Lemarié--Rieusset
Talk 3 MSS25

Rost groups and Chow groups

After recollections on classical Chow groups, we will introduce Rost groups and show that the four basic maps induce maps on Rost groups. We will also present the localization long exact sequence which is an important tool (and will be used in the following talk to compute the Rost groups of $\mathbb{A}^n_F \setminus \{0\}$).

I Chow groups (Sources: 3264 & All That by Eisenbud and Harris & Intersection theory 2nd ed. by Fulton) Ch.1

Let F be a field and X be an F-scheme.

Def: The group of cycles (resp. k-cycles) on X, denoted $Z(X)$ (resp. $Z_k(X)$), is the free abelian group generated by the set S (resp. S_k) of reduced irreducible closed subschemes (integral) of X (resp. of dim. k): $Z(X) = \bigoplus_{\alpha \in S} \mathbb{Z}\alpha$ (resp. $Z_k(X) = \bigoplus_{\alpha \in S_k} \mathbb{Z}\alpha$).

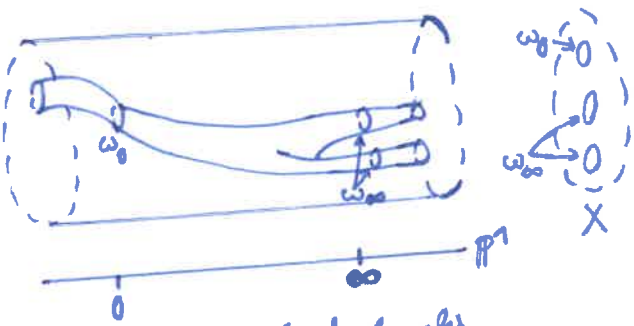
Def: The cycle $[Y]$ of a closed subscheme Y of X is: $[Y] = \sum_{i \in I} \ell_{\mathcal{O}_{Y,\xi_{Y_i}}}(\mathcal{O}_{Y,\xi_{Y_i}}) Y_i$ where $\{Y_i, i \in I\}$ is the set of irreducible components of Y_{red} (the reduced scheme ass. to Y).

Two cycles are rationally equivalent if there is a rationally parametrised family of cycles interpolating between them. More formally:

Def: Let $Rat(X)$ be the subgroup of $Z(X)$ generated by the $[\Phi \cap (\{t_0\} \times X)] - [\Phi \cap (\{t_1\} \times X)]$ with $t_0, t_1 \in \mathbb{P}^1$ and Φ an integral closed subscheme of $\mathbb{P}^1 \times X$ not contained in any fibre $\{t\} \times X$. Two cycles are rationally equivalent if their difference is in $Rat(X)$.
The Chow group of X is $CH(X) = Z(X)/Rat(X)$ (also denoted $A(X)$).

Prop: The Chow group is graded by dimension: $CH(X) = \bigoplus_k CH_k(X)$ (also denoted $A_k(X)$).

Rk: If X is smooth and equidimensional then we denote $CH^c(X) := CH_{\dim(X)-c}(X)$.



Rationally param. family of cycles interpolating between ω_0 and ω_∞

Def: If W is a $(k+1)$-dim. integral closed subscheme of X and r is a non-zero rational function on W then the divisor $[\mathrm{div}(r)]$ of r is: $[\mathrm{div}(r)] = \sum_V \mathrm{ord}_V(r) V$ with V ranging through codim. 1 integral closed subschemes of W and $\mathrm{ord}_V(r)$ the order of r at V ("zeros", "poles").

Prop: • $Rat(X)$ is the subgroup of $Z(X)$ gen. by the $[\mathrm{div}(r)]$ (for all k) and is graded: $Rat(X) = \bigoplus_k Rat_k(X)$ with $Rat_k(X)$ the subgroup of $Z_k(X)$ gen. by the $[\mathrm{div}(r)]$ for this k.
• $CH_k(X) = Z_k(X)/Rat_k(X)$.

II Rost groups (Source: Chow groups with coeff. by Rost)

Last talk, Rost's cycle complex was defined: $C_*(X; M) = \cdots \to C_{p+1}(X;M) \xrightarrow{d_{p+1}} C_p(X;M) \to \cdots$ (M: cycle module), with $C_{p+1}(X;M) = \bigoplus_{x \in X_{(p+1)}} M(K(x))$, $C_p(X;M) = \bigoplus_{y \in X_{(p)}} M(K(y))$

with $(d_{p+1})^x_y = \partial^x_y = \begin{cases} \sum_{\tilde{y}} c_{K(\tilde{y})/K(y)} \circ \partial_{\tilde{y}} & \text{if } y \in \overline{\{x\}}^{(1)} \text{ (and } \tilde{y} \text{ ranges through the points of the normalization of } \overline{\{x\}} \text{ over } y) \\ 0 & \text{if } y \notin \overline{\{x\}}^{(1)} \end{cases}$

($\widetilde{\overline{\{x\}}} \to \overline{\{x\}}$, $\tilde{y} \mapsto y$, induces $\varphi: K(y) \to K(\tilde{y})$ and $c_{K(\tilde{y})/K(y)} = \varphi^*: M(K(\tilde{y})) \to M(K(y))$; $\partial_{\tilde{y}} = \partial_v : M(K(x)) \to M(K(\tilde{y}))$, where $K(x) = \mathrm{Frac}(\mathcal{O}_{\widetilde{\overline{\{x\}}},\tilde{y}})$ and $\mathcal{O}_{\widetilde{\overline{\{x\}}},\tilde{y}}$ is a DVR (v))

Def: The Rost complex $C_*(X; M, q)$, $q \in \mathbb{Z}$, is: $\cdots \to C_{p+1}(X;M,q) \xrightarrow{d_{p+1}} C_p(X;M,q) \to \cdots$ with $C_{p+1}(X;M,q) = \bigoplus_{x \in X_{(p+1)}} M_{q+p+1}(K(x))$ and $C_p(X;M,q) = \bigoplus_{y \in X_{(p)}} M_{q+p}(K(y))$

(d_{p+1} as above; $C_*(X;M) = \bigoplus_{q \in \mathbb{Z}} C_*(X;M,q)$)

Ex: $C_p(X;K^M,-p)=\bigoplus_{y\in X_{(p)}} \underbrace{K^M_{-p+p}(K(y))}_{\mathbb{Z}}=Z_p(X)$; $\forall q<p$ $C_q(X;K^M,-p)=\bigoplus_{y\in X_{(q)}} K^M_{\underbrace{-p+q}_{<0}}(K(y))=0 \rightsquigarrow d_{q+1}=0$

$$C_{p+1}(X;K^M,-p)=\bigoplus_{x\in X_{(p+1)}} K^M_{-p+p+1}(K(x))=\bigoplus_{x\in X_{(p+1)}} K(x)^*$$

$d_{p+1}: C_{p+1}(X;K^M,-p)\to C_p(X;K^M,-p)$ is built from $\partial: \{u\pi^m\}\mapsto m$ the order (e.g. $\{x=0\}$ in $\mathbb{A}^1$-$\mathrm{Spec}(F[x])$ $\{ux^m\}\mapsto m$ order as a zero (if $m\geq 0$) or as a pole (if $m<0$))

↳ it's the divisor map so that its image is $\mathrm{Rat}_p(X)$

Thus, $CH_p(X)=\mathrm{Ker}(d_p)/\mathrm{Im}(d_{p+1})$.

Def: $A_p(X,M,q)=\mathrm{Ker}(d_p: C_p(X;M,q)\to C_{p-1}(X;M,q))/\mathrm{Im}(d_{p+1}: C_{p+1}(X;M,q)\to C_p(X;M,q))$ is the Rost group (or Chow group) of p-dim. cycles in X ass. to the cycle module M and the integer q.

$A_p(X;M)=\mathrm{Ker}(d_p: C_p(X;M)\to C_{p-1}(X;M))/\mathrm{Im}(d_{p+1}: C_{p+1}(X;M)\to C_p(X;M))$ is the Rost group (or Chow group) of p-dim. cycles in X with coefficients in M.

Rk: $A_p(X;K^M,-p)=CH_p(X)$.

III The four basic maps (Source: Chow groups with coefficients by Rost)

1 Boundary maps

Let (Y,i,X,j,U), or (Y,X,U) for short, be a boundary triple, i.e. $i:Y\hookrightarrow X$ and $j: U=X\setminus \overbrace{i(Y)}^{Y \text{ by abuse}}\hookrightarrow X$.

$C_p(X;M)=C_p(Y;M)\oplus C_p(U;M)$
$C_p(X;M,q)=C_p(Y;M,q)\oplus C_p(U;M,q)$
$(\partial^U_Y)^x_y=\partial^x_y$ if $x\in U$ and $y\in Y$

; if $y\in Y$ and $\eta\in U$ then $\partial^y_\eta=0$ since $\eta\notin\overline{\{y\}}^{(1)}\subset Y$ (closed) ($U=X\setminus Y$)

$\Rightarrow C_*(Y;M)$ is a subcomplex of $C_*(X;M)$ of quotient complex $C_*(U;M)$

($C_*(Y,M,q)$ is a subcomplex of $C_*(X,M,q)$ of quotient complex $C_*(U,M,q)$)

$$\sum_{x,y,\eta\in X}\partial^y_\eta\circ\partial^x_y=\underbrace{\sum_{x,y,\eta\in Y}\partial^y_\eta\circ\partial^x_y}_{=0 \text{ since } d_Y\circ d_Y=0}+\sum_{x\in U,\, y,\eta\in Y}\partial^y_\eta\circ\partial^x_y+\sum_{x,y\in U,\,\eta\in Y}\partial^y_\eta\circ\partial^x_y+\underbrace{\sum_{x,y,\eta\in U}\partial^y_\eta\circ\partial^x_y}_{=0 \text{ since } d_U\circ d_U=0}=0 \text{ since } d_X\circ d_X=0$$

hence $\sum_{x\in U,\,y,\eta\in Y}\partial^y_\eta\circ\partial^x_y=-\sum_{x,y\in U,\,\eta\in Y}\partial^y_\eta\circ\partial^x_y$ i.e. $d_Y\circ\partial^U_Y=-\partial^U_Y\circ d_U$. Thus, the boundary map ∂^U_Y induces a map on the homology of these complexes:

$\partial^U_Y(\mathrm{Ker}\, d^U_p)\subset \mathrm{Ker}\, d^Y_{p-1}$
$\partial^U_Y(\mathrm{Im}\, d^U_{p+1})\subset \mathrm{Im}\, d^Y_p$
$\partial^U_Y: A_p(U;M)\to A_{p-1}(Y;M)$ ✓
$\partial^U_Y: A_p(U;M,q)\to A_{p-1}(Y;M,q)$ ✓

Furthermore, we get the localization long exact sequence:

$$\cdots\to A_p(X;M)\xrightarrow{\text{quotient complex}} A_p(U;M)\xrightarrow{\partial^U_Y} A_{p-1}(Y;M)\xrightarrow{\text{subcomplex}} A_{p-1}(X;M)\to\cdots$$

(commonly known as the "exact triangle" theorem) and: $\cdots\to A_p(X;M,q)\xrightarrow{\text{qu. c.}} A_p(U;M,q)\xrightarrow{\partial^U_Y} A_{p-1}(Y;M,q)\xrightarrow{\text{subc.}} A_{p-1}(X;M,q)\to\cdots$

2 Multiplication with units

Let $a_1,\dots,a_n\in\mathcal{O}_X(X)^*$. For each $x\in X$, $a_1(x),\dots,a_n(x)\in K(x)^*$.

$\{a_1,\dots,a_n\}^x_y=\begin{cases}\{a_1(x),\dots,a_n(x)\}\times - & \text{if } x=y\\ 0 & \text{otherwise}\end{cases}$; $\{a_1,\dots,a_n\}=\{a_1\}\circ\cdots\circ\{a_n\}$.

If X is an F-scheme (with F a field), this turns each $C_p(X;M)$ into a $K^M_*(F)$-module (via $F^*\subset\mathcal{O}_X(X)^*$).

Let $a\in\mathcal{O}_X(X)^*$.

$$(d_X\circ\{a\})^x_y=\partial^x_y(\{a(x)\}\times -)\underset{\uparrow}{=}-\{a(y)\}\times\partial^x_y=(-\{a\}\circ d_X)^x_y \quad\text{thus } d_X\circ\{a\}=-\{a\}\circ d_X.$$

R3e: $\forall$ val. v on F, v-unit u, $\rho\in M(F)$
(A3e in Ronald's talk) $\partial_v(\{u\}\cdot\rho)=-\{\overline{u}\}\cdot\partial_v(\rho)$

Hence $d_X\circ\{a_1,\dots,a_n\}=(-1)^n\{a_1,\dots,a_n\}\circ d_X$. Thus
$\{a_1,\dots,a_n\}: A_p(X;M)\to A_p(X;M)$ ✓
$\{a_1,\dots,a_n\}: A_p(X;M,q)\to A_p(X;M,q+n)$ ✓

③ Proper push-forward

Let $f: X \to Y$ over B with X and Y of finite-type over a field.

$f_*: C_*(X;M) \to C_*(Y;M)$ is given by $(f_*)^x_y = \begin{cases} \varphi^* & \text{if } y = f(x) \text{ and } \varphi: K(y) \to K(x) \text{ all. to } f \text{ is finite} \\ 0 & \text{otherwise} \end{cases}$

We want to show that if f is proper then f_* commutes with the differentials.

Assume f is proper. $\quad \delta(f_*): C_p(X,M) \to C_{p-1}(Y,M)$

Let $\delta(f_*) = d_Y \circ f_* - f_* \circ d_X$. Let $x \in X_{(p)}$ and $y \in Y_{(p-1)}$. Let $\eta = f(x) \in Y_{(q)}$.

ⓐ If $y \notin \overline{\{\eta\}}$: $(d_Y \circ f_*)^x_y = 0$ since $(f_*)^x_b = 0$ if $b \neq \eta$ and $(d_Y)^\eta_y = 0$ (since $y \notin \overline{\{\eta\}}^{(1)}$)

and $(f_* \circ d_X)^x_y = 0$ since $(d_X)^x_b = 0$ if $b \notin \overline{\{x\}}^{(1)}$ and $\underbrace{(f_*)^b_y = 0 \text{ if } y \neq f(b)}$

hence $(\delta(f_*))^x_y = 0$. $\quad$ and $f(\overline{\{x\}}^{(1)}) \subset \overline{\{\eta\}}$

ⓑ If $y = \eta$: $(d_Y \circ f_*)^x_y = 0$ (as above)

and $(f_* \circ d_X)^x_y = 0$ by RC (reciprocity for curves): $\overline{\{x\}}$ being a proper curve over $K(y)$ (via $f|_{\overline{\{x\}}}^{\operatorname{Spec}(K(y))}$ $\leftarrow$ properness is local on the target) and M being a cycle module over B, $c \circ d = 0$

with $d = d_X \Big|_{M(K(x))}^{\bigoplus_{b \in \overline{\{x\}}^{(1)}} M(K(b))}$ and $c = f_* \Big|_{\bigoplus_{b \in \overline{\{\eta\}}^{(1)}} M(K(b))}^{M(K(y))}$.

ⓒ If $y \in \overline{\{\eta\}}$ and $y \neq \eta$: $\overset{\Rightarrow q \geq p}{q = p}$ and $\overset{\Rightarrow y \in \overline{\{\eta\}}^{(1)}}{\varphi: K(\eta) \to K(x)}$ all. to f is finite

we may replace Y with $\overline{\{\eta\}}$ and X with $\overline{\{x\}}$.

Let $g: \widetilde{\overline{\{x\}}} \to \overline{\{x\}}$ be the normalisation and $\tilde{x}$ be its generic point (so that $g(\tilde{x}) = x$).

Let $h: \widetilde{\overline{\{\eta\}}} \to \overline{\{\eta\}}$ be the normalisation and $\tilde{\eta}$ be its generic point (so that $h(\tilde{\eta}) = \eta$).

$\exists!\, \tilde{f}: \widetilde{\overline{\{x\}}} \to \widetilde{\overline{\{\eta\}}}$ s.t.

$$\begin{tikzcd} \widetilde{\overline{\{x\}}} \arrow[r, "g"] \arrow[d, "\tilde{f}"'] \arrow[dr, phantom, "\circlearrowleft"] & \overline{\{x\}} \arrow[d, "f"] \\ \widetilde{\overline{\{\eta\}}} \arrow[r, "h"'] & \overline{\{\eta\}} \end{tikzcd}$$

and $\tilde{f}$ is proper.

Let us show that $(\delta(f_*))^x_y \circ \underbrace{(g_*)^{\tilde{x}}_x}_{\text{iso.}} = \sum_{\tilde{y} \in \widetilde{\overline{\{\eta\}}}^{(1)}} (h_*)^{\tilde{y}}_y \circ \underbrace{(\delta(\tilde{f}_*))^{\tilde{x}}_{\tilde{y}}}_{\substack{\text{we will show} \\ \text{that } = 0}}$.

$$(\delta(f_*))^x_y \circ (g_*)^{\tilde{x}}_x = (d_Y)^\eta_y \circ (f_*)^x_\eta \circ (g_*)^{\tilde{x}}_x - \sum_{s \in \overline{\{x\}}^{(1)}} (f_*)^s_y \circ (d_X)^x_s \circ (g_*)^{\tilde{x}}_x$$

$\forall$ finite φ, ψ $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$ $\leftarrow$ R1b

$$= (d_Y)^\eta_y \circ (h_*)^{\tilde{\eta}}_\eta \circ (\tilde{f}_*)^{\tilde{x}}_{\tilde{\eta}} - \sum_s (f_*)^s_y \circ (d_X)^x_s \circ (g_*)^{\tilde{x}}_x$$

since h is the normalisation $\widetilde{\overline{\{\eta\}}} \to \overline{\{\eta\}}$ $\to \delta(h_*)|M(\tilde{\eta}) = 0$

$$= \sum_{\tilde{x} \in \widetilde{\overline{\{\eta\}}}^{(1)}} (h_*)^{\tilde{x}}_y \circ (d_{\widetilde{\overline{\{\eta\}}}})^{\tilde{\eta}}_{\tilde{x}} \circ (\tilde{f}_*)^{\tilde{x}}_{\tilde{\eta}} - \sum_s (f_*)^s_y \circ (d_X)^x_s \circ (g_*)^{\tilde{x}}_x$$

since g is the normalisation $\widetilde{\overline{\{x\}}} \to \overline{\{x\}}$ $\to \delta(g_*)|M(\tilde{x}) = 0$

$$= \sum_{\tilde{x}} (h_*)^{\tilde{x}}_y \circ (d_{\widetilde{\overline{\{\eta\}}}})^{\tilde{\eta}}_{\tilde{x}} \circ (\tilde{f}_*)^{\tilde{x}}_{\tilde{\eta}} - \sum_{\tilde{s} \in \widetilde{\overline{\{x\}}}^{(1)}} (f_*)^{g(\tilde{s})}_y \circ (g_*)^{\tilde{s}}_{g(\tilde{s})} \circ (d_{\widetilde{\overline{\{x\}}}})^{\tilde{x}}_{\tilde{s}}$$

$$\overset{\text{R1b}}{=} \sum_{\tilde{x}} (h_*)^{\tilde{x}}_y \circ (d_{\widetilde{\overline{\{\eta\}}}})^{\tilde{\eta}}_{\tilde{x}} \circ (\tilde{f}_*)^{\tilde{x}}_{\tilde{\eta}} - \sum_{\tilde{s}} (h_*)^{\tilde{f}(\tilde{s})}_y \circ (\tilde{f}_*)^{\tilde{s}}_{\tilde{f}(\tilde{s})} \circ (d_{\widetilde{\overline{\{x\}}}})^{\tilde{x}}_{\tilde{s}}$$

$$= \sum_{\tilde{y} \in \widetilde{\overline{\{\eta\}}}^{(1)}} (h_*)^{\tilde{y}}_y \circ (\delta(\tilde{f}_*))^{\tilde{x}}_{\tilde{y}}.$$

Note that every $\tilde{s} \in \widetilde{\overline{\{x\}}}$ such that $\tilde{f}(\tilde{s}) = \tilde{y} \in \widetilde{\overline{\{\eta\}}}^{(1)}$ is in $\widetilde{\overline{\{x\}}}^{(1)}$ so that $\mathcal{O}_{\widetilde{\overline{\{x\}}}, \tilde{s}}$ is a valuation ring (and $\mathcal{O}_{\widetilde{\overline{\{\eta\}}}, \tilde{y}}$ is a valuation ring). This, together with the properness of $\tilde{f}$ and R3b ($\forall\, \varphi: F \to E$ finite and val. v on F $\partial_v \circ \varphi^* = \sum_{\substack{w \text{ val. on } E \\ \text{extending } v}} \varphi_{wv}^* \circ \partial_w$ where $\varphi_{wv}: K(v) \to K(w)$ is induced by φ) imply that $(\delta(\tilde{f}_*))^{\tilde{x}}_{\tilde{y}} = 0$, so that $(\delta(f_*))^x_y = 0$.

Thus $\begin{cases} f_*: A_p(X;M) \to A_p(Y;M) \checkmark \\ f_*: A_p(X;M,q) \to A_p(Y,M,q) \checkmark \end{cases}$

④ Flat pull-back

Let $g : Y \to X$ be a morphism over B and $\mathcal{A}$ be a coherent sheaf on Y.

Let $s \geqslant s(g) := \max(\{p-q,\ y \in Y_{(p)},\ g(y) \in X_{(q)}\})$.

$[\mathcal{A}, g, s] : C_p(X; M) \to C_{p+s}(Y, M)$ is given by $([\mathcal{A},g,s])^x_y = \begin{cases} \overbrace{[\mathcal{A},g]^x_y}^{\in \mathbb{Z} \text{ (defined last talk)}} \varphi_* & \text{if } g(y) = x \text{ and } \varphi : K(x) \to K(y) \text{ is ass. to } g \\ 0 & \text{otherwise} \end{cases}$

If Y and X are of finite-type over a field and g is of relative dimension $s \in \mathbb{Z}$, i.e. all its fibres are either empty or equidimensional of dimension s, then $g^* := [\mathcal{O}_Y, g, s]$.

We want to show that if $\mathcal{A}$ is flat over X then $[\mathcal{A}, g, s]$ commutes with the differentials (so that if g is flat then g^* commutes with the differentials).

Assume $\mathcal{A}$ is flat over X.

Let $\delta = d_Y \circ [\mathcal{A}, g, s] - [\mathcal{A}, g, s] \circ d_X$. Let $x \in X_{(p)}$ and $y \in Y_{(p+s-1)}$. Let $\eta = g(y) \in X_{(q)}$.

ⓐ If $\eta \notin \overline{\{x\}}$: $([\mathcal{A},g,s] \circ d_X)^x_y = 0$ since $[\mathcal{A},g,s]^t_y = 0$ if $t \neq \eta$ and $(d_X)^x_\eta = 0$ (since $\eta \notin \overline{\{x\}}^{(1)}$)

and $(d_Y \circ [\mathcal{A},g,s])^x_y = 0$ since $(d_Y)^b_y = 0$ if $y \notin \overline{\{b\}}^{(1)}$ and $[\mathcal{A},g,s]^x_b = 0$ if $x \neq g(b)$ and $g(\overline{\{b\}}^{(1)}) \subset \overline{\{x\}}$

hence $\delta^x_y = 0$.

ⓑ If $\eta = x$: $([\mathcal{A},g,s] \circ d_X)^x_y = 0$ (as above)

and $(d_Y \circ [\mathcal{A},g,s])^x_y = 0$ by R3c (A3c in Linda's talk): $\forall \varphi : E \to F$ and v val. on F which is trivial on E, $\partial_v \circ \varphi_* = 0$ and the fact that: $\forall u \in Y$ s.t. $g(u) = x$ $\forall$ val. v on $K(u)$ with centre y (i.e. such that $\mathcal{O}_v$ dominates $\mathcal{O}_{Y,y}$, i.e. $\mathcal{O}_{Y,y} \subset \mathcal{O}_v$ and $m_{Y,y} = m_v \cap \mathcal{O}_{Y,y}$) v is trivial on $K(x)$

hence $\delta^x_y = 0$.

ⓒ If $\eta \in \overline{\{x\}}$ and $\eta \neq x$ ($\Rightarrow q \leq p-1$): note that $s \geqslant p+s-1-q$ i.e. $q \geqslant p-1$ hence $q = p-1$ i.e. $\eta \in X_{(p-1)}$.

We may replace X with $\overline{\{x\}}$ and Y with $\overline{\{y\}}$.

Let $f : \widetilde{\overline{\{x\}}} \to \overline{\{x\}}$ be the normalisation and $\tilde{x}$ be its generic point (so that $f(\tilde{x}) = x$).

Let us consider the pull-back diagram:

$$\begin{tikzcd} \widetilde{\overline{\{y\}}} \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\overline{\{x\}}} \arrow[d, "f"] \\ \overline{\{y\}} \arrow[r, "g"'] & \overline{\{x\}} \end{tikzcd}$$

These are all schemes of finite-type over a field.

f is proper hence f' is proper $\Rightarrow$ f_* and f'_* commute with the differentials (see ③ above)

$s \geqslant s(g')$ (actually $s = s(g')$)

(□) $[\mathcal{A}, g, s] \circ f_* = f'_* \circ [(f')^* \mathcal{A}, g', s]$ by the following lemma

<u>Lemma:</u> If
$$\begin{tikzcd} U \arrow[r, "g_2"] \arrow[d, "f_2"'] & Z \arrow[d, "f_1"] \\ V \arrow[r, "g_1"'] & T \end{tikzcd}$$
is a pull-back diagram of schemes of finite-type over a field, $\mathcal{B}$ is a coherent sheaf on V and $\lambda \geqslant s(g), s(g')$ then:

$[\mathcal{B}, g_1, \lambda] \circ (f_1)_* = (f_2)_* \circ [f_2^* \mathcal{B}, g_2, \lambda]$.

In particular if g_1 is flat then $g_1^* \circ (f_1)_* = (f_2)_* \circ g_2^*$.

<u>Proof:</u> It follows from R1c (A1c in Linda's talk): $\psi_* \circ \varphi^* = \sum_{p \in \operatorname{Spec}(\underbrace{L \otimes_F E}_{R})} \operatorname{length}(R_{(p)})\, (\varphi_p)^* \circ (\psi_p)_*$ with $\varphi : F \to E$ finite and $\psi : F \to L$, $\varphi_p : L \to R/p$, $\psi_p : E \to R/p$ canonical.

$$\underbrace{\delta^x_y \circ (f_*)^{\tilde{x}}_x}_{\text{iso.}} = (d_Y \circ [\mathcal{A},g,s] \circ f_* - [\mathcal{A},g,s] \circ d_X \circ f_*)^{|M(K(y))}_{|M(K(\tilde{x}))}$$

$= (d_Y \circ f'_* \circ [(f')^* \mathcal{A}, g', s] - [\mathcal{A},g,s] \circ d_X \circ f_*)^{|-}_{|-}$ by (□) above

$= (f'_* \circ d_{\widetilde{\overline{\{y\}}}} \circ [(f')^* \mathcal{A}, g', s] - [\mathcal{A},g,s] \circ d_X \circ f_*)^{|-}_{|-}$ since f'_* commutes with the diff.

$= (\text{———} - [\mathcal{A},g,s] \circ f_* \circ d_{\widetilde{\overline{\{x\}}}})^{|-}_{|-}$ since f_* commutes with the diff.

$= (\text{———} - f'_* \circ [(f')^* \mathcal{A}, g', s] \circ d_{\widetilde{\overline{\{x\}}}})^{|-}_{|-}$ by (□) above

$= (f'_*)^{\tilde{y}}_y \circ (d_{\widetilde{\overline{\{y\}}}} \circ [(f')^* \mathcal{A}, g', s] - [(f')^* \mathcal{A}, g', s] \circ d_{\widetilde{\overline{\{x\}}}})^{|M(K(\tilde{y}))}_{|M(K(\tilde{x}))}$

It is therefore enough to show that $\left(d_{\overline{\{y\}}} \circ [(f')^* \mathcal{A}, g', s] - [(f')^* \mathcal{A}, g', s] \circ d_{\overline{\{x\}}}\right)^{|M(\overline{\{y\}})|}_{|M(\overline{\{x\}})|} = 0$ in order to show that $\delta \hat{y} = 0$. Since $\overline{\{x\}}$ is normal, this follows from:

- R1a (A1a) : $\forall \varphi, \psi \quad (\psi \circ \varphi)_* = \psi_* \circ \varphi_*$
- R2d : $\forall \varphi$ finite $\varphi^* \circ \varphi_* = \deg(\varphi)\, \mathrm{Id}$ which follows from R2c (A2c) applied to $y = 1 \in \underbrace{K_0^M(E)}_{\simeq \mathbb{Z}}$
  R2c (A2c) : $\forall \varphi, F \to E$ finite, $y \in K_*^M(E), \rho \in M(F)$ $\varphi^*(y \cdot \varphi_*(\rho)) = \varphi^*(y) \cdot \rho$ $(\varphi^*(1) = \deg(\varphi))$
- R3a : $\forall \varphi, E \to F$ $\forall v$ val on F which restricts to a non-trivial val w on E with ramification index e (i.e. $m_w \mathcal{O}_v = m_v^e$) $\partial_v \circ \varphi_* = e \cdot \overline{\varphi}_* \circ \partial_w$ with $\overline{\varphi} : K(w) \to K(v)$ canonical
- the flatness of $\mathcal{A}$ over X
- tedious algebraic computations (see the bottom of page 353 and page 354 in Rost's article).

Thus $\begin{cases} [\mathcal{A}, g, s] : A_p(X; M) \to A_{p+s}(Y; M) \checkmark \\ [\mathcal{A}, g, s] : A_p(X; M, q) \to A_{p+s}(Y; M, q-s) \checkmark \end{cases}$

In particular, if g is flat:
$\begin{cases} g^* : A_p(X; M) \to A_{p+s}(Y; M) \checkmark \\ g^* : A_p(X; M, q) \to A_{p+s}(Y; M, q-s) \checkmark \end{cases}$ with s the relative dim. of g

<u>P.S.</u> : There are other results in Section 4 of Rost's article but they will not be used in this seminar (which skips Sections 6, 7 and 8 of Rost's article) except for Lemma 4.5 which will be stated and proved in the following talk.