Lecture 5: Deformation to the normal bundle

§ The deformation space

Def

Let $Y \overset{i_Y}{\hookrightarrow} X$ be a closed immersion in Sch_k ← finite type separated k-schemes
and let $\mathcal{I} = \mathcal{I}_Y \subset \mathcal{O}_X$ be the ideal sheaf.

We have the graded $\mathcal{O}_X$ algebra $\bigoplus \mathcal{I}^n \subset \mathcal{O}_X$ and the graded $\mathcal{O}_Y$-algebra $\bigoplus \mathcal{I}^n/\mathcal{I}^{n+1}$, $\mathcal{O}_X/\mathcal{I} = \mathcal{O}_Y$. $n \geq 0$

1) The normal cone of Y in X, N_YX is the Y-scheme

$$N_YX := \mathrm{Spec}_{\mathcal{O}_Y} \bigoplus \mathcal{I}^n/\mathcal{I}^{n+1},$$

with $N_YX \to Y$ induced by $\mathcal{O}_Y \to \bigoplus \mathcal{I}^n/\mathcal{I}^{n+1}$

2) The blow-up of X along Y, Bl_YX is $\mathrm{Proj}_{\mathcal{O}_X} \bigoplus_{n\geq 0} \mathcal{I}^n$. We have the cartesian diagram

$$\begin{tikzcd}
E := \mathrm{Proj}_{\mathcal{O}_Y} \bigoplus \mathcal{I}^n/\mathcal{I}^{n+1} \arrow[r, hook, "i_E"'] \arrow[d] & Bl_YX \arrow[d, "\pi"] & Bl_YX \arrow[l, hook', "j_Y"'] \arrow[d, "\wr"] \\
Y \arrow[r, hook, "i_Y"'] & X & X \setminus Y \arrow[l, hook', "j_Y"']
\end{tikzcd}$$

E is the exceptional divisor of π, a Cartier divisor on Bl_YX

3) The deformation space $D(X,Y) \to X\times\mathbb{A}^1$ is the open subscheme $Bl_{Y\times 0} X\times\mathbb{A}^1 \setminus Bl_{Y\times 0} X\times 0$ of $Bl_{Y\times 0} X\times\mathbb{A}^1 \to X\times\mathbb{A}^1$, where the closed immersion $Bl_{Y\times 0} X\times 0 \hookrightarrow Bl_{Y\times 0} X\times\mathbb{A}^1$ is induced by the closed immersion $i_0 : X\times 0 \hookrightarrow X\times\mathbb{A}^1$

<u>Remark</u>) Explicitly: Write $\mathbb{A}^1 = \mathrm{Spec}\, k[t]$. Then the ideal sheaf $\mathcal{I}_{Y\times 0} \subset \mathcal{O}_{X\times\mathbb{A}^1}$ is $(\mathcal{I}, t)$, the ideal sheaf generated by $p_X^{-1}\mathcal{I}$ and $p_{\mathbb{A}^1}^{-1}(t)$. Then

$$Bl_{Y\times 0} X\times\mathbb{A}^1 = \mathrm{Proj}_{\mathcal{O}} \bigoplus (\mathcal{I},t)^n$$

and the closed immersion $Bl_{Y\times 0} X\times 0 \overset{X\times\mathbb{A}^1}{\cong} Bl_Y X \hookrightarrow Bl_{Y\times 0} X\times\mathbb{A}^1$ is induced by the surjection $\bigoplus (\mathcal{I},t)^n \to \bigoplus \mathcal{I}^n$ with kernel $(t_1)\mathcal{O}_{X\times\mathbb{A}^1}$, where $t_1 = t \in (\mathcal{I},t)^1 \subset \bigoplus_{n\geq 0}(\mathcal{I},t)^n$

2) $D(X,Y) \to X\times\mathbb{A}^1$ is affine with

$$D(X,Y) = \mathrm{Spec}_{\mathcal{O}_{X\times\mathbb{A}^1}}\left[\bigoplus_{n\geq 0}(\mathcal{I},t)^n[t_1^{-1}]\right]_0$$

for $X = \mathrm{Spec}\,A$ affine, Y defined by $I\subset A$,

$$D(X,Y) = \mathrm{Spec}\ \underbrace{A[t] + \sum I^n t^{-n}}_{\text{a sub alg}}$$

$$A[t,t^{-1}]$$

<u>Prop</u> 1) We have a cartesian diagram

$$\begin{tikzcd}
 & N_X Y \arrow[r, hook, "i"] \arrow[d] & D(X,Y) \arrow[d, "\mu"] & X\times(\mathbb{A}^1\setminus 0) \arrow[l, hook', "j"'] \arrow[d, "="] \\
Y\times 0 \arrow[ur, hook] & X\times 0 \arrow[r, hook, "i_0"] & X\times\mathbb{A}^1 & X\times(\mathbb{A}^1\setminus 0) \arrow[l, hook', "j_0"']
\end{tikzcd}$$

s

2) μ is flat

3) If $Y\subset X$ is a regular embedding then $N_X Y\to Y$ is a vector bundle, normal bundle

<u>Proof</u> of 1 We have the cartesian diagram

$$\begin{tikzcd}
E \arrow[r, hook] \arrow[d] & \mathrm{Bl}_{Y\times 0} X\times\mathbb{A}^1 \arrow[d] & X\times\mathbb{A}^1\setminus Y\times 0 \arrow[l, hook'] \arrow[d, equal] \\
Y\times 0 \arrow[r, hook] & X\times\mathbb{A}^1 & X\times\mathbb{A}^1\setminus Y\times 0 \arrow[l, hook']
\end{tikzcd}$$

and $E \cap Bl_{Y\times 0} X\times 0 \supset \tilde{E}$

$$\begin{tikzcd} \tilde{E} \arrow[r, hook] \arrow[d] & Bl_{Y\times 0} X\times 0 \arrow[r, hook] \arrow[d] & (X \setminus Y)\times 0 \arrow[d, equal] \\ Y\times 0 \arrow[r, hook] & X\times 0 \arrow[r, hookleftarrow] & (X\setminus Y)\times 0 \end{tikzcd}$$

gives cartesian diagram

$$\begin{tikzcd} & E\setminus\tilde{E} \arrow[r, hook] \arrow[d] \arrow[dl] & D(X,Y) \arrow[r, hookleftarrow] \arrow[d] & X\times(\mathbb{A}^1\setminus 0) \arrow[d, equal] \\ Y\times 0 \arrow[r, hook] & X\times 0 \arrow[r, hook] & X\times \mathbb{A}^1 \arrow[r, hookleftarrow] & X\times(\mathbb{A}^1\setminus 0) \end{tikzcd}$$

$$E\setminus \tilde{E} \hookrightarrow E = \mathrm{Proj}\left(\bigoplus_n \frac{(\mathcal{I},t)^n}{(\mathcal{I},t)^{n+1}}\right)$$

$$\| $$

$$\mathrm{Spec}_{\mathcal{O}_Y}\left[\left(\bigoplus \frac{(\mathcal{I},t)^n}{(\mathcal{I},t)^{n+1}}\right)[t_1^{-1}]\right]_0 \qquad \frac{(\mathcal{I},t)^n}{(\mathcal{I},t)^{n+1}} = \frac{\bigoplus_{j=0}^{n} \mathcal{I}^j t_1^{n-j}}{\bigoplus_{j=0}^{n} \mathcal{I}^{j+1} t_1^{n-j} \oplus \mathcal{I}^0 t_1^{n+1}}$$

$$\| \qquad \qquad \bigoplus_{j=0}^{n} \frac{\mathcal{I}^j}{\mathcal{I}^{j+1}} t_1^j \ (\mathrm{mod}\, t)$$

$$\mathrm{Spec}\left(\bigoplus \mathcal{I}^n/\mathcal{I}^{n+1}\right) = N_Y X$$

(Matsumura Thm 49)

2) flatness: $\forall f$: $A[t] \oplus \bigoplus I^n t^{-n} \ni$ is a non-zero divisor

and $(A[t] \oplus \bigoplus I^n t^{-n})[t^{-1}] = A[t, t^{-1}]$

3) Suppose $X = \operatorname{Spec} A$, $A \supset I$, $Y = V(I)$
and $I = (f_1, \dots, f_r)$ regular sequence
$\bar{f}_i = $ im of f_i in I/I^2. Then

i) I/I^2 is a free A/I module with basis $\bar{f}_1, \dots, \bar{f}_r$
rank r

ii) $\operatorname{Sym}^* I/I^2 \xrightarrow{\sim} \bigoplus_{n \geq 0} I^n/I^{n+1}$

(eg Matsumura - Comm Alg)

Globally this says $\mathcal{J}/\mathcal{J}^2$ is a locally free $\mathcal{O}_Y$-module
and $\operatorname{Sym}^* \mathcal{J}/\mathcal{J}^2 \xrightarrow{\sim} \bigoplus_{n \geq 0} \mathcal{J}^n/\mathcal{J}^{n+1}$

$\Rightarrow N_Y X = \operatorname{Spec}_{\mathcal{O}_Y} \operatorname{Sym}^* \mathcal{J}/\mathcal{J}^2$ and
the sheaf of sections of $N_Y X$ is $(\mathcal{J}/\mathcal{J}^2)^\vee$

$\Rightarrow N_Y X \to Y$ is a vector bundle

<u>Remark</u> The closed immersion $Y \times \mathbb{A}^1 \hookrightarrow X \times \mathbb{A}^1$
defines the closed immersion $Y \times \mathbb{A}^1 = \operatorname{Bl}_{Y \times 0} Y \times \mathbb{A}^1 \hookrightarrow \operatorname{Bl}_{Y \times 0} X \times \mathbb{A}^1$

$\operatorname{Bl}_{Y \times 0} Y \times \mathbb{A}^1 \subset D(X,Y) \Leftarrow \operatorname{Spec}_{\mathcal{O}_{X \times \mathbb{A}^1}} \bigoplus (I,t)^n \quad \operatorname{Proj}_{\mathcal{O}_{Y \times \mathbb{A}^1}} \bigoplus \mathcal{O}_{Y \times \mathbb{A}^1} t^n \quad \operatorname{Proj} \bigoplus (\mathcal{O},t)^n$

§ The double deformation space

<u>Def</u> Let $Z \overset{i_Y^Z}{\hookrightarrow} Y \overset{i_X^Y}{\hookrightarrow} X$ be closed immersions. We have

$Z \times \mathbb{A}^1 \hookrightarrow Y \times \mathbb{A}^1 \hookrightarrow D(X, Y)$. Define

$$\begin{array}{c} \overline{D}(X, Y, Z) = D(D(X,Y), Z \times \mathbb{A}^1) \\ \downarrow \\ X \times \mathbb{A}^1 \times \mathbb{A}^1 \end{array}$$

<u>Remark</u> We have

$$\underset{Z \times \mathbb{A}^1 \times 0}{Bl} \underset{U \text{ open}}{(Bl_{Y \times 0}\, X \times \mathbb{A}^1)} \times \mathbb{A}^1$$

$$\underset{Z \times \mathbb{A}^1 \times 0}{Bl} \underset{U \text{ open}}{D(X, Y)} \times \mathbb{A}^1 = Bl_{Z \times \mathbb{A}^1 \times 0} (Bl_{Y \times 0}\, X \times \mathbb{A}^1 \setminus Bl_{Y \times 0}\, X \times 0) \times \mathbb{A}^1$$

$$\overline{D}(X, Y, Z) = Bl_{Z \times \mathbb{A}^1 \times 0}\, D(X,Y) \times \mathbb{A}^1 \setminus Bl_{Z \times \mathbb{A}^1 \times 0}\, D(X,Y) \times 0$$

Prop 1) $\bar{D}(X,Y,Z)\big|_{X\times\mathbb{A}^1\times(\mathbb{A}^1\setminus 0)} = D(X,Y)\times(\mathbb{A}^1\setminus 0)$

2) $\bar{D}(X,Y,Z)\big|_{X\times(\mathbb{A}^1\setminus 0)\times\mathbb{A}^1} \cong (\mathbb{A}^1\setminus 0)\times D(X,Z)$ (really $= p_{13}^{*} D(X,Z)$)

3) $\bar{D}(X,Y,Z)\big|_{X\times 0\times\mathbb{A}^1} = D(N_Y X, s_Y(Z))$

where $s_Y : Y \to N_Y X$ is the 0-section

4) $\bar{D}(X,Y,Z)\big|_{X\times\mathbb{A}^1\times 0} = N_{Z\times\mathbb{A}^1} D(X,Y) \hookleftarrow N_Z X\times(\mathbb{A}^1\setminus 0)$

$$\begin{tikzcd}
N_Z N_Y X \arrow[r, hook] \arrow[dr] & \bar{D}(X,Y,Z)\big|_{X\times\mathbb{A}^1\times 0} \arrow[d] & N_Z X\times(\mathbb{A}^1\setminus 0) \arrow[l, hook'] \arrow[d] \\
& X\times 0 \arrow[r] & X\times\mathbb{A}^1 & X\times(\mathbb{A}^1\setminus 0) \arrow[l, hook'] \\
& & t &
\end{tikzcd}$$

$Z\times(\mathbb{A}^1\setminus 0)$

Proof (1) $\bar{D} = D(D(X,Y), Z\times\mathbb{A}^1) \Rightarrow$

$\bar{D}\big|_{X\times\mathbb{A}^1\times(\mathbb{A}^1\setminus 0)} = D(X,Y)\times(\mathbb{A}^1\setminus 0)$

(2) $\bar{D}|_{X\times(\mathbb{A}^1\setminus 0)\times\mathbb{A}^1} = D(D(X,Y)|_{Y\times(\mathbb{A}^1\setminus 0)}, Z\times(\mathbb{A}^1\setminus 0))$

$$= D(X\times(\mathbb{A}^1\setminus 0), Z\times(\mathbb{A}^1\setminus 0))$$

$$= p_{13}^{*} D(X,Z)$$

$$\cong (\mathbb{A}^1\setminus 0)\times D(X,Z)$$

(3) Since $Bl_{Z\times\mathbb{A}^1\times 0}(D(X,Y)\times\mathbb{A}^1) \to X\times\mathbb{A}^1\times\mathbb{A}^1$ is flat

$\cup$

$\bar{D}$

we have

$$Bl_{Z\times\mathbb{A}^1\times 0}(D(X,Y)\times\mathbb{A}^1)|_{X\times 0\times\mathbb{A}^1} \overset{\text{open}}{\supset} \bar{D}|_{X\times 0\times\mathbb{A}^1}$$

$$= Bl_{Z\times 0}(D(X,Y)|_0\times\mathbb{A}^1)$$

$$= Bl_{Z\times 0}(N_Y X\times\mathbb{A}^1) \supset \bar{D}|_{X\times 0\times\mathbb{A}^1}$$

and same for $Bl_{Z\times\mathbb{A}^1\times 0}(D(X,Y)\times 0)$ & $Z\times 0 \subset N_Y X\times\mathbb{A}^1$

is $s_Y(Z)\times 0$

$$\Rightarrow \bar{D}|_{X\times 0\times\mathbb{A}^1} = Bl_{Z\times 0}(N_Y X\times\mathbb{A}^1)\setminus Bl_{Z\times 0}(N_Y X\times 0)$$

$$= D(N_Y X, s_Y Z).$$

(4) $\bar{D}(X, Y, Z)\big|_{X \times \mathbb{A}^1 \setminus 0} = D(D(X,Y), Z \times \mathbb{A}^1)\big|_{X \times \mathbb{A}^1 \setminus 0}$

$= N_{Z \times \mathbb{A}^1} D(X,Y)$

§ The specialization map. Take $Y \overset{i}{\hookrightarrow} X$ a closed immersion, giving the def diagram

$$\begin{array}{ccccccc} N_Y X & \hookrightarrow & D(X,Y) & \hookleftarrow & X \times (\mathbb{A}^1 \setminus 0) & \xrightarrow{\pi} & X \\ \downarrow & & \downarrow & & \| & & \\ 0 & \hookrightarrow & X \times \mathbb{A}^1 & \hookleftarrow & X \times (\mathbb{A}^1 \setminus 0) & & \end{array}$$

<u>Def</u> Define the specialization map
$$J(i) = J(X, Y) : X \longrightarrow N_Y X$$

as the composition
$$X \xrightarrow{\pi^*} X \times (\mathbb{A}^1 \setminus \{0\}) \xrightarrow{\{t\}} X \times (\mathbb{A}^1 \setminus \{0\}) \xrightarrow{\partial} N_Y X$$

<u>Note</u> • $J(i) : C_*(X; M) \to C_*(N_Y X, M)$ is a map of complexes of bidegree $(0,0)$

• for $M = K_*^M$ the induced map
$$\underset{\substack{\| \\ CH_p(X)}}{A_p(X, K_*^M, -p)} \to A_p(N_Y X, -p) = CH_p(N_Y X)$$
is Fulton's "specialization to the normal cone

Note Rost's convention for the closed immersion

$$N_X Y = \operatorname{Spec}_{\mathcal{O}_Y} \bigoplus_{n\geq 0} \mathcal{I}_Y^n/\mathcal{I}_Y^{n+1} \underset{i}{\overset{\sim}{\hookrightarrow}} D(X,Y) = \operatorname{Spec}_{\mathcal{O}_X[t]} \mathcal{O}_X[t] \oplus \sum \mathcal{I}_Y^n t^{-n}$$

has $i^*: \mathcal{I}_Y^n t^{-n} \to \mathcal{I}_Y^n/\mathcal{I}_Y^{n+1}$ defined as

$$i^*(a t^{-n}) = (-1)^n \operatorname{res}(a) \quad \operatorname{res}: \mathcal{I}^n \to \mathcal{I}^n/\mathcal{I}^{n+1}$$

and $\tilde{i}^*: \mathcal{O}_X[t] \to \mathcal{O}_Y = \mathcal{O}_X/\mathcal{I}_Y$ by $t \to 0$, $\mathcal{O}_X \xrightarrow{\text{res}} \mathcal{O}_Y$

"to avoid signs appearing later on"

Lemma (11.1) Let $Y \underset{i}{\hookrightarrow} X$ be a closed immersion with the 0-sect. $s_Y: Y \to N_Y X$. Then

$$J(X,Y) \circ i_* = s_{Y*} \qquad (4.1)(1)-(3)$$

pf $\quad J(X,Y) \circ i_*$ is the composite $(4.4)(1), (4.3)$

$$Y \xrightarrow{p_Y^*} Y \times \mathbb{A}^1 \setminus \{0\} \xrightarrow{\{t\}} Y \times \mathbb{A}^1 \setminus \{0\} \xrightarrow{\partial} Y \xrightarrow{s_{Y*}} N_Y X \qquad \partial \circ \{t\} \circ p_Y^* = \operatorname{id}$$

A description of $J(i)$ "at a generic point"

We extend to essentially finite type k-schemes

Let $\mathcal{O}$ be a dvr (ess. of finite type$/k$) with max ideal $m = (\pi)$ quotient field F res field κ, valuation v

$X = \operatorname{Spec}\mathcal{O} \supset Y = \operatorname{Spec}\kappa$. Then $N_Y X = \operatorname{Spec}\kappa[\bar{\pi}]$

$\bar{\pi} = \operatorname{res}(\pi) \in m/m^2$ and we have

$$N_Y X \underset{i}{\hookrightarrow} D(X,Y) = \operatorname{Spec}\mathcal{O}[t, \pi t^{-1}] \hookrightarrow \operatorname{Spec}\mathcal{O}[t, t^{-1}]$$

Let $\mathcal{D} = \mathcal{O}_{D(X,Y), N_Y X}$, a dvr with fract field $F(t)$ residue field E, giving a valuation w on $F(t)$

Let x = gen pt of $X \in X^{(0)}$, $E = \kappa(\bar{\pi}) = \kappa(e)$, $e \in E^{(0)}$

We have $J(X,Y) : C_\cdot(X, M) \to C_\cdot(N_Y X, M)$

$$\begin{array}{ccc} & \cup & & \cup \\ J^0 : & M(x) = M(F) & \to & M(e) = M(E) \end{array}$$

Lemma

(11.2) $J^0 = r_{E/\kappa} \circ s^v_\pi + \{\bar{\pi}\} \cdot r_{E/\kappa} \circ \partial_v$

Pf $N_Y X \subset D(X,Y) = \operatorname{Spec}\mathcal{O}[t, \pi t^{-1}]$ is a principal divisor with ideal $(t)\mathcal{O}[t, \pi t^{-1}]$

$$X \times_U (\mathbb{A}^1 \setminus 0) = \operatorname{Spec} \mathcal{O}[t, t^{-1}]$$

$$\operatorname{Spec} F \times (\mathbb{A}^1 \setminus 0) = \operatorname{Spec} F[t, t^{-1}] \supset \operatorname{Spec} F(t)$$

Then $$J^0 = \partial_\omega \circ \{t\} \circ r_{F(t)/F}$$

<u>Note</u> E = residue field $\kappa(\omega)$ for the valuation ω and for $i : N_Y X \hookrightarrow D(X, Y)$ we have $i^*(-\pi t^{-1}) = \bar{\pi} \in m/m^2$

$\in \mathcal{O}^\times$

so $\{t\} = \{t \cdot (-\pi t^{-1})\} - \{-\pi t^{-1}\}$

$\{-\pi\}$

Then for $\rho \in M(F)$ we have (compatibilities §4)

$$\partial_\omega \circ \{t\} \circ r_{F(t)/F}(\rho) = \partial_\omega \circ \{-\pi\} \circ r_{F(t)/F}(\rho) - \partial_\omega \{-\pi t^{-1}\} r_{F(t)/F}(\rho)$$

$$= r_{E/\kappa} \partial_\nu (\{-\pi\} \rho) + \{\bar{\pi}\} r_{E/\kappa} \partial_\nu (\rho)$$

$$= r_{E/\kappa} s^\nu_\kappa (\rho) + \{\bar{\pi}\} r_{E/\kappa} \partial_\nu (\rho) \quad \square$$

Lemma

(11.3) Let $g: X' \to X$ be flat,

$$\begin{tikzcd} Y' \arrow[r, hook] \arrow[d, "\bar g"'] & X' \arrow[d, "g"] \\ Y \arrow[r, hook] & X \end{tikzcd}$$ cartes.

gives $N(g): N_{Y'}X' = N_Y X \times_Y Y' \to N_Y X$

Then $N(g)^* \circ J(X,Y) = J(X',Y') \circ g^*$ | by $X' \times_X D(X,Y) = D(X',Y')$.

Pf g flat $\Rightarrow D(X',Y') \cong X' \times_X D(X,Y)$

$$\begin{tikzcd} D(X',Y') \arrow[r, "\cong"] \arrow[dr] & X' \times_X D(X,Y) \arrow[d, "p_2"] \\ & D(X,Y) \end{tikzcd}$$

$$\begin{tikzcd} N_{Y'}X' = Y' \times_Y N_Y X \arrow[r, hook] \arrow[d, "N(g)"'] \arrow[dr, phantom, "\ulcorner"] & D(X',Y') \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X' \times (\mathbb{A}^1 \setminus 0) \arrow[l, hook'] \arrow[r, "\pi'"] \arrow[d, "g \times id"] & X' \arrow[d, "g"] \\ N_Y X \arrow[r, hook] & D(X,Y) & X \times (\mathbb{A}^1 \setminus 0) \arrow[l, hook'] \arrow[r, "\pi"] & X \end{tikzcd}$$

$$\Rightarrow N(g)^* \circ \partial \circ \{t\} \circ \pi^{*\vee} \overset{(4,4)(2)}{=} \partial' \circ (g \times id)^* \circ \{t\} \circ \pi^*$$

$N(g)^* \circ J(X,Y)$

$$\overset{(4,3)}{=} \partial' \circ \{t\} \circ (g \times id)^* \circ \pi^*$$

$$= \partial' \circ \{t\} \circ \pi'^* \circ g^*$$

$$= J(X',Y') \circ g^* \quad \square$$

(11.4) Suppose we have a diagram
$$\begin{tikzcd} Y \arrow[r, hook] & X \arrow[d, "p"] \\ & W \end{tikzcd}$$
with p flat of rel. dim d. Suppose that the composition
$$\underbrace{N_Y X \to Y \hookrightarrow X \xrightarrow{p} W}_{g}$$
is also flat of rel. dim d. Then
$$J(X,Y)\circ p^* = g^* : W \rightsquigarrow N_Y X$$

Pf: Let $\pi_W : W\times(\mathbb{A}^1\setminus 0)\to W$ be the projections
$\pi_X : X\times(\mathbb{A}^1\setminus 0)\to X$, $\mu : D(X,Y)\to X\times\mathbb{A}^1$ structure map
and let f be the composition
$$D(X,Y)\xrightarrow{\mu} X\times\mathbb{A}^1 \xrightarrow{p\times id} W\times\mathbb{A}^1$$

$$\begin{tikzcd} X\times(\mathbb{A}^1\setminus 0) \arrow[r, "p\times id"] \arrow[d, "\pi_X"] \arrow[dr, phantom, "\ulcorner"] & W\times(\mathbb{A}^1\setminus 0) \arrow[d, "\pi_W"] \\ X \arrow[r] & W \end{tikzcd}$$

Then
$$J(X,Y)\circ p^* = \partial^{X\times(\mathbb{A}^1\setminus 0)}_{N_Y X}\circ\{t\}\circ\pi_X^*\circ p^*$$
$$= \partial\circ\{t\}\circ(p\times id)^*\circ\pi_W^* \qquad (§4)$$

f is flat and $f|N_Y X = g$

$$\begin{tikzcd} N_Y X \arrow[r, hook] \arrow[d] \arrow[dd, bend left, "g"] & D(X,Y) \arrow[dd, "f"] \arrow[ddl, phantom, "\ulcorner"] & X\times(\mathbb{A}^1\setminus 0) \arrow[l, hook'] \arrow[dd, "p\times id"] \arrow[ddl, phantom, "\ulcorner"] \\ Y\times 0 \hookrightarrow X\times 0 \arrow[d] & & \\ W\times 0 \arrow[r, hook] & W\times\mathbb{A}^1 & W\times(\mathbb{A}^1\setminus 0) \arrow[l, hook'] \end{tikzcd}$$
$$\underbrace{\qquad\qquad\qquad}_{\partial'}$$

$$J(X,Y)\circ p^* = \partial\circ\{t\}\circ(p\times id)^*\circ\pi_W^*$$
$$\overset{(4.3)}{=} \partial\circ(p\times id)^*\circ\{t\}\circ\pi_W^*$$
$$\overset{(4.4)(2)}{=} g^*\partial'\circ\{t\}\circ\pi_W^*$$
$$= g^* \quad \text{by Lemma 9.5 } (\partial'\circ\{t\}\circ\pi_W^* = id) \qquad \square$$

(11.5) Given a commutative diagram

$$\begin{tikzcd} & X' \arrow[d, "p"] \\ Y \arrow[ur, hook, "i'"] \arrow[r, hook, "i"] & X \end{tikzcd}$$

with p smooth and i, i' regular embeddings, we have the induced surjection of vector bundles on Y, $N(p) : N_Y X' \to N_Y X$. Then

$$J(X', Y) \circ p^* = Np^* \circ J(X, Y)$$

Pf. Let $\mathcal{I}_Y \subset \mathcal{O}_X$, $\mathcal{I}'_Y \subset \mathcal{O}_{X'}$ be the ideal sheaves. The map $p^* : \mathcal{O}_X \to \mathcal{O}_{X'}$ has $p^* \mathcal{I}_Y \subset \mathcal{I}'_Y$, since

$$\begin{tikzcd} \mathcal{O}_{D(X',Y)} \arrow[r, phantom, "="] & \mathcal{O}_{X'}[t] \oplus \bigoplus_{n \geq 1} \mathcal{I}'^{\,n}_Y t^{-n} \\ \mathcal{O}_{D(X,Y)} \arrow[u, "D(p)^*"] \arrow[r, phantom, "="] & \mathcal{O}_X[t] \oplus \bigoplus_{n \geq 1} \mathcal{I}_Y t^{-n} \arrow[u, "p^* \quad p^*"'] \end{tikzcd}$$

we have a commutative diagram

$$\begin{tikzcd} N_Y X' \arrow[r, hook] \arrow[d, "N(p)"] & D(X',Y) \arrow[d, "D(p)"] & X' \times (\mathbb{A}^1 \setminus 0) \arrow[l, hook'] \arrow[d, "p \times id"] \arrow[r, "p_{X'}"] & X' \arrow[d, "p"] \\ N_Y X \arrow[r, hook] & D(X,Y) & X \times (\mathbb{A}^1 \setminus 0) \arrow[l, hook'] \arrow[r, "p_X"] & X \end{tikzcd}$$

The left-hand square is cartesian since both $N_Y X$ and $N_Y X'$ are principal divisors in $D(X,Y)$, $D(X',Y)$ defined by t.

<u>Note</u> $D(p)$ is flat: Let $J \subset \mathcal{O}_{D(X,Y)}$ be an ideal sheaf

- $J \otimes_{\mathcal{O}} \mathcal{O}'$ f.gen $\mathcal{O}'$-module $\Rightarrow \bigcap_n (t^n)(J \otimes \mathcal{O}') = 0$ (Krull intersection theorem)

(Matsumura, thm. 49) $\rightsquigarrow$

- $\times t : \mathcal{O}' \to \mathcal{O}'$ is injective
- $\mathcal{O}/(t) \to \mathcal{O}'/(t)$ is flat ($N_Y X' \to N_Y X$)

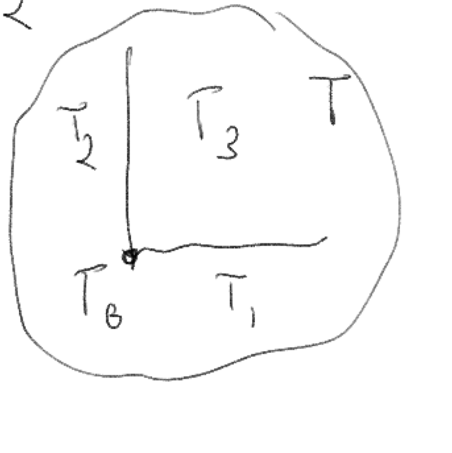
Thm:

$$
\begin{aligned}
J(X',Y)\circ p^* &= \partial_Y^{X'} \circ \{t\} \circ p_{X'}^* \circ p^* \\
\overset{(4.1)}{=}\ & \partial_Y^{X'} \circ \{t\} (p \times id)^* \circ p_X^* \\
\overset{(4.3)}{=}\ & \partial_Y^{X'} \circ (p \times id)^* \circ \{t\} \circ p_X^* \\
\overset{(4.4)(2)}{=}\ & N(p)^* \circ \partial_Y^{X} \circ \{t\} \circ p_X^* \quad \text{(flatness)} \\
&= N(p)^* \circ J(X,Y) \quad \square
\end{aligned}
$$

(11.6) Let T be a k-scheme with closed subschemes T_1, T_2. Let

- $T_0 = T_1 \cap T_2$ • $T_i' = T_i \setminus T_0$; $i = 1, 2$
- $T_3 = T \setminus (T_1 \cup T_2)$

b d



This gives boundary maps

$\partial_1^3 : T_3 \to \tilde{T}_1 \iff \tilde{T}_1 \hookrightarrow T \setminus T_2 \hookleftarrow T_3$

$\partial_2^3 : T_3 \to \tilde{T}_2 \iff \tilde{T}_2 \hookrightarrow T \setminus T_1 \hookleftarrow T_3$

$\partial_0^i : \tilde{T}_i \to T_0 \iff T_0 \hookrightarrow T_i \hookleftarrow \tilde{T}_i \qquad i = 1, 2$

Then there is a homotopy h of bidegree $(-1,-1)$

$$0 \underset{h}{\sim} \partial_0^1 \partial_1^3 + \partial_0^2 \partial_2^3 : T_3 \to T_0 \; ; \quad \delta(h) = \partial_0^1 \partial_1^3 + \partial_0^2 \partial_2^3$$

<u>proof</u> Write

$$C_\bullet(T, M) = C_\bullet(T_0, M) \oplus C_\bullet(\tilde{T}_1, M) \oplus C_\bullet(\tilde{T}_2, M) + C_\bullet(T_3, M)$$

This decomposes the differential d_T as

$$d_T = \sum_{i \leq j} d_i^j \qquad d_i^j : C_\bullet(\overset{(\sim)}{T_j}, M) \to C_\bullet(\overset{(\sim)}{T_i}, M)$$

where $d_i^i = d_{\overset{(\sim)}{T_i}}$ and $d_1^3 = \partial_1^3$, $d_2^3 = \partial_2^3$, $d_0^2 = \partial_0^2$, $d_0^1 = \partial_0^1$

Then looking at the component $C(T_3, M) \to C(T_0, M)$ of $0 = d_T \circ d_T$, we have

$$0 = (d_T d_T)_0^3 = \Big[\Big(\sum_{j \geq i} d_i^j\Big) \circ \Big(\sum_{j \geq i} d_i^j\Big)\Big]_0^3$$

$$= d_0^1 d_1^3 + d_0^2 d_2^3 + d_1^0 d_0^3 + d_0^3 d_3^3$$

$$= \partial_0^1 \partial_1^3 + \partial_0^2 \partial_2^3 + d_{T_0} d_0^3 + d_0^3 d_{T_3}$$

$$= \partial_0^1 \partial_1^3 + \partial_0^2 \partial_2^3 + \delta(d_0^3)$$

so take $h = -d_0^3$ $\square$

§ Some useful lemmas

Let $Z \hookrightarrow Y \hookrightarrow X$ be regular closed immersion. Recall

$$D(D(X,Y), Z\times \mathbb{A}^1)\Big|_{\mathbb{A}^1\times\{0\}} = N_{Z\times\mathbb{A}^1} D(X,Y)$$

$\hookleftarrow (s=0)$

and we have the cartesian diagram

$$\begin{tikzcd}
N_Z N_Y X \arrow[r, hook] \arrow[d] & N_{Z\times\mathbb{A}^1} D(X,Y) \arrow[d] & N_Z X \times (\mathbb{A}^1\setminus\{0\}) \arrow[l, hook'] \arrow[d] \\
0 \arrow[r, hook] & \mathbb{A}^1 & \mathbb{A}^1\setminus\{0\} \arrow[l, hook'] \\
 & & t
\end{tikzcd}$$

giving the boundary maps $\partial : N_Z X \times (\mathbb{A}^1\setminus\{0\}) \to N_Z N_Y X$

and projection $\pi : N_Z X \times (\mathbb{A}^1 \setminus\{0\}) \to N_Z X$

<u>Define</u> $\tilde{J}(N_Z X, N_Z N_Y X) = \partial \circ \{t\} \circ \pi^*$

Lemma 1 Let $Z \hookrightarrow Y \hookrightarrow X$ be regular embeddings

Take $\tilde{T} = \tilde{D} = D(D(X,Y), Z \times \mathbb{A}^1)$, $T_1 = \tilde{D}|_{\{0\} \times \mathbb{A}^1} = D(N_Y X, s_Y(Z))$

$T_2 = \tilde{D}|_{\mathbb{A}^1 \times \{0\}} = D(N_Z X, N_Z Y)$, $T_0 = T_1 \cap T_2$

and let $\pi : T_3 \to X$ be the projection, & write

$\mathbb{A}^1 \times \mathbb{A}^1 = \operatorname{Spec} k[t,s]$. Then

$\partial_0^1 \partial_1^3 \circ \{t,s\} \circ \pi^* = J(N_Y X, Z) \circ J(X,Y)$

$\partial_0^3 \circ \partial_2^3 \circ \{s,t\} \circ \pi^* = \tilde{J}(N_Z X, N_Z N_Y X) \circ J(X,Z)$ are homotopic,

as maps $C.(X,M) \to C.(N_Z N_Y X, M)$

Pf Let $\pi_s = \pi_{13} : X \times (\mathbb{A}^1 \setminus \{0\}) \times (\mathbb{A}^1 \setminus \{0\}) \to X \times (\mathbb{A}^1 \setminus \{0\}) = X \times \operatorname{Spec} k[s,s^{-1}]$

$\pi_t = \pi_{12} : X \times (\mathbb{A}^1 \setminus \{0\}) \times (\mathbb{A}^1 \setminus \{0\}) \to X \times (\mathbb{A}^1 \setminus \{0\}) = X \times \operatorname{Spec} k[t,t^{-1}]$

$\pi^s : X \times \operatorname{Spec} k[s,s^{-1}] \to X$ be the

$\pi^t : X \times \operatorname{Spec} k[t,t^{-1}] \to X$ projections

$\pi_{N_Y X} : N_Y X \times (\mathbb{A}^1 \setminus \{0\}) \to N_X Y$

$\pi_{N_Z X} : N_Z X \times (\mathbb{A}^1 \setminus \{0\}) \to N_Z X$

We have

$$\partial_0^1 \partial_1^3 \circ \{t,s\} \circ \pi^* = \partial_0^1 \circ \{s\} \circ \partial_1^3 \circ \{t\} \circ \pi_s^* \circ \pi^{s*}$$

$$= \partial_0^1 \circ \{s\} \circ \tilde{J}(X \times \{(\mathbb{A}^1,0)\}, Y \times (\mathbb{A}^1,0)) \circ \pi^{s*}$$

(11.3) $\quad = \partial_0^1 \circ \{s\} \circ \pi_{N_Y X}^* \circ \tilde{J}(X,Y)$

$\bar{D}|_{t=0} = D(N_Y X, Z) \quad = \tilde{J}(N_Y X, Z) \circ \tilde{J}(X,Y)$

and

$$\partial_0^2 \partial_2^3 \circ \{s,t\} \circ \pi^* = \partial_0^2 \circ \{t\} \circ \partial_2^3 \circ \{s\} \circ \pi_t^* \circ \pi^{t*}$$

$$= \partial_0^2 \circ \{t\} \circ \tilde{J}(X \times (\mathbb{A}^1,0), Z \times (\mathbb{A}^1,0)) \circ \pi^{t*}$$

(11.3) $\quad = \partial_0^2 \circ \{t\} \circ \pi_{N_Z X}^* \circ \tilde{J}(X,Z)$

$\bar{D}|_{s=0} = N_{Z \times \mathbb{A}^1} D(X,Y) \quad = \tilde{\tilde{J}}(N_Z X, N_Z N_Y X) \circ \tilde{J}(X,Z)$ $\square$

<u>Lemma 2</u> Let $Z \hookrightarrow Y \hookrightarrow X$ be regular embedding. Let $\pi^{N_Z X}: N_Z X \to Z$, $\pi^{N_Z N_Y X}: N_Z N_Y X \to Z$ be the projections.

Then $\tilde{\tilde{J}}(N_Z X, N_Z N_Y X) \circ \pi^{N_Z X *} = \pi^{N_Z N_Y X *}$