$$\partial^U_Y : \underset{\substack{\| \\ \bigoplus\limits_{\substack{x \in U_{(p)} \\ \| \\ X_{(p)} \cap U}} \Pi(x)}}{C_p(U;\Pi)} \longrightarrow \underset{\substack{\| \\ \bigoplus\limits_{\substack{y \in Y_{(p)} \\ \| \\ X_{(p)} \cap Y}} \Pi(y)}}{C_{p-1}(Y;M)}$$

→ indeed the dimension doesn't depend on the ambient space!

s.t. $(\partial^U_Y)^x_y = \partial^x_y$. → the same that define the differential of the Rost complex!

All these operations that we defined, for the moment are just group homs btw the Rost complexes. In the following talk we will see some compatibilities with the differential d (they commute or anticommute with d, depending on a notion of sign).
This tells that they are also morphisms of chain complexes, and so will induce morphisms on homology groups, which we will also see in the next talk.