TALK 4: HOMOTOPY INVARIANCE FOR ROST GROUPS [Ros, §9]

Reference:

[Ros] Rost "Chow groups with coefficients"

Recall from previous talks:

- Given X a scheme/k (scheme of finite type/k) and a cycle module M over k, we defined the associated Rost complex

$$(C_\bullet(X;M), d).$$

$$C_p(X;M) = \bigoplus_{x\in X_{(p)}} M(x) \quad \Big(= \bigoplus_{j\in\mathbb{Z}} M_j(x) \Big)$$

and its $\mathbb{Z}$-grading induced by the $\mathbb{Z}$-grading of M

$$C_\bullet(X;M) = \bigoplus_{j\in\mathbb{Z}} C_\bullet(X;M,j).$$

$$C_p(X;M,j) = \bigoplus_{x\in X_{(p)}} M_{p+j}(x) \quad (\cup)$$

The homology groups are the Rost groups

$$A_p(X;M) := H_p(C_\bullet(X;M)) \qquad A_p(X;M,j) := H_p(C_\bullet(X;M,j))$$

They are a generalization "with coefficients" of the Chow groups $CH_p(X)$: taking the cycle module given by Milnor K-theory $M = K^M_*$, we have that

$$CH_p(X) \cong A_p(X; K^M_*, -p)$$

- We defined 4 basic maps between Rost complexes (f^*, g_*, $\{a_1,\dots,a_n\}$, ∂^U_Y) and proved that:

Remember we don't need the hp. of flat/proper to define the maps, but we need them to have morph. of complexes!

I) for any $f: Y \to X$ flat morphism of schemes/k of constant relative dimension s, the pullback map

$$f^*: C_\bullet(X;M,j) \to C_{\bullet+s}(X;M,j-s)$$

commutes with differentials ($f^* \circ d = d\circ f^*$)

II) for any $g: X \to Y$ proper morphism of schemes/k, the pushforward map

$$g_*: C_\bullet(X;M,j) \to C_\bullet(Y,M,j)$$

commutes with differentials. ($g_* \circ d = d\circ g_*$)

III) for any $a\in \mathcal{O}(X)^\times$, X scheme/k, the multiplication by unit map

$$\{a\}: C_\bullet(X;M,j) \to C_\bullet(X;M,j+1)$$

anticommutes with differentials. ($\{a\}\circ d = -d\circ\{a\}$)

Notice that it follows: $\{a_1,\dots,a_n\}\circ d = (-1)^n d\circ\{a_1,\dots,a_n\}$

b/c $\{a_1,\dots,a_n\} = \{a_1\}\circ\dots\circ\{a_n\} : C_\bullet(X;M,j) \to C_\bullet(X;M,j+n)$

IV) for any boundary triple (X, i, Y, j, U), the boundary map

$$\partial^U_Y : C_\bullet(U; \Pi, j) \longrightarrow C_{\bullet-1}(Y; \Pi, j)$$

anticommutes with differentials. $(\partial^U_Y \circ d = - d \circ \partial^U_Y)$

The (anti) commutativity with differentials allows to have induced morphisms on the homology groups, that is, btw the Rost groups.

In this talk we are interested in the morphism of schemes/k given by a vector bundle of rank n

$$\pi : V \longrightarrow X.$$

Since it is a flat morphism with constant relative dimension n, we have the induced pullback map btw the Rost groups:

$$\pi^* : A_p(X; \Pi, j) \longrightarrow A_{p+n}(V; \Pi, j-n)$$

For $\Pi = K^M_*$ and $j = -p$, this is the pullback map btw the Chow groups

$$\pi^* : CH_p(X) \longrightarrow CH_{p+n}(V).$$

In classical intersection theory, this is proved to be an isomorphism (is the homotopy invariance property for Chow groups, see Fulton "Intersection theory", Thm 3.3).

The aim of this talk is to prove the analogous property for Rost groups, that is, the pullback map

$$\pi^* : A_p(X; \Pi, j) \longrightarrow A_{p+n}(V; \Pi, j-n)$$

is an isomorphism.

This is [Ros, Prop. 8.6] and it is first proved by working directly on Rost groups. A second proof, that is the one that we present today, instead works at the level of Rost complex, by showing that π^* is an homotopy equivalence of complexes. This is useful for computations!

To do it, we have to construct an homotopy inverse with homotopies.

We do it in progressively more general cases:

§1. for the trivial vector bundle of rank 1 $\pi : \mathbb{A}^1 \times X \to X$.

§2. by induction, for the trivial vector bundle of rank n $\pi : \mathbb{A}^n \times X \longrightarrow X$.

§3. by glueing on boundary triples, in the general case.

b/c we need it for knots!

Then, we will compute an example that will be also useful in the next talks: the Rost groups of $\mathbb{A}^n \setminus 0$.

But first we give precise definition and the statement of what we want to prove.

§0. Preliminary definitions

Given X, Y schemes/k and a graded group hom

$$\alpha : C_\bullet(X; n, j) \longrightarrow C_{\bullet+r}(Y; n, j+s)$$

this means that $\forall p \in \mathbb{Z}$, α restricts to

$$\alpha : C_p(X; n, j) \longrightarrow C_{p+r}(Y; n, j+s)$$

We will always consider maps of the this kind and we denote them by $\alpha : X \bullet\!\!\rightarrow Y$.

We define the SIGN of α

$$\mathrm{sgn}(\alpha) := (-1)^{r+s}.$$

Notice that the 4 basic maps are of this kind and

$$\mathrm{sgn}(f^*) = (-1)^{s-s} = 1 \qquad \mathrm{sgn}(g_*) = (-1)^{0+0} = 1$$

$$\mathrm{sgn}(\{a_1, \dots, a_n\}) = (-1)^{0+n} = (-1)^n \qquad \mathrm{sgn}(\partial^u_y) = (-1)^{-1+0} = -1$$

We define $\delta(\alpha) := d \circ \alpha + \mathrm{sgn}(\alpha)\, \alpha \circ d$.

Notice that in the last talk we proved that δ of the 4 basic maps (in the flat case for pullback and proper case for pushforward) is $=0$.

Intuitively, having $\delta(\alpha) = 0$ means that α induces morphisms on Rost groups.

we mean that α and β are of the kind above!

Notice that given $X \overset{\alpha}{\bullet\!\!\longrightarrow} Y \overset{\beta}{\bullet\!\!\longrightarrow} Z$, then also the composition is a map of the kind we consider $\beta \circ \alpha : X \bullet\!\!\rightarrow Z$, and δ satisfies a Leibnitz rule

$$\delta(\beta \circ \alpha) = \delta(\beta) \circ \alpha + \mathrm{sgn}(\beta)\, \beta \circ \delta(\alpha).$$

Now we give the precise definition and the statement.

Def: Given a map $\alpha : X \multimap Y$ s.t. $\delta(\alpha)=0$, we say that α is a STRONG HOMOTOPY EQUIVALENCE if there exist maps $r : Y \multimap X$ and $H : Y \multimap Y$ s.t.

(1) $\delta(r)=0$ "r induces morphisms btw Rost groups"

(2) $r\circ\alpha = id_X = (id_X)_*$ "r is a retraction of α"

(3) $H\circ\alpha=0$ → this is a mysterious condition that we will need in §3!

(4) $\delta(H) = id_Y - \alpha\circ r$ "H is an homotopy btw id_Y and $\alpha\circ r$"

The pair (r,H) is called an h-DATA for α.

we don't ask for the existence of a homotopy btw id_X and $r\circ\alpha$! That's why α is called STRONG

In this talk we want to prove:

Prop: Let X be a scheme/k and $\pi : V \to X$ an algebraic vector bundle. Then $\pi^* : X \multimap V$ is a strong homotopy equivalence.

Notice that $\alpha : X \multimap Y$ being a strong homotopy equivalence with h-data (r,H) implies that α is an homotopy equivalence of complexes with quasi-inverse r. Hence it induces isomorphisms on the Rost groups.

Cor: In the same hp of the last proposition, the induced morphisms on Rost groups

$$\pi^* : A_p(X;M,j) \to A_{p+n}(V;M,j-n)$$

are isomorphism.

Now, we prove the proposition. The proof uses some compatibilities of the 4 basic maps from [Ros, §4.] that we haven't proved (we just did compatibility with differentials). I'll mention them when I need.

§1. Case $\pi : \mathbb{A}^1 \times X \longrightarrow X$

We define r and H and we prove that they are an h-data for π^*.

$$r : \mathbb{A}^1 \times X \xrightarrow{j^*} (\mathbb{A}^1 \setminus 0) \times X \xrightarrow{\{-\frac{1}{t}\}} (\mathbb{A}^1 \setminus 0) \times X \xrightarrow{\partial} X$$

$$H : \mathbb{A}^1 \times X \xrightarrow{p_2^*} (\mathbb{A}^1 \times \mathbb{A}^1 \setminus \Delta) \times X \xrightarrow{\{s-t\}} (\mathbb{A}^1 \times \mathbb{A}^1 \setminus \Delta) \times X \xrightarrow{p_{1*}} \mathbb{A}^1 \times X$$

where
- $j : (\mathbb{A}^1 \setminus 0 \hookrightarrow \mathbb{A}^1) \times X$ is the open immersion has rel dim 0!
- $t \in \mathbb{Z}[t] = \mathcal{O}(\mathbb{A}^1) \rightsquigarrow -\frac{1}{t} \in \mathcal{O}(\mathbb{A}^1)^\times \rightsquigarrow -\frac{1}{t} \in \mathcal{O}(\mathbb{A}^1 \times X)^\times$
  via $\mathcal{O}(\mathbb{A}^1) \longrightarrow \mathcal{O}(\mathbb{A}^1 \times X)$
- ∂ is the boundary map for $(\infty \hookrightarrow \mathbb{P}^1 \setminus 0) \times X$
  we mean for the boundary triple given by this closed immersion and its open complement
  we already compose with this iso!
  $\partial_\infty : (\mathbb{A}^1 \setminus 0) \times X = \mathbb{P}^1 \setminus \{0, \infty\} \times X \longrightarrow \infty \times X \cong X$
- $p_1 : (\mathbb{A}^1 \times \mathbb{A}^1 \setminus \Delta \longrightarrow \mathbb{A}^1) \times X$ is the projection on the 1st comp.
- $p_2 : (\mathbb{A}^1 \times \mathbb{A}^1 \setminus \Delta \longrightarrow \mathbb{A}^1) \times X$ " " " " 2nd "
  has rel. dim 1!
- $s-t \in \mathbb{Z}[s,t] = \mathcal{O}(\mathbb{A}^1 \times \mathbb{A}^1)$ is a global parameter defining $\Delta = V(s-t) \rightsquigarrow s-t \in \mathcal{O}(\mathbb{A}^1 \times \mathbb{A}^1 \setminus \Delta)^\times \rightsquigarrow s-t \in \mathcal{O}((\mathbb{A}^1 \times \mathbb{A}^1 \setminus \Delta) \times X)^\times$
  via $\mathcal{O}(\mathbb{A}^1 \times \mathbb{A}^1 \setminus \Delta) \longrightarrow \mathcal{O}((\mathbb{A}^1 \times \mathbb{A}^1 \setminus \Delta) \times X)$

Notice that (omitting X and π)

$r : C_p(\mathbb{A}^1, j) \longrightarrow C_p(\mathbb{A}^1 \setminus 0, j) \longrightarrow C_p(\mathbb{A}^1 \setminus 0, j+1) \longrightarrow C_{p-1}(\mathbb{A}^1, j+1)$

$H : C_p(\mathbb{A}^1, j) \longrightarrow C_{p+1}(\mathbb{A}^1 \times \mathbb{A}^1 \setminus \Delta, j-1) \longrightarrow C_{p+1}(\mathbb{A}^1 \times \mathbb{A}^1 \setminus \Delta, j) \longrightarrow C_{p+1}(\mathbb{A}^1, j)$

We prove the properties of h-data:

(1) $(\delta(r) = 0)$

Since j is flat, r is composition of maps with $\delta = 0$.
By Leibnitz rule for δ, this implies that $\delta(r) = 0$.

(2) $\left(r \circ \pi^* = \mathrm{id}_X \right)$

We use a lemma about compatibility of the 4 basic maps:

Lemma: [Ros, Lemma 4.5]

Let • $f : Y \to X$ be a smooth morphism of schemes/k of constant relative dimension 1
• $\sigma : X \to Y$ a section of f $\quad (f \circ \sigma = \mathrm{id}_X)$
• $t \in \mathcal{O}(Y)^\times$ a global parameter defining the closed subscheme $\sigma(X) \subsetneq Y$

Then
$$\partial \circ \{t\} \circ \tilde{f}^* = \mathrm{id}_X$$
where • ∂ is the boundary map for $\sigma(X) \hookrightarrow Y$
• $\tilde{f} := f|_{Y \setminus \sigma(X)} : Y \setminus \sigma(X) \longrightarrow X$

$\partial : Y \setminus \sigma(X) \longmapsto \sigma(X) \xrightarrow{\tilde{\sigma}_*^{-1}} X$

actually, to get id_X, we have to pre-compose with $\tilde{\sigma}_*^{-1}$ which is the iso given by the pushforward map of the iso $\tilde{\sigma} : X \xrightarrow{\simeq} \sigma(X)$, which is σ restricted to image

We apply it to

$Y \rightsquigarrow (\mathbb{P}^1 \setminus 0) \times X$

$X \rightsquigarrow X$

$f \rightsquigarrow (\mathbb{P}^1 \setminus 0) \times X \to X$ projection on X

$\sigma \rightsquigarrow \sigma : (\infty \hookrightarrow \mathbb{P}^1 \setminus 0) \times X$ $\qquad \sigma(X) \xleftarrow[\simeq]{\tilde{\sigma}} X$

$t \rightsquigarrow -\frac{1}{t} \in \mathcal{O}((\mathbb{P}^1 \setminus 0) \times X)$, it defines $\infty \times X \subsetneq (\mathbb{P}^1 \setminus 0) \times X$

$\rightsquigarrow$ image of $-\frac{1}{t} \in \mathcal{O}(\mathbb{A}^1 \setminus 0) \subset \mathcal{O}(\mathbb{P}^1 \setminus 0)$ along $\mathcal{O}((\mathbb{P}^1 \setminus 0) \times X) \to \mathcal{O}(\mathbb{P}^1 \setminus 0)$

Notice that then $\tilde{f}$ is

$$\begin{tikzcd} (\mathbb{P}^1 \setminus \{0,\infty\}) \times X = (\mathbb{A}^1 \setminus 0) \times X \arrow[rr] \arrow[dr, "j"'] & & X \\ & \mathbb{A}^1 \times X \arrow[ur, "\pi"'] & \end{tikzcd}$$

projection on X

and that ∂ is ∂.

Hence, by the lemma,
$$\partial \circ \{-\tfrac{1}{t}\} \circ (\pi \circ j)^* = \mathrm{id}_X$$
$$\| $$
$$\underbrace{\partial \circ \{-\tfrac{1}{t}\} \circ j^*}_{= r} \circ \pi^*$$

(3) ($H \circ \pi^* = 0$)

We can see that the composition is 0 by reasoning on dimensions.

The composition is:

$$\begin{tikzcd} H: \mathbb{A}^1 \times X \arrow[r, "p_2^*"] & (\mathbb{A}^1 \times \mathbb{A}^1 \setminus \Delta) \times X \arrow[r, "\{s-t\}"] & (\mathbb{A}^1 \times \mathbb{A}^1 \setminus \Delta) \times X \arrow[r, "p_{1*}"] & \mathbb{A}^1 \times X \\ X \arrow[u, "\pi^*"] \end{tikzcd}$$

Let $x \in X_{(p)}$.

Let $\xi \in \mathbb{A}^1 \times \{x\}$ be the generic point. Then $\xi \in ((\mathbb{A}^1 \setminus 0) \times X)_{(p+1)}$.

Let $\xi' \in (\mathbb{A}^1 \times \mathbb{A}^1 \setminus \Delta) \times \{x\}$ be the " " . Then $\xi' \in ((\mathbb{A}^1 \times \mathbb{A}^1 \setminus \Delta) \times X)_{(p+2)}$.

Then $w := p_2(\xi') \in \mathbb{A}^1 \times \{x\}$ is the " " . Then $w \in (\mathbb{A}^1 \times X)_{(p+1)}$.

b/c p_2 takes the generic point into the generic point!

Following the def of the 4 basic maps, we see that the composition is s.t.

$$\begin{array}{ccccccc} \Pi(\xi) \subset C_{p+1}(\mathbb{A}^1 \times X; \Pi) & \xrightarrow{p_2^*} & C_{p+2}((\mathbb{A}^1 \times \mathbb{A}^1 \setminus \Delta) \times X; \Pi) & \xrightarrow{\{s-t\}} & C_{p+2}((\mathbb{A}^1 \times \mathbb{A}^1 \setminus \Delta) \times X; \Pi) & \xrightarrow{p_1^*} & C_q(\mathbb{A}^1 \times X, M), \quad q \leq p+1 \\ \uparrow \pi^* & & \cup & & \cup & & \cup \\ & & \Pi(\xi') & & \Pi(\xi') & & \Pi(w) \\ M(x) \subset C_p(X; \Pi) & & & & & & \end{array}$$

But then the last map is 0 b/c the pushforward map keeps the dimension.

So the composition is 0.

Notice that $H \neq 0$ b/c we could start with $y \in (\mathbb{A}^1 \times X)_{(p)}$ s.t. $\pi(y) = x \in X_{(p)}$, so that H is given by

$$\begin{array}{ccccccc} C_p(\mathbb{A}^1 \times X; \Pi) & \xrightarrow{p_2^*} & C_{p+1}((\mathbb{A}^1 \times \mathbb{A}^1 \setminus \Delta) \times X; \Pi) & \xrightarrow{\{s-t\}} & C_{p+1}((\mathbb{A}^1 \times \mathbb{A}^1 \setminus \Delta) \times X; \Pi) & \xrightarrow{p_1^*} & C_{p+1}(\mathbb{A}^1 \times X; \Pi) \\ \cup & & \cup & & \cup & & \cup \\ \Pi(y) & & \Pi(\xi') & & \Pi(\xi') & & \Pi(w) \end{array}$$

(4) $(\delta(H) = \mathrm{id}_{\mathbb{A}^1\times X} - \pi^*\circ r)$

By Leibnitz rule for δ (iteratively) we get:

$$\delta(H) = \delta(p_{1*})\circ\{s-t\}\circ p_2^* + \underbrace{\mathrm{sgn}(p_{1*})}_{=1}\, p_{1*}\circ\delta(\{s-t\}\circ p_2^*) =$$

$$= \delta(p_{1*})\circ\{s-t\}\circ p_2^* + p_{1*}\circ\underbrace{\delta(\{s-t\})}_{=0}\circ p_2^* + \mathrm{sgn}\{s-t\}\, p_{1*}\circ\{s-t\}\circ\underbrace{\delta(p_2^*)}_{=0} =$$

b/c p_2^* is flat

$$= \delta(p_{1*})\circ\{s-t\}\circ p_2^*$$

Notice $\delta(p_{1*})\neq 0$ b/c p_1 is <u>not</u> proper!

We need to rewrite $\delta(p_{1*})$. We consider the factorization of p_1

$$p_1 : (\mathbb{A}^1\times\mathbb{A}^1\setminus\Delta)\times X \xrightarrow{q} \mathbb{A}^1\times\mathbb{P}^1\times X \xrightarrow{\overline{p}_1} \mathbb{A}^1\times X$$

where
- $q : (\mathbb{A}^1\times\mathbb{A}^1\setminus\Delta \hookrightarrow \mathbb{A}^1\times\mathbb{P}^1)\times X$ is the open immersion
- $\overline{p}_1 : (\mathbb{A}^1\times\mathbb{P}^1 \to \mathbb{A}^1)\times X$ is the projection on the 1^{st} component

By functoriality of the pushforward map (see [Ros, Prop 4.1]), we get the factorization of p_{1*}

$(g'\circ g)_* = g'_*\circ g_*$

$$p_{1*} : (\mathbb{A}^1\times\mathbb{A}^1\setminus\Delta)\times X \overset{q_*}{\bullet\!\!\longrightarrow} \mathbb{A}^1\times\mathbb{P}^1\times X \overset{\overline{p}_{1*}}{\bullet\!\!\longrightarrow} \mathbb{A}^1\times X$$

Then, using again Leibnitz formula, we get

$$\delta(p_{1*}) = \underbrace{\delta(\overline{p}_{1*})}_{=0}\circ q_* + \underbrace{\mathrm{sgn}(\overline{p}_{1*})}_{=1}\,\overline{p}_{1*}\circ\delta(q_*) =$$

b/c $\overline{p}_1$ is proper

$$= \overline{p}_{1*}\circ\delta(q_*)$$

We go on rewriting $\delta(q_*) = d\circ q_* - \underbrace{\mathrm{sgn}(q_*)}_{=1}\, q_*\circ d$.

Notice that we have the decomposition

$$(\mathbb{A}^1\times\mathbb{P}^1 = \underbrace{\mathbb{A}^1\times\mathbb{A}^1\setminus\Delta}_{\text{open}} \sqcup \underbrace{\mathbb{A}^1\times\infty \sqcup \Delta}_{\text{closed}})\times X$$

and denote by

$i_\Delta : (\Delta \hookrightarrow \mathbb{A}^1\times\mathbb{P}^1)\times X$, $i_\infty : (\mathbb{A}^1\times\infty \hookrightarrow \mathbb{A}^1\times\mathbb{P}^1)\times X$ the 2 closed immersions

$j_\Delta : (\mathbb{A}^1\times\mathbb{P}^1\setminus\Delta \hookrightarrow \mathbb{A}^1\times\mathbb{P}^1)\times X$, $j_\infty : (\mathbb{A}^1\times\mathbb{P}^1\setminus\mathbb{A}^1\times\infty = \mathbb{A}^1\times\mathbb{A}^1 \hookrightarrow \mathbb{A}^1\times\mathbb{P}^1)\times X$ the open complements

and by ∂_Δ and ∂_∞ the corresponding boundary maps.

$\partial_\Delta : (\mathbb{A}^1\times\mathbb{P}^1\setminus\Delta)\times X \bullet\!\!\longrightarrow \Delta\times X$, $\partial_\infty : \mathbb{A}^1\times\mathbb{A}^1\times X = \mathbb{A}^1\times(\mathbb{P}^1\setminus\infty)\times X \bullet\!\!\longrightarrow \mathbb{A}^1\times\infty\times X$.

So, we have the decomposition

$$C_p((\mathbb{A}^1\times\mathbb{P}^1)\times X; M) \cong \begin{array}{l} \xleftarrow{q_*} C_p((\mathbb{A}^1\times\mathbb{A}^1\setminus\Delta)\times X; M) \\ \oplus \\ \xleftarrow{i_{\Delta *}} C_p(\Delta\times X; M) \\ \oplus \\ \xleftarrow{i_{\infty *}} C_p((\mathbb{A}^1\times\infty)\times X; M) \end{array}$$

Given a point $w \in ((\mathbb{A}^1 \times \mathbb{A}^1 \setminus \Delta) \times X)_{(p)}$ denote by z its closure in $\mathbb{A}^1 \times \mathbb{P}^1 \times X$. Then we have:

$$
\begin{tikzcd}
C_p((\mathbb{A}^1\times\mathbb{A}^1\setminus\Delta)\times X;\Pi) \supset M(w) \arrow[r, "d"] \arrow[d, "id = q_*"'] \arrow[dr, "q_*\circ d"] & \bigoplus\limits_{t \in z^{(1)} \cap (\mathbb{A}^1\times\mathbb{A}^1\setminus\Delta)\times X} \Pi(t) \arrow[d, "q_*"] & \subset C_{p-1}((\mathbb{A}^1\times\mathbb{A}^1\setminus\Delta)\times X;\Pi) \\
C_p(\mathbb{P}^1\times\mathbb{A}^1\times X;\Pi) \supset M(w) \arrow[r, "d"] \arrow[dr, "i_{\Delta *}\circ\partial_\Delta"'] \arrow[ddr, "i_{\infty *}\circ\partial_\infty"'] & \boxed{\bigoplus\limits_{t \in z^{(1)} \cap (\mathbb{A}^1\times\mathbb{A}^1\setminus\Delta)\times X} \Pi(t)} & \subset C_{p-1}((\mathbb{A}^1\times\mathbb{A}^1\setminus\Delta)\times X;\Pi) \\
& \oplus\ \bigoplus\limits_{t \in z^{(1)} \cap \Delta} \Pi(t) & \subset C_{p-1}(\Delta\times X;\Pi) \\
& \oplus\ \bigoplus\limits_{t \in z^{(1)} \cap \mathbb{A}^1\times\infty} \Pi(t) & \subset C_{p-1}((\mathbb{A}^1\times\infty)\times X;\Pi)
\end{tikzcd}
$$

q_* lands inside here

$$\Big\} = C_{p-1}(\mathbb{P}^1\times\mathbb{A}^1\times X;\Pi)$$

the components are these bc d and $\partial_\Delta, \partial_\infty$ have components defined in the same way. But we need to compose with the pullback and pushforward maps to see $\partial_\Delta, \partial_\infty$ inside the big space

This diagram tells that

$$d \circ q_* = q_* \circ d + i_{\Delta *} \circ \partial_\Delta + i_{\infty *} \circ \partial_\infty$$

that is,

Notice that to be precise we should precompose here with pushforward map of $\mathbb{A}^1 \times \mathbb{A}^1 \setminus \Delta \to \mathbb{A}^1 \times \mathbb{P}^1 \setminus \Delta$, $\mathbb{A}^1 \times \mathbb{A}^1 \setminus \Delta \to \mathbb{A}^1 \times \mathbb{A}^1$

$$\delta(q_*) = d \circ q_* - q_* \circ d = i_{\Delta *} \circ \partial_\Delta + i_{\infty *} \circ \partial_\infty$$

but we omit it since our components are the identity (as for q_*)

Putting together the equalities found until now we get

$$\delta(H) = \delta(p_{1*}) \circ \{s-t\} \circ p_2^* =$$

$$= \bar{p}_{1*} \circ \delta(q) \circ \{s-t\} \circ p_2^* =$$

$$= \underbrace{\bar{p}_{1*} \circ i_{\Delta *} \circ \partial_\Delta \circ \{s-t\} \circ p_2^*}_{\text{I}} + \underbrace{\bar{p}_{1*} \circ i_{\infty *} \circ \partial_\infty \circ \{s-t\} \circ p_2^*}_{\text{II}}$$

We now show $\text{I} = id_{\mathbb{A}^1 \times X}$ and $\text{II} = -\pi^* \circ r$, concluding the proof.

I We apply again [Ros, Lemma 4.5] to

$Y \rightsquigarrow \mathbb{A}^1 \times \mathbb{A}^1 \times X$

$X \rightsquigarrow \mathbb{A}^1 \times X$

$f \rightsquigarrow (\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1) \times X$ projection on the 1^{st} component

$\sigma \rightsquigarrow \delta : (\mathbb{A}^1 \to \mathbb{A}^1 \times \mathbb{A}^1) \times X$ the diagonal $\sigma(\mathbb{A}^1) \overset{\tilde{\delta}}{\underset{\cong}{\leftarrow}} \mathbb{A}^1$

$t \rightsquigarrow s-t \in \mathcal{O}(\mathbb{A}^1 \times \mathbb{A}^1 \times X)$, it defines $\Delta \times X \subseteq \mathbb{A}^1 \times \mathbb{A}^1 \times X$

$\hookrightarrow$ image of $s-t \in \mathcal{O}(\mathbb{A}^1 \times \mathbb{A}^1)$ along $\mathcal{O}(\mathbb{A}^1 \times \mathbb{A}^1) \to \mathcal{O}(\mathbb{A}^1 \times \mathbb{A}^1 \times X)$

Notice that then $\tilde{f}$ is

$$p_2 : (\mathbb{A}^1 \times \mathbb{A}^1 \setminus \Delta \longrightarrow \mathbb{A}^1) \times X$$

and ∂ is $\underline{\bar{p}_{1*} \circ i_{\Delta *}} \circ \partial_\Delta$

recall that this is in fact $X \xrightarrow[\simeq]{\tilde{\vartheta}_*^{-1}} \vartheta(X) \xrightarrow{\partial} Y \setminus \vartheta(X)$

$= (\bar{p}_1 \circ i_\Delta)_*$ is the $\tilde{\vartheta}_*^{-1}$

in fact $\bar{p}_1 \circ i_\Delta : (\Delta \longrightarrow \mathbb{A}^1) \times X$ is an iso inverse of $\tilde{\vartheta} : (\mathbb{A}^1 \longrightarrow \Delta) \times X$

Hence, by the lemma

$$\bar{p}_{1*} \circ i_{\Delta *} \circ \partial_\Delta \circ \{s-t\} \circ p_2^* = id_{\mathbb{A}^1 \times X}$$

Idea: we need to replace $\{s-t\}$ with $\{-\frac{1}{t}\} = -\{-t\}$. We need to restrict to an open subset:

(II) Let $W := \mathbb{A}^1 \times \mathbb{P}^1 \setminus (\Delta \cup \mathbb{A}^1 \times 0)$

$(\mathbb{A}^1 \times \mathbb{A}^1 \setminus \Delta) \times X \supset U := W \setminus (\mathbb{A}^1 \times \infty) = \mathbb{A}^1 \times \mathbb{P}^1 \setminus (\Delta \cup \mathbb{A}^1 \times 0 \cup \mathbb{A}^1 \times \infty)$

it lands in $(\mathbb{A}^1 \setminus 0) \times X$!

$\tilde{p}_2 : U \times X \hookrightarrow (\mathbb{A}^1 \times \mathbb{A}^1 \setminus \Delta) \times X \longrightarrow (\mathbb{A}^1 \setminus 0) \times X$ restriction of p_2 at $U \times X$

$\tilde{\partial}_\infty$ the boundary map for $\mathbb{A}^1 \times \infty \times X \hookrightarrow W \times X$ $\tilde{\partial}_\infty : U \times X \longrightarrow \mathbb{A}^1 \times \infty \times X$

We have that

$$\partial_\infty \circ \{s-t\} \circ p_2^* = \tilde{\partial}_\infty \circ \{s-t\} \circ \tilde{p}_2^* \circ j^*$$

this is given by $s-t \in \mathcal{O}(U \times X)^\times$, image of $s-t \in \mathcal{O}((\mathbb{A}^1 \times \mathbb{A}^1 \setminus \Delta) \times X)^\times$ along $\mathcal{O}((\mathbb{A}^1 \times \mathbb{A}^1 \setminus \Delta) \times X) \longrightarrow \mathcal{O}(U \times X)$

because, following the definition of p_2^*, given a point $w \in \mathbb{A}^1 \times X$, the generic point ξ of $p_2^{-1}(w) \subset (\mathbb{A}^1 \times \mathbb{A}^1 \setminus \Delta) \times X$ is also inside $U = \mathbb{A}^1 \times \mathbb{A}^1 \setminus (\Delta \cup \mathbb{A}^1 \times 0)$. So, we can restrict to $U \times X$.

Then, consider $u := \frac{s-t}{-t} \in \mathcal{O}(W \times X)^\times$. it is a unit b/c W is $\mathbb{A}^1 \times \mathbb{P}^1$ where we removed Δ and $\mathbb{A}^1 \times 0$!

We have that

$$\tilde{\partial}_\infty \circ \{s-t\} \circ \tilde{p}_2^* = \tilde{\partial}_\infty \circ \{-t\} \circ \tilde{p}_2^* + \tilde{\partial}_\infty \circ \{u\} \circ \tilde{p}_2^* =$$

$\{s-t\} = \{-tu\} = \{-t\} + \{u\}$

we use a compatibility result btw boundary maps and multiplication by global unit: (see [Ros, lemma 4.3]):

$$\partial_Y^U \circ \{j^*(a)\} = -\{i^*(a)\} \circ \partial_Y^U$$

where $a \in \mathcal{O}(X)^\times$

Apply it to $X \rightsquigarrow W \times X$, $Y \rightsquigarrow \mathbb{A}^1 \times \infty \times X$, $a \rightsquigarrow u$

$$= \tilde{\partial}_\infty \circ \{-t\} \circ \tilde{p}_2 - \underbrace{\underset{1}{\{u\}} \circ \tilde{\partial}_\infty \circ \tilde{p}_2^*}_{=0} =$$

$$= \tilde{\partial}_\infty \circ \{-t\} \circ \tilde{p}_2^*$$

is the multiplication by 0!

where $\mathcal{O}(W \times X) \longrightarrow \mathcal{O}(\mathbb{A}^1 \times \infty \times X)$

$u \longmapsto u = \frac{s-t}{-t} = 1 - \underset{\substack{\parallel \\ 1}}{\frac{s}{t}}$

we still call it u

b/c $t = \infty$ on $\mathbb{A}^1 \times \infty \times X$

Finally, we have:

analogously to I, $\overline{p}_{1*} \circ i_{\infty*} = (\overline{p}_1 \circ i_\infty)_*$ is an iso b/c $\overline{p}_1 \circ i_\infty : \mathbb{A}^1 \times \infty \times X \xrightarrow{\cong} \mathbb{A}^1 \times X$. With a bit of abuse, we can stop writing it.

$$\overline{p}_{1*} \circ i_{\infty*} \circ \partial_\infty \circ \{s-t\} \circ p_2^* = \partial_\infty \circ \{s-t\} \circ p_2^*$$

1st equality: we restrict to U

$$= \tilde{\partial}_\infty \circ \{s-t\} \circ \tilde{p}_2^* \circ j^*$$

2nd equality: we replace $\{s-t\}$ with $\{-t\}$

$$= \tilde{\partial}_\infty \circ \{-t\} \circ \tilde{p}_2^* \circ j^* =$$

we use a compatibility of pullback with multiplication by units (see [Ros, Lemma 4.3]):

$g^* \circ \{a\} = \{g^* a\} \circ g^*$

Apply to $g \rightsquigarrow \tilde{p}_2$, $a \rightsquigarrow -t$

$$= \tilde{\partial}_\infty \circ \tilde{p}_2^* \circ \{-t\} \circ j^* =$$

$$= \pi^* \circ \partial \circ \{-t\} \circ j^* =$$

we use a compatibility result btw flat pullback and boundary maps (see [Ros, Prop 4.4]):

Given

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d, "\overline{h}"'] & X \arrow[d, "h"] & U = X \setminus Y \arrow[l, hook'] \arrow[d, "\overline{\overline{h}}"] \\ Y' \arrow[r, hook] & X' & U' = X' \setminus Y \arrow[l, hook'] \end{tikzcd}, \quad h \text{ flat}$$

then $\overline{h}^* \circ \partial_{Y'}^{U'} = \partial_Y^U \circ \overline{\overline{h}}^*$

Apply to:

$$\begin{tikzcd} \mathbb{A}^1 \times X \cong \mathbb{A}^1 \times \infty \times X \arrow[r, hook] \arrow[d, "\pi"] & W \times X \arrow[d, "\text{projection on 2nd comp.}"] & U \times X \arrow[l, hook'] \arrow[d, "\tilde{p}_2"] \\ X \cong \infty \times X \arrow[r, hook] & (\mathbb{P}^1 \setminus 0) \times X & \mathbb{A}^1 \times X \arrow[l, hook'] \end{tikzcd}$$

these are the isos we are omitting!

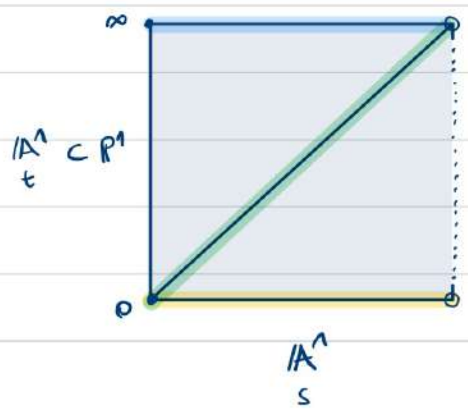
$-\{-\frac{1}{t}\} = -\{-t^{-1}\} = -(-\{-t\}) = \{-t\}$

$$= -\pi^* \circ \underbrace{\partial \circ \{-\tfrac{1}{t}\} \circ j^*}$$

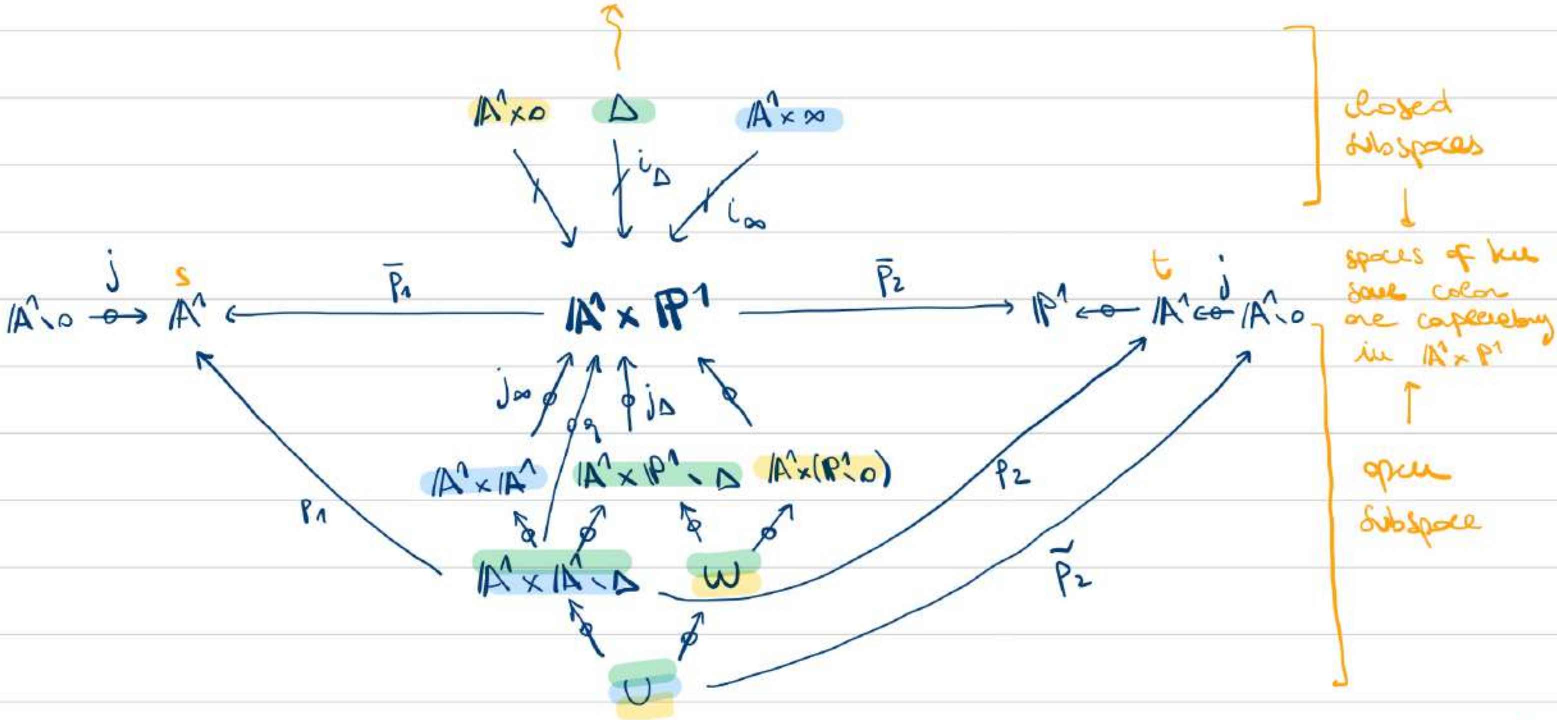
def of r

$$= -\pi^* \circ r$$

Summary of the morphisms involved.



Notice that the Δ is closed in $\mathbb{A}^1\times\mathbb{P}^1$ b/c $\Delta_\infty \subsetneq \mathbb{P}^1\times\mathbb{P}^1$ is closed, so $\Delta = \Delta_\infty \cap \mathbb{A}^1\times\mathbb{P}^1$ is closed in $\mathbb{A}^1\times\mathbb{P}^1$. (diagonal of $\mathbb{P}^1$)



and apply $\_\times X$

§2. Case $\pi : \mathbb{A}^n\times X \longrightarrow X$

Denote $\pi^n_Y : \mathbb{A}^n\times Y \longrightarrow Y$.

We construct an h-dda for π^*_X by induction.

The base step for $n=1$ is §1.

Now assume that we have constructed an h-dda (r^n_Y, H^n_Y) for any $\pi^{n\,*}_Y$. We want to construct an h-dda (r^{n+1}_X, H^{n+1}_X) for $\pi^{n+1\,*}_X$.

Notice that π^{n+1}_X factors as

$$\begin{tikzcd} \mathbb{A}^n\times Y = \mathbb{A}^{n+1}\times X \arrow[rr, "\pi^{n+1}_X"] \arrow[dr, "\pi^n_Y = \pi^n_{\mathbb{A}^1\times X}"'] & & X \\ & \mathbb{A}^1\times X =: Y \arrow[ur, "\pi"'] & \end{tikzcd}$$

notation of §1!

By inductive hp we have the h-dda for $\pi^{n\,*}_Y : (r^n_Y, H^n_Y)$, and we also have the h-dda for $\pi^* : (r, H)$.

We define

$$r^{n+1}_X := r\circ r^n_Y$$
$$H^n_X := H^n_Y + \pi^{n\,*}_Y \circ H \circ r^n_Y$$

$H : \mathbb{A}^1\times X \longrightarrow \mathbb{A}^1\times X$

$r^n_Y : \mathbb{A}^{n+1}\times X = \mathbb{A}^n\times Y \longrightarrow Y = \mathbb{A}^1\times X$

$H^n_Y : \mathbb{A}^{n+1}\times X = \mathbb{A}^n\times Y \longrightarrow \mathbb{A}^n\times Y = \mathbb{A}^{n+1}\times X$

$\pi^{n\,*}_Y : \mathbb{A}^1\times X = Y \longrightarrow \mathbb{A}^n\times Y = \mathbb{A}^{n+1}\times X$

Checking that properties (1)–(4) are satisfied is very straitforward. The most complicated thing one has to do is apply leibnitz rule for δ in (4).

§3. General case

To treat the general case we need the following def.

Def: Given $\pi : V \to X$ a vector bundle, a COORDINATION $\tau = (X_i, \tau_i)$ of π is the datum of a finite sequence of closed subsets of X

$$\emptyset = X_0 \subsetneq X_1 \subsetneq \dots \subsetneq X_\ell = X$$

with trivializations on the adjacent open complements

$$\begin{tikzcd}
V|_{X_{i+1} \setminus X_i} \arrow[r, "\simeq"] \arrow[d, "\pi|_{X_{i+1}\setminus X_i}"'] & \mathbb{A}^n \times (X_{i+1} \setminus X_i) \arrow[dl, "\text{projection}"] \\
X_{i+1} \setminus X_i &
\end{tikzcd}$$

where n is the rank of π

Notice that, under our hp, any vector bundle admits a coordination b/c the schemes we are considering are noetherian.
(take $U \subset X$ trivializing open subset and take $X \setminus U \subsetneq X$. Repeat this with $X \setminus U$. This process ends by noetherianity of X).

Also notice that $\pi|_{X_1} = \pi|_{X_1 \setminus \emptyset}$ is a trivial vector bundle, so, by §2, we can construct an h-data for $\pi|_{X_1}^*$.
Also $\pi|_{X_2 \setminus X_1}$ is trivial, so we can construct an h-data for $\pi|^*_{X_2 \setminus X_1}$.
The idea is to glue them and iterate the process until we arrive to construct an h-data for π^*.
So, we reduce to consider the following situation:

$$\begin{tikzcd}
V_U \arrow[r, hook] \arrow[d, "\pi_U"'] \arrow[dr, phantom, "\lrcorner", very near start] & V \arrow[d, "\pi"] & V_Y \arrow[l, hook'] \arrow[d, "\pi_Y"] \arrow[dl, phantom, "\llcorner", very near start] \\
U \arrow[r, hook] & X & Y = X \setminus U \arrow[l, hook']
\end{tikzcd}$$

and we assume that we have h-data (r_U, H_U) for π_U^*
and (r_Y, H_Y) for π_Y^*

Let ∂ be the boundary map associated to $V_Y \hookrightarrow V$

We define

$$r = \begin{pmatrix} r_Y & -r_Y\circ\partial\circ H_U \\ 0 & r_U \end{pmatrix} \qquad H = \begin{pmatrix} H_Y & -H_Y\circ\partial\circ H_U \\ 0 & H_U \end{pmatrix}$$

where the matrix notation refers to the decompositions of Morse complexes

$$C_*(V;\Lambda) = C_*(V_Y;\Lambda) \oplus C_*(V_U;\Lambda)$$
$$C_*(X;\Lambda) = C_*(Y;\Lambda) \oplus C_*(U;\Lambda)$$

we first put the closed and then the open!

Also in this case the proofs are straightforward.
It's worth to check just (4) to see why we ask for the mysterious condition (3) in the def of strong homotopy equivalence:

we can see from §1. and following the other definitions that $sgn(H) = -1$

$$\delta(H) = d\circ H + H\circ d =$$

$$= \begin{pmatrix} d_{V_Y} & \partial \\ 0 & d_{V_U} \end{pmatrix} \circ \begin{pmatrix} H_Y & -H_Y\circ\partial\circ H_U \\ 0 & H_U \end{pmatrix} + \begin{pmatrix} H_Y & -H_Y\circ\partial\circ H_U \\ 0 & H_U \end{pmatrix} \circ \begin{pmatrix} d_{V_Y} & \partial \\ 0 & d_{V_U} \end{pmatrix} =$$

$$= \begin{pmatrix} d_{V_Y}\circ H_Y & -\boxed{d_{V_Y}\circ H_Y}\circ\partial\circ H_U + \partial\circ H_U \\ 0 & d_{V_U}\circ H_U \end{pmatrix} +$$

$$+ \begin{pmatrix} H_Y\circ d_{V_Y} & H_Y\circ\partial - H_Y\circ\partial\circ H_U\circ d_{V_U} \\ 0 & H_U\circ d_{V_U} \end{pmatrix} =$$

$$= \begin{pmatrix} d_{V_Y}\circ H_Y & H_Y\circ\underbrace{d_{V_Y}\circ\partial}_{=\partial\circ d_{V_U}}\circ H_U - \cancel{\partial\circ H_U} + \pi_Y^*\circ r_Y\circ\partial\circ H_U + \cancel{\partial\circ H_U} \\ 0 & d_{V_U}\circ H_U \end{pmatrix} +$$

$$+ \begin{pmatrix} H_Y\circ d_{V_Y} & \cancel{H_Y\circ\partial} + H_Y\circ\partial\circ d_{V_U}\circ H_U - \cancel{H_Y\circ\partial} + \overbrace{\underbrace{H_Y\circ\underbrace{\partial\circ\pi_U^*}_{=\pi_Y^*\circ\partial}}_{H_Y\circ\pi_Y^*=0}\circ r_U}^{=0} \\ 0 & H_U\circ d_{V_U} \end{pmatrix} =$$

HERE WE USE THE CONDITION (3)!

$$\delta(H_Y) = d_{V_Y}\circ H_Y + H_Y\circ d_{V_Y} \quad\to\quad \boxed{d_{V_Y}\circ H_Y} = -H_Y\circ d_{V_Y} + id_{V_Y} - \pi_Y^*\circ r_Y$$
$$\| $$
$$id_{V_Y} - \pi_Y^*\circ r_Y$$

analogously $H_U\circ d_{V_U} = -d_{V_U}\circ H_U + id_{V_U} - \pi_U^*\circ r_U$

$$= \begin{pmatrix} d_{V_Y} \circ H_Y + H_Y \circ d_{V_Y} & \pi_Y^* \circ r_Y \circ \partial \circ H_U \\ 0 & d_{V_U} \circ H_U + d_{V_U} \circ H_U \end{pmatrix} =$$

$$= \begin{pmatrix} \delta(H_Y) & \pi_Y^* \circ r_Y \circ \partial \circ H_U \\ & \delta(H_U) \end{pmatrix} =$$

$$= \begin{pmatrix} id_{V_Y} - \pi_Y^* \circ r_Y & \pi_Y^* \circ r_Y \circ \partial \circ H_U \\ 0 & id_{V_U} - \pi_U^* \circ r_U \end{pmatrix} =$$

$$= \begin{pmatrix} id_{V_Y} & 0 \\ 0 & id_{V_U} \end{pmatrix} - \begin{pmatrix} \pi_Y^* & 0 \\ 0 & \pi_U^* \end{pmatrix} \circ \begin{pmatrix} r_Y & r_Y \circ \partial \circ H_U \\ 0 & r_U \end{pmatrix} =$$

$$= id_V - \pi^* \circ r$$

<u>ex</u>: Computation of $A_p(\mathbb{A}^n \setminus 0; M)$.
Use the long exact sequence of localization for $0 \hookrightarrow \mathbb{A}^n$

$$A_p(0;M) \to A_p(\mathbb{A}^n;M) \to A_p(\mathbb{A}^n \setminus 0;M) \to A_{p-1}(0;M) \to \dots$$

The Rost complex of the point Spec k is concentrated in deg 0:

$$0 \to M(k) \to 0 \to 0 \dots$$

So $A_p(0;M) \cong \begin{cases} M(k) & p=0 \\ 0 & \text{else} \end{cases}$

By homotopy invariance, we deduce that

$$A_p(\mathbb{A}^n;M) \cong \begin{cases} M(k) & p=n \\ 0 & \text{else} \end{cases}$$

$\boxed{n \geq 2}$ Since $n \geq 2$, so $n-1 \geq 1$, then

$$\underset{\substack{\| \\ 0}}{A_n(0;M)} \to \underset{\substack{\| \\ M(k)}}{A_n(\mathbb{A}^n;M)} \xrightarrow{\cong} A_n(\mathbb{A}^n \setminus 0;M) \to \underset{\substack{\| \\ 0}}{A_{n-1}(0;M)}$$

We also have, since $n \geq 1$,

$$\underset{\substack{\| \\ 0}}{A_1(\mathbb{A}^n;M)} \to A_1(\mathbb{A}^n \setminus 0;M) \xrightarrow{\cong} \underset{\substack{\| \\ M(k)}}{A_0(0;M)} \to \underset{\substack{\| \\ 0}}{A_0(\mathbb{A}^n;M)}$$

For other $p \neq 1, n$, we have

$$\underset{\substack{\| \\ 0}}{A_p(\mathbb{A}^n;M)} \to \underset{\substack{\| \\ 0}}{A_p(\mathbb{A}^n \setminus 0;M)} \to \underset{\substack{\| \\ 0}}{A_{p-1}(0;M)}$$

So, for $n \geq 2$

$$A_p(\mathbb{A}^n \setminus 0;M) = \begin{cases} M(k) & p = 0, n \\ 0 & \text{else}. \end{cases}$$

$\boxed{n = 1}$ For any $p \neq 1$ we have

$$\underset{\substack{\| \\ 0}}{A_p(\mathbb{A}^1;M)} \to \underset{\substack{\| \\ 0}}{A_p(\mathbb{A}^1 \setminus 0;M)} \to \underset{\substack{\| \\ 0}}{A_{p-1}(0;M)}$$

and the SES

$$\underset{\substack{\| \\ 0}}{A_1(0;M)} \to \underset{\substack{\| \\ M(k)}}{A_1(\mathbb{A}^1,M)} \to A_1(\mathbb{A}^1 \setminus 0,M) \xrightarrow{\partial} \underset{\substack{\| \\ M(k)}}{A_0(0,M)} \to \underset{\substack{\| \\ 0}}{A_0(\mathbb{A}^1;M)}.$$

Notice that this SES splits, because:

Let $j: \mathbb{A}^1 \setminus 0 \hookrightarrow \mathbb{A}^1$ be the open inversion

$p: \underset{\substack{\| \\ \operatorname{Spec} k[t]}}{\mathbb{A}^1} \to \operatorname{Spec} k \cong 0$ be the structure morphism

Consider

$$\{t\} \circ j^* \circ p^* : A_0(0;M) \to A_1(\mathbb{A}^1 \setminus 0;M).$$

It is a section of the SES b/c

$$\partial \circ \{t\} \circ j^* \circ p^* = \partial \circ \{t\} \circ (p \circ j)^* = \mathrm{id}_0$$

by Lemma 4.5 applied to

$f \rightsquigarrow p: \mathbb{A}^1 \to 0$

$t \rightsquigarrow t$

$\rightsquigarrow \tilde{f}$ is $p \circ j$

∂ is ∂

We conclude that

$$A_p(\mathbb{A}^1 \setminus 0;M) \cong \begin{cases} M(k) \oplus M(k) & p = 1 \\ 0 & \text{else}. \end{cases}$$